Equilibrium Analysis and Social Optimization of a Selectable Single or Time-Based Batch Service

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Keywords: Queueing Game, Mode Choice, Batch Service, Clearing System, Nash Equilibrium, Social Optimization, Social Optimal Pricing.

Abstract: In current transportation systems, a common road is shared by multiple types of vehicles with different capacities. To consider this phenomenon, we propose a model in which customers can strategically select a single or batch service, and then receive a common service in a single-server queue with exponential service times. Customers potentially arrive at the system according to a Poisson process and choose whether to join the queue directly or wait for a batch service. The batch service commences periodically according to a Poisson process and the capacity of the batch is determined by a geometric distribution. The optimization of such a model has not been studied despite being an important social issue. We derive the unique equilibrium strategy of customers, socially optimal strategy, and socially optimal relationship of fees for both services. Furthermore, we demonstrate that these optimal fees exhibit linear relationships. In terms of practical application, this system will allow us to consider the effects of road congestion on transportation platforms.

1 INTRODUCTION

In modern society, batch service systems, such as buses or trains, are frequently employed in transportation, where customers are transported in groups. Recently, batch service queueing models with strategic customers have been studied to obtain insights from an economic perspective.

There are generally two types of batch service queues. The first is a system in which the service is conducted using a constant batch (see, e.g., (Bountali and Economou, 2017; Bountali and Economou, 2019a; Bountali and Economou, 2019b)). The other is a system in which customers are served in a batch periodically according to a given interval distribution, called a clearing system (see, for example, (Economou and Manou, 2013; Manou et al., 2014; Manou et al., 2017)).

Furthermore, an interesting topic in batch service queues is modeling the routing behavior of strategic customers. Several studies have been conducted on the strategic choice among infinite server systems with batches and single-server systems of a single service (Calvert, 1997; Afimeimounga et al., 2005; Afimeimounga et al., 2010; Chen et al., 2012; Wang and Ziedins, 2018). These studies have presented interesting Downs–Thomson and Braess-type paradoxes through equilibrium analyses of the model.

In this study, we propose a model in which customers select a single service or time-based (clearing system-type) batch service strategically, and both types of customers receive services in a common single-server queue (see the detailed explanation in Section 2). Note that customers who choose the batch service receive a common service as one batch. From an application perspective, different types of vehicles, for example, cars and buses, often coexist on the same road. Some studies have attempted to model traffic flow using queueing theory (see, e.g., (Van Woensel and Vandaele, 2007)). In these studies, part of a road is modeled as a service station, which is a single-server queuing model.

The model proposed in this study can be used to study the trade-off problem between the total time in a common queue and the waiting time for batch service customers. As more customers choose a single service, traffic congestion is induced in the common queue. However, as more customers choose the batch service, the total waiting time for batch service of all customers accumulates.

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Nakamura, A. and Phung-Duc, T.
Equilibrium Analysis and Social Optimization of a Selectable Single or Time-Based Batch Service.
DOI: 10.5220/0012323800003639

In *Proceedings of the 13th International Conference on Operations Research and Enterprise Systems (ICORES 2024)*, pages 92-100
ISBN: 978-989-758-681-1; ISSN: 2184-4372

Based on the above, this study aimed to determine the pricing policy for the two types of services to achieve a socially optimal state. As relevant research on the pricing of multiple types of services, several studies have been conducted on queues with priorities that can be purchased by additional fees for taking over ordinary customers (see, for example, Chapter 4 in (Hassin and Haviv, 2003), and (Afeche and Mendelson, 2004; Li et al., 2020)). Both social optimization and revenue maximization in priority queues have been considered in (Afeche and Mendelson, 2004). However, to the best of our knowledge, the optimization of a model in which customers can select a single or batch service under pricing control has not been studied, although it is a socially important issue.

The remainder of this paper is organized as follows. First, we describe the model setting in Section 2. Subsequently, the equilibrium strategy of customers is derived in Section 3, and the socially optimal strategy and pricing policy are discussed in Section 4. Furthermore, we present the findings of some numerical results in Section 5. Finally, some concluding remarks are given in Section 6.

2 SETTING OF THE MODEL

This section presents the detailed setting of the selectable model for single and batch services. Customers arrive at the system according to a Poisson process at a rate λ. If a customer chooses the single service, then the customer immediately joins a single server queue whose service time is exponentially distributed with parameter μ. The batch corresponds to a bus system having a finite capacity. If the batch service is selected, then the customer must wait in the waiting room until the batch service commences. We assume that the interval of occurrences of the batch service follows an exponential distribution with rate γ and that the capacity for the batch is X, which follows with probability $0 < \alpha < 1$:

$$q_c := \mathrm{P}(X = c) = (1-\alpha)^{c-1}\alpha, \qquad c = 1,2,\cdots,$$

for the tractability of the probability generating function (PGF). When the batch service commences, at most X customers from the head in the waiting room visit the same server with a single service and receive the common service (the service time also follows an exponential distribution with rate μ). We assume that unserved customers (owing to capacity) immediately abandon the system.

From an application perspective of the transportation system, a single server is considered to be a service station on the road, as proposed in (Van Woensel and Vandaele, 2007). In (Van Woensel and Vandaele, 2007), part of the road is regarded as a service station, which is a single-server queueing model, and traffic congestion is expressed by the performance measures of the service station.

For stability of the system in any case, we assume $\mu > \lambda + \gamma$. We define the fees for single and batch services as τ_S and τ_B, respectively. The reward for receiving service in the common server is R, and the time cost per unit time is C. We also make the following assumptions regarding τ_S and τ_B:

Assumption 1.

$$0 \le \tau_S \le R - \frac{C}{\mu-\gamma} \text{ and } R - \frac{C}{\mu-\gamma} \ge 0.$$

Assumption 2.

$$0 \le \tau_B \le R - \frac{C}{\mu-\lambda-\gamma} - \frac{C}{\gamma},$$

and

$$R - \frac{C}{\mu-\lambda-\gamma} - \frac{C}{\gamma} \ge 0.$$

These assumptions yield positiveness of fees and avoid situations in which no one obtains positive utility by receiving each type of service. Naturally, we assume the following:

Assumption 3. $\tau_S \ge \tau_B$.

If $\tau_B \le \tau_S$, then choosing a batch service does not yield the best response (see the detailed proof of Theorem 1). Assumption 3 excludes this case, except for $\tau_S = \tau_B$ (note that we include $\tau_S = \tau_B$ to consider the case in which fees are not imposed; that is, $\tau_S = \tau_B = 0$). This model is a two-stage game between the monopolist and customers; that is, in the first stage of the game, the monopolist selects admission fees for a single service τ_S and batch service τ_B. In the second stage, the customers determine the strategy (p_S, p_B), where p_S and p_B denote the joining probabilities for a single service and batch service, respectively. We assume that the system is unobservable and that balking is not allowed; that is, $p_S + p_B = 1$ always holds true.

Under the condition that customers follow strategy (p_S, p_B), the mean waiting time for batch service $\mathrm{E}[W_B]$ and the mean sojourn time in the common queue $\mathrm{E}[S]$ can be calculated as follows:

$$\mathrm{E}[W_B] = \frac{1}{\gamma}, \qquad \text{and} \qquad \mathrm{E}[S] = \frac{1}{\mu - (p_S\lambda + \gamma)}.$$

Therefore, the expected total time for batch service customers who can receive the service, $\mathrm{E}[T_B]$, is given by

$$\mathrm{E}[T_B] = \frac{1}{\mu - (p_S\lambda + \gamma)} + \frac{1}{\gamma}.$$

The expected total time for all customers (including customers who abandon the batch service) is calculated as

$$\mathrm{E}[T] = \frac{p_S}{\mu - (p_S\lambda + \gamma)} + p_B\left(\frac{P_{batch}(p_B)}{\mu - (p_S\lambda + \gamma)} + \frac{1}{\gamma}\right),$$

where $P_{batch}(p_B)$ denotes the ratio of the expected number of customers served in a service cycle to the expected number of customers joining the batch service waiting room. According to (Manou et al., 2014), $P_{batch}(p_B)$ is given by:

$$P_{batch}(p_B) = 1 - \Pi\left(\frac{\lambda p_B}{\lambda p_B + \gamma}\right),$$

where $\Pi(z)$ denotes the PGF of the batch size distribution.

$$\Pi(z) = \sum_{c=1}^{\infty} q_c z^c = \frac{\alpha z}{1 - z(1-\alpha)}.$$

Therefore, $P_{batch}(p_B)$ in the present model is calculated as

$$P_{batch}(p_B) = \frac{\gamma}{\alpha\lambda p_B + \gamma}.$$

3 EQUILIBRIUM STRATEGY

Theorem 1 presents the Nash equilibrium strategy for customers.

Theorem 1. *Under Assumptions 1 and 2, the unique equilibrium strategy for customers* (p_S^e, p_B^e) *is*

- *Case 1:* $\tau_S \le \tau_B + \frac{C}{\gamma}$. *Then, a unique equilibrium strategy* (p_S^e, p_B^e) *exists:*
$$(p_S^e, p_B^e) = (1, 0).$$
- *Case 2:* $\tau_B + \frac{C}{\gamma} \le \tau_S \le \tau_B\frac{\gamma}{\alpha\lambda+\gamma} - \frac{\alpha\lambda}{\alpha\lambda+\gamma}\frac{C}{\mu-\gamma} + \frac{\alpha\lambda}{\alpha\lambda+\gamma}R + \frac{C}{\gamma}$. *Then, a unique equilibrium strategy* $(p_S^e, 1-p_S^e)$ *exists where*
$$p_S^e = \frac{-A_1 - \sqrt{(A_1)^2 - 4A_0A_2}}{2A_2}, \tag{1}$$
$$A_2 = \left(R - \tau_S + \frac{C}{\gamma}\right)\alpha\lambda^2,$$
$$A_1 = -(R-\tau_S)(\alpha\lambda+\gamma)\lambda - (R-\tau_S)(\mu-\gamma)\alpha\lambda + C\alpha\lambda + (R-\tau_B)\gamma\lambda - \frac{C}{\gamma}\lambda(\alpha\lambda+\gamma) - \frac{C}{\gamma}(\mu-\gamma)\alpha\lambda,$$
$$A_0 = R\alpha\lambda(\mu-\gamma) - \tau_S(\mu-\gamma)(\alpha\lambda+\gamma) + \tau_B\gamma(\mu-\gamma) - C\alpha\lambda + \frac{C}{\gamma}(\mu-\gamma)(\alpha\lambda+\gamma).$$
- *Case 3:* $\tau_S \ge \tau_B\frac{\gamma}{\alpha\lambda+\gamma} - \frac{\alpha\lambda}{\alpha\lambda+\gamma}\frac{C}{\mu-\gamma} + \frac{\alpha\lambda}{\alpha\lambda+\gamma}R + \frac{C}{\gamma}$. *Then, a unique equilibrium strategy* (p_S^e, p_B^e) *exists:*
$$(p_S^e, p_B^e) = (0, 1).$$

Proof. Given that the dominant customer adopts strategy (p_S, p_B), the expected utility for a tagged customer who adopts strategy (p'_S, p'_B) is given by

$$U((p'_S, p'_B);(p_S, p_B)) = p'_S\left(R - \tau_S - \frac{C}{\mu - (p_S\lambda+\gamma)}\right) + p'_B\left((R-\tau_B)P_{batch}(p_B) - \frac{CP_{batch}(p_B)}{\mu-(p_S\lambda+\gamma)} - \frac{C}{\gamma}\right). \tag{2}$$

We then find that the tagged customer must solve the problem

$$\max_{(p'_S,p'_B)\in([0,1],[0,1])} U((p'_S, p'_B);(p_S, p_B))$$

under $p_S + p_B = 1$ and $p'_S + p'_B = 1$. Here, it is obvious that $U((p'_S, p'_B);(p_S, p_B))$ is linear with respect to p'_S and p'_B. Therefore, the tagged customer bases his decision on the following two quantities:

$$S_S^{ind}(p_S) = (R-\tau_S) - \frac{C}{\mu-(p_S\lambda+\gamma)},$$
$$S_B^{ind}(p_S, p_B) = (R-\tau_B)P_{batch}(p_B) - \frac{CP_{batch}(p_B)}{\mu-(p_S\lambda+\gamma)} - \frac{C}{\gamma}. \tag{3}$$

The set of best responses against (p_S, p_B), that is, $BR(p_S, p_B)$, is given by

$$BR(p_S,p_B) = \begin{cases} (1,0), & \text{if } S_S^{ind}(p_S) \ge S_B^{ind}(p_S,p_B), \\ (0,1), & \text{if } S_B^{ind}(p_S,p_B) \le S_S^{ind}(p_S), \\ (a,1-a), & \text{if } S_S^{ind}(p_S) = S_B^{ind}(p_S,p_B), \end{cases} \tag{4}$$

where $a \in [0,1]$. We can confirm the equilibrium strategy using the following procedure:

Strategy $(1,0)$ is an equilibrium strategy if and only if $(1,0) \in BR(1,0)$; that is, $S_S^{ind}(1) \ge S_B^{ind}(1,0)$, which reduces to

$$R - \tau_S \ge \frac{C}{\mu-\lambda-\gamma} + \left((R-\tau_B) - \frac{C}{\mu-\lambda-\gamma}\right) - \frac{C}{\gamma}. \tag{5}$$

Strategy $(0,1)$ is an equilibrium strategy if and only if $(0,1) \in BR(0,1)$; that is, $S_S^{ind}(0) \le S_B^{ind}(0,1)$, which reduces to

$$R - \tau_S \le \frac{C}{\mu-\gamma} + \left((R-\tau_B) - \frac{C}{\mu-\gamma}\right)\frac{\gamma}{\alpha\lambda+\gamma} - \frac{C}{\gamma}. \tag{6}$$

Here, using Assumption 1, we can easily find that

$$\begin{aligned}
&\frac{C}{\mu-\gamma}+\left(R-\tau_S-\frac{C}{\mu-\gamma}+\frac{C}{\gamma}\right)\frac{\alpha\lambda+\gamma}{\gamma}\\
&\quad-\frac{C}{\mu-\lambda-\gamma}-\left(R-\tau_S-\frac{C}{\mu-\lambda-\gamma}+\frac{C}{\gamma}\right)\\
&>\frac{C}{\mu-\gamma}+\left(R-\tau_S-\frac{C}{\mu-\gamma}+\frac{C}{\gamma}\right)\\
&\quad-\frac{C}{\mu-\lambda-\gamma}-\left(R-\tau_S-\frac{C}{\mu-\lambda-\gamma}+\frac{C}{\gamma}\right)\\
&=0,
\end{aligned} \tag{7}$$

which implies that τ_B satisfying both (5) and (6) simultaneously does not exist. Therefore, the equilibrium strategies $(1,0)$ and $(0,1)$ cannot simultaneously be in equilibrium.

Finally, the strategy $(p_S^e, 1-p_S^e)$ is an equilibrium strategy if and only if $(p_S^e, 1-p_S^e) \in BR(p_S^e, 1-p_S^e)$, i.e., $S_S^{ind}(p_S^e) = S_B^{ind}(p_S^e, 1-p_S^e)$. This necessary and sufficient condition is equivalent to p_S^e satisfying the following condition:

$$\begin{aligned}
&(R-\tau_S)-\frac{C}{\mu-p_S\lambda-\gamma}\\
&=\left((R-\tau_B)-\frac{C}{\mu-p_S\lambda-\gamma}\right)\frac{\gamma}{\alpha\lambda(1-p_S)+\gamma}-\frac{C}{\gamma},\\
&\Longrightarrow A_2p_S^2+A_1p_S+A_0=0,
\end{aligned} \tag{8}$$

where A_0, A_1, and A_2 are defined in Theorem 1. Subsequently, let $F(p_S^e)$ denote

$$\begin{aligned}
F(p_S^e) &= (R-\tau_S)-\frac{C}{\mu-p_S^e\lambda-\gamma}\\
&\quad-\left((R-\tau_B)-\frac{C}{\mu-p_S^e\lambda-\gamma}\right)\frac{\gamma}{\alpha\lambda(1-p_S^e)+\gamma}+\frac{C}{\gamma}.
\end{aligned} \tag{9}$$

Then, we obtain

$$\begin{aligned}
F'(p_S^e) &= -\frac{\lambda C}{(\mu-p_S^e\lambda-\gamma)^2}\frac{\alpha\lambda(1-p_S^e)}{\alpha\lambda(1-p_S^e)+\gamma}\\
&\quad-\left((R-\tau_B)-\frac{C}{\mu-p_S^e\lambda-\gamma}\right)\frac{\gamma\alpha\lambda}{(\alpha\lambda(1-p_S^e)+\gamma)^2}.
\end{aligned} \tag{10}$$

Assumption 2 indicates that $F'(p_S^e) < 0$ is true for $0 \le p_S^e \le 1$. Therefore, $F(p_S^e) = 0$ has a unique root as (1) within $0 \le p_S^e \le 1$ if $F(0) \ge 0$ and $F(1) \le 0$; that is, the following is satisfied:

$$\begin{aligned}
&\tau_B-\frac{\alpha C}{\mu-\lambda-\gamma}+\alpha R+\frac{C}{\gamma}\le\tau_S\\
&\le\tau_B\frac{\gamma}{\alpha\lambda+\gamma}-\frac{\alpha(\gamma+\lambda)}{\alpha\lambda+\gamma}\frac{C}{\mu-\gamma}+\frac{\alpha(\gamma+\lambda)}{\alpha\lambda+\gamma}R+\frac{C}{\gamma}.
\end{aligned} \tag{11}$$

□

Remark 1. *When $\tau_S = \tau_B$ holds, the unique equilibrium strategy (p_S^e, p_B^e) is $(p_S^e, p_B^e) = (1,0)$. This is because (5) holds true if $\tau_S = \tau_B$. This result is natural because batch-service customers must wait, and there is a possibility that they are not served while they do not occur on single-service customers. Therefore, if fees are not imposed (i.e., $\tau_S = \tau_B = 0$), the unique equilibrium is to choose the single service. However, as the fee for the batch service becomes less than that for the single service, the equilibrium shifts in favor of the batch service being used more.*

Moreover, we obtain the following theorem:

Theorem 2. *The model is an avoid-the-crowd (ATC) type.*

Proof. The increase in utility when a customer chooses the single service (compared to the batch service) is given by $F(p_S^e)$. As has already shown, $F(p_S^e)$ decreases monotonically within $0 \le p_S^e \le 1$, from which the ATC property follows. □

Remark 2. *A comparison of Theorems 2 and Theorem 4.5 in (Hassin and Haviv, 2003) is of interest. The latter theorem shows that an unobservable single-server queue with a priority (an additional fee) is of the follow-the-crowd (FTC) type. Note that multiple equilibria often exist in an FTC setting, and this trend also applies to basic unobservable queues with priority. In comparison, our model has a unique equilibrium; therefore, this is more tractable for considering the social optimization problem in Section 4.*

The reasons for these differences are as follows. In the priority queue, an ordinary (not buying priority) customer may be overtaken by successive priority customers after their decision is completed. Thus, customers tend to buy more priorities to avoid this situation as the arrival rate increases. On the other hand, batch service customers (lower fee and longer waiting time) are never overtaken after their batch enters the queue. In addition, recall that batch services occur periodically; therefore, it is guaranteed that batch service customers can line the queue within a certain amount of time, as long as the capacity is not exceeded.

4 SOCIAL OPTIMIZATION

Next, we consider the social planner's point of view. First, we present Theorem 3:

Theorem 3. *For the social planner's admission problem, a unique socially optimal strategy (p_S^s, p_B^s) is given by*

- *Case 1:*
$$R \le \frac{C}{\mu-\gamma} + \frac{C}{1-\frac{\gamma^2}{(\alpha\lambda+\gamma)^2}}\left(\frac{\gamma}{\alpha\lambda+\gamma}\frac{\lambda}{(\mu-\gamma)^2}-\frac{1}{\gamma}\right).$$
Then, a unique socially optimal strategy (p_S^s, p_B^s) *exists:*
$$(p_S^s, p_B^s) = (0, 1).$$
- *Case 2:*
$$R \ge \frac{C}{\mu-\gamma} + \frac{C}{1-\frac{\gamma^2}{(\alpha\lambda+\gamma)^2}}\left(\frac{\gamma}{\alpha\lambda+\gamma}\frac{\lambda}{(\mu-\gamma)^2}-\frac{1}{\gamma}\right),$$
and
$$\frac{\lambda}{(\mu-\lambda-\gamma)^2}-\frac{1}{\gamma} \ge 0.$$
Then, a unique socially optimal strategy $(p_S^s, 1-p_S^s)$ *exists where* p_S^s *is a unique solution of*
$$\frac{\lambda(\alpha\lambda(1-p_S^s)+\gamma)^2-\lambda\gamma^2}{(\alpha\lambda(1-p_S^s)+\gamma)^2}\left(R-\frac{C}{\mu-p_S^s\lambda-\gamma}\right)- \frac{\lambda p_S^s\{\alpha\lambda(1-p_S^s)+\gamma\}+\lambda(1-p_S^s)\gamma}{\alpha\lambda(1-p_S^s)+\gamma}\frac{\lambda C}{(\mu-p_S^s\lambda-\gamma)^2} + \frac{\lambda C}{\gamma} = 0. \tag{12}$$
- *Case 3:* $\frac{\lambda}{(\mu-\lambda-\gamma)^2}-\frac{1}{\gamma} \le 0$. *Then, a unique socially optimal strategy* (p_S^s, p_B^s) *exists:*
$$(p_S^s, p_B^s) = (1, 0).$$

Proof. Assuming that all customers follow strategy (p_S, p_B), The mean number of single-service customers and batch-service customers, $\mathrm{E}[C_S]$ and $\mathrm{E}[C_B]$, in the common queue is given by
$$\mathrm{E}[C_S] = \frac{p_S\lambda}{\mu-p_S\lambda-\gamma}, \qquad \mathrm{E}[C_B] = \frac{\gamma}{\mu-p_S\lambda-\gamma}\mathrm{E}[B],$$
respectively, where $\mathrm{E}[B]$ is the mean number of customers (who choose the batch service) in a batch,
$$\mathrm{E}[B] = \frac{p_B\lambda}{\gamma}P_{batch}(p_B).$$
According to Little's law, the mean number of waiting customers for the batch service is given by
$$\mathrm{E}[N_B] = \frac{p_B\lambda}{\gamma}.$$
Therefore, social welfare per unit time is given by
$$S^{soc}(p_S, p_B) = \lambda\left(p_S + p_B\frac{\gamma}{\alpha\lambda p_B+\gamma}\right)\left(R-\frac{C}{\mu-p_S\lambda-\gamma}\right) - \frac{p_B\lambda C}{\gamma}. \tag{13}$$

Using the normalization condition $p_S + p_B = 1$, $S^{soc}(p_S, p_B)$ can be rewritten as
$$S^{soc}(p_S) = \lambda p_S\left(R-\frac{C}{\mu-p_S\lambda-\gamma}\right) + \frac{\lambda(1-p_S)\gamma}{\alpha\lambda(1-p_S)+\gamma}\left(R-\frac{C}{\mu-p_S\lambda-\gamma}\right) - \frac{(1-p_S)\lambda C}{\gamma}. \tag{14}$$

Here, the social planner must solve for $\max_{p_S\in[0,1]} S^{soc}(p_S)$. We obtain the following differentiation form for p_S.
$$\frac{\partial S^{soc}(p_S)}{\partial p_S} = \frac{\lambda(\alpha\lambda(1-p_S)+\gamma)^2-\lambda\gamma^2}{(\alpha\lambda(1-p_S)+\gamma)^2}\left(R-\frac{C}{\mu-p_S\lambda-\gamma}\right) - \frac{\lambda p_S\{\alpha\lambda(1-p_S)+\gamma\}+\lambda(1-p_S)\gamma}{\alpha\lambda(1-p_S)+\gamma}\frac{\lambda C}{(\mu-p_S\lambda-\gamma)^2} + \frac{\lambda C}{\gamma}, \tag{15}$$
$$\frac{\partial^2 S^{soc}(p_S)}{\partial^2 p_S} = -\left(1-\frac{\gamma^2}{(\alpha\lambda(1-p_S)+\gamma)^2}\right)\frac{2\lambda^2 C}{(\mu-p_S\lambda-\gamma)^2} - \frac{2\lambda^3 p_S C}{(\mu-p_S\lambda-\gamma)^3} - \frac{2\alpha\lambda^2\gamma^2}{(\alpha\lambda(1-p_S)+\gamma)^3}\left(R-\frac{C}{\mu-p_S\lambda-\gamma}\right) - \frac{\lambda(1-p_S)\gamma}{\alpha\lambda(1-p_S)+\gamma}\frac{2\lambda^2 C}{(\mu-p_S\lambda-\gamma)^3}.$$
From Assumption 1, it is clear that $\frac{\partial^2 S^{soc}(p_S)}{\partial^2 p_S} < 0$ holds for $0 \le p_S \le 1$, indicating the concavity of $S^{soc}(p_S)$. Hence, the socially optimal strategy (p_S^s, p_B^s) becomes $(0,1)$ if $\left.\frac{\partial S^{soc}(p_S)}{\partial p_S}\right|_{p_S=0} \le 0$, i.e.,
$$R \le \frac{C}{\mu-\gamma} + \frac{C}{1-\frac{\gamma^2}{(\alpha\lambda+\gamma)^2}}\left(\frac{\gamma}{\alpha\lambda+\gamma}\frac{\lambda}{(\mu-\gamma)^2}-\frac{1}{\gamma}\right), \tag{16}$$
the socially optimal strategy (p_S^s, p_B^s) becomes $(1,0)$ if $\left.\frac{\partial S^{soc}(p_S)}{\partial p_S}\right|_{p_S=1} \ge 0$, i.e.,

$$\frac{\lambda}{(\mu-\lambda-\gamma)^2}-\frac{1}{\gamma}\leq 0, \tag{17}$$

and the socially optimal (p_S^s, p_B^s) becomes $(p_S^s, 1-p_S^s)$ where p_S^s is a unique solution of (15) in the other case. Here, due to Assumption 1, we have

$$0\leq\frac{\gamma}{\alpha\lambda+\gamma}\frac{\lambda}{(\mu-\gamma)^2}-\frac{1}{\gamma},$$

if (16) holds. As the following:

$$\frac{\gamma}{\alpha\lambda+\gamma}\frac{\lambda}{(\mu-\gamma)^2}\leq\frac{\lambda}{(\mu-\lambda-\gamma)^2}$$

is clear, the inequalities (16) and (17) cannot be satisfied simultaneously. Thus, the strategy becomes the unique strategy.

□

Remark 3. *From Theorem 3, we consider that the socially optimal probability for a single service increases as R increases as long as* $\lambda/(\mu-\lambda-\gamma)^2-1/\gamma\geq 0$ *is satisfied. When R is small, the cost of the waiting time, i.e., C, is considered to be significant. Thus, in this case, it is socially optimal for customers to wait for batch services and to prevent traffic congestion in the common queue. On the other hand, when R is high, the risk of batch service in which customers may not be served because of the capacity is considered to be significant because unserved customers obtain zero reward. Therefore, choosing the single service becomes socially optimal. In the case of* $\lambda/(\mu-\lambda-\gamma)^2-1/\gamma\leq 0$*, the arrival of buses is too infrequent which results in choosing the single service with probability 1 becomes social optimal.*

Based on Theorems 1 and 3, we consider the socially optimal relationship between τ_S and τ_B. To this end, we put forth the following assumption:

Assumption 4. *If multiple patterns exist in the relationship between* τ_S *and* τ_B*, the social planner adopts the relationship such that the difference between* τ_S *and* τ_B *is the smallest.*

From Assumption 4, we obtain Theorem 4.

Theorem 4. *The socially optimal relationship between the admission fees (*τ_S^s *and* τ_B^s*) for single and batch services satisfies the following linear equation:*

- *Case 1:*
$$R\leq\frac{C}{\mu-\gamma}+\frac{C}{1-\frac{\gamma^2}{(\alpha\lambda+\gamma)^2}}\left(\frac{\gamma}{\alpha\lambda+\gamma}\frac{\lambda}{(\mu-\gamma)^2}-\frac{1}{\gamma}\right).$$
Then, τ_S^s *and* τ_B^s *satisfy*
$$\tau_S^s=\frac{\gamma}{\alpha\lambda+\gamma}\tau_B^s-\frac{\alpha\lambda}{\alpha\lambda+\gamma}\frac{C}{\mu-\gamma}+\frac{\alpha\lambda}{\alpha\lambda+\gamma}R+\frac{C}{\gamma}.$$
- *Case 2:*
$$R\geq\frac{C}{\mu-\gamma}+\frac{C}{1-\frac{\gamma^2}{(\alpha\lambda+\gamma)^2}}\left(\frac{\gamma}{\alpha\lambda+\gamma}\frac{\lambda}{(\mu-\gamma)^2}-\frac{1}{\gamma}\right),$$
and
$$\frac{\lambda}{(\mu-\lambda-\gamma)^2}-\frac{1}{\gamma}\geq 0.$$
Then, τ_S^s *and* τ_B^s *satisfy*
$$\tau_S^s=T_1\tau_B^s+T_2,$$
where
$$T_1=\frac{\gamma}{\alpha\lambda(1-p_S^s)+\gamma},$$
$$T_2=R-\frac{C}{\mu-p_S^s\lambda-\gamma}+\frac{C}{\gamma}-\left(R-\frac{C}{\mu-p_S^s\lambda-\gamma}\right)\frac{\gamma}{\alpha\lambda(1-p_S^s)+\gamma}.$$
- *Case 3:* $\dfrac{\lambda}{(\mu-\lambda-\gamma)^2}-\dfrac{1}{\gamma}\leq 0$*. Then,* τ_S^s *and* τ_B^s *satisfy*
$$\tau_S^s=\tau_B^s+\frac{C}{\gamma}.$$

Proof. In Case 1, the socially optimal strategy is $(0,1)$. Thus, the social planner sets a fee such that the equilibrium strategy becomes $(0,1)$ under Assumption 4. The same argument holds true for Case 3. In Case 2, the socially optimal strategy is $(p_S^s, 1-p_S^s)$, where p_S^s is a unique solution to (12). Therefore, we obtain the result by substituting p_S^s into the equation for the equilibrium strategy, that is, (1). □

Remark 4. *From Theorem 4, we find that that* τ_S^s/τ_B^s *(the amount of* τ_S^s *relative to* τ_B^s*) decreases as R increases as long as* $\lambda/(\mu-\lambda-\gamma)^2-1/\gamma\geq 0$ *is satisfied. This can be explained as follows. When R is large, the single service is socially preferable, as discussed in Remark 3. Therefore, it is optimal for the social planner to set* τ_S^s *relatively low and induce customers to use the single service more often, and vice versa.*

5 NUMERICAL EXAMPLES

This section provides some numerical results for the social welfare and socially optimal relationship between the fees for single and batch services. Note that all experiments were conducted under the condition that Assumptions 1, 2, and 3 are satisfied. We show some examples under that $\lambda/(\mu-\lambda-\gamma)^2-1/\gamma\geq 0$ is satisfied. The detailed parameter settings are described in each figure caption.

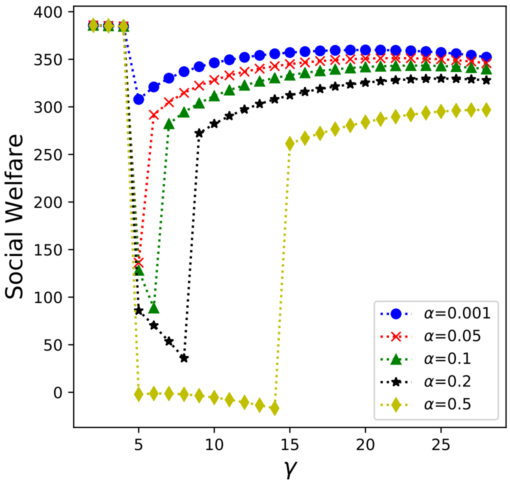

Figure 1: Social welfare for γ and α. The other parameters are $\lambda = 10, \mu = 40, R = 40, C = 40, \tau_S = 20, \tau_B = 10$.

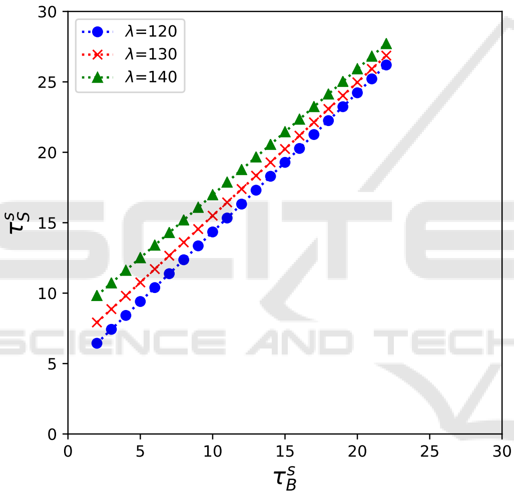

Figure 2: τ_S^s vs. τ_B^s for λ. The other parameters are $\mu = 160, \alpha = 0.1, \gamma = 10, R = 40, C = 20$.

5.1 Social Welfare

In this subsection, the results for Social Welfare, i.e., (13), are presented.

Figure 1 depicts Social Welfare with regard to the batch service, that is, the rate of the exponential distribution for the interval of the batch service occurrences γ, and the geometric distribution of the capacity α. Here, the expected batch size becomes $1/\alpha$ because of the mean of the geometric distribution. Thus, the expected batch size increases as α decreases.

Figure 1 illustrates an intriguing tendency. The graphs behave as decreasing, increasing, and decreasing again, against γ. This trend can be interpreted as follows. When γ is low, choosing the batch service rarely becomes the equilibrium strategy. Therefore, although γ is increased slightly, this only just induces the congestion level of the common queue (although few customers choose the batch service), and Social Welfare is decreased. When γ reaches a certain level, customers often start to use the batch service, and Social Welfare increases. However, when γ increases further, its effect on the congestion of the common queue becomes stronger than the total reward for batch service customers.

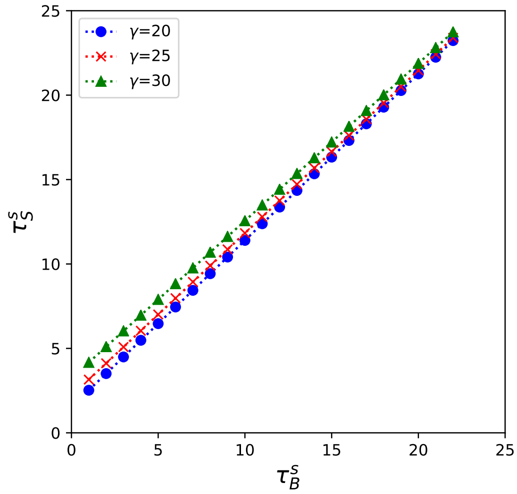

Figure 3: τ_S^s vs. τ_B^s for γ. The other parameters are $\lambda = 25, \mu = 100, \alpha = 0.2, R = 40, C = 40$.

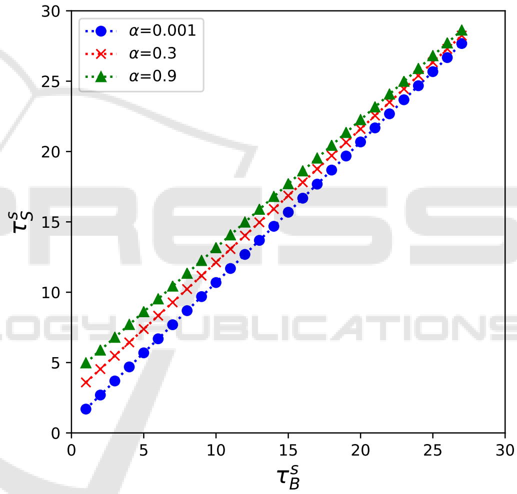

Figure 4: τ_S^s vs. τ_B^s for α. The other parameters are $\lambda = 25, \mu = 60, \gamma = 30, R = 40, C = 20$.

5.2 Relationship Between Fees of Single and Batch Services

In the following section, we discuss the socially optimal relationship between fees for single and batch services; that is, Theorem 4. Figures 2–4 show τ_S^s when the horizontal axis represents τ_B^s.

We observe that both fees become closer as τ_B^s increases. This is natural from the form of Theorem 4. Fees become dominant in the linear equations as they increase. Recall that balking is not allowed in this model. This assumption may allow even large fees to become socially optimal. Therefore, the development of a model in which balking is allowed would be meaningful in future work.

It is confirmed in Figure 2 that τ_S^s becomes larger as λ increases for the same τ_B^s. This result implicitly suggests that the batch service becomes socially preferable when the arrival rate is high, and it is better for the social planner to set τ_B^s low to induce customers to use the batch service. It follows that congestion in the common queue becomes significant if too many customers use the single service.

Figures 3 and 4 show the results for the batch service parameters. As the maximum throughput of the batch service increases, that is, γ increases or α decreases, a lower cost of τ_B^s becomes socially preferable. This naturally implies that it is socially optimal for customers to use more batch services when the performance of the batch service is superior; thus, the social planner must set τ_B^s lower.

6 CONCLUSION

In this study, we proposed a model in which customers can probabilistically select a single or batch service. A novel feature of this model is that both customer types join a common queue. From an application perspective, this setting enables us to consider the effects of road congestion on transportation platforms.

We proved the existence of a unique equilibrium strategy in this model. Moreover, we derived a socially optimal strategy and demonstrated that using batch services to alleviate the congestion of the common queue is socially desirable compared to the equilibrium strategy. Based on these results, we derived a socially optimal relationship between the fees for both services. Interestingly, these optimal fees exhibit a linear relationship.

In addition, we presented several numerical examples. In particular, the social welfare for the batch service parameters (γ and α) in Figure 1 shows a unique tendency. It can be interpreted from Figure 1 that increasing the frequency and capacity of the batch service does not necessarily lead to better results. Overall, the main managerial findings are summarized in Remarks 1–4 and Section 5.

As potential extensions of this study, a model with balking or an (partially) observable scheme can be considered. In addition, the model should be further studied under more general assumptions regarding batch service occurrences and capacity.

ACKNOWLEDGEMENTS

This work was supported by JSPS KAKENHI Nos. 21K11765, 18K18006, and JST SPRING No. JPMJSP2124. In addition, this study was supported by F-MIRAI: R&D Center for Frontiers of MIRAI in Policy and Technology, the University of Tsukuba, and Toyota Motor Corporation Collaborative R&D Center.

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