Dynamic Modeling and Effective Inventory Management for Uncertain Perishable Supply Chains with non Synchronized Internal Dynamics

Beatrice Ietto[1,2][a] and Valentina Orsini[3][b]
[1]*DIMA, UNIVPM, Ancona, Italy*
[2]*Einstein Center Digital Future, Berlin, Germany*
[3]*DII, UNIVPM, Ancona,Italy*

Keywords: Supply Chain Management, Supply Chain Modeling, Inventory Control, Perishable Goods, Robust MPC.

Abstract: The problem we consider is to define an efficient management strategy for a periodic-review production-inventory system whose dynamics is characterized by various elements of complexity. These elements characterize the vast majority of practical situations and consist of: 1) perishable goods with unknown deterioration rate, 2) an uncertain future customer demand with oscillations that are not statistically modelable, 3) non-synchronized operations of goods stocking and dispatching inside each review period. With reference to this problem we define a general model of supply chain dynamics capturing the mentioned complex features and encompassing, as particular cases, other models proposed in the literature. We also provide a coherent two criteria based evaluation of the Bullwhip Effect (BE) and a Replenishment Policy (RP) resulting from a min-max formulation of Model Predictive Control (MPC) approach. The RP maximizes the satisfied customer demand and contains the BE within precise limits independent of the customer demand randomness.

1 INTRODUCTION

An effective control strategy of inventory level in Supply Chains (SC) requires solving the challenging problem of reconciling the following Control Specifications (CSs): CS1) maximize the satisfied customer demand (this is the primary goal), CS2) avoid overstocking, CS3) mitigate the Bullwhip Effect (BE). The high conflictuality of these CSs calls for an optimality criterion. The widely acknowledged success achieved by MPC in this regard is mainly due to the ability to take into account the physical limitations of the system, and to the receding horizon nature of the control law that is continuously adapted according to the incoming observations (Rossiter and Bishop, 2004). A thorough list of the very numerous MPC based techniques dealing with different aspects of the inventory problem in supply chain can be found in the surveys (Sarimveis et al., 2008; Ivanov et al., 2018).

In this vast literature, only a few authors (see e.g. (Gaggero and Tonelli, 2015; Taparia et al., 2020; Lejarza and Baldea, 2020; Hipolito et al., 2022)) considered the presence of deteriorating goods in the inventory system, despite the acknowledged importance of the topic (Li et al., 2010; Chaudary et al., 2018). Several alternatives to the MPC approach have been proposed in (Ignaciuk, 2014; Ignaciuk, 2015; Pan and Li, 2015; Leśniewsky and Bartoszewicz, 2020; Cholodowicz and Orlowski, 2023) and references therein.

All the previous contributions (MPC and non-MPC) dealing with perishable goods assume an exactly known deterioration rate. Unfortunately, this simplistic assumption is not satisfied in the most cases due to unstable and variable storage conditions (Chaudary et al., 2018). Another important issue regards the "Internal Dynamics" (ID). With this syntagm we define the sequence of OPerations (OP) inside each review period: OP1) updating the inventory value; OP2) receiving goods from manufacturer; OP3) dispatching goods to the customer; OP4) placing a replenishment order. A free planning of OP2 and OP3 could be unrealizable due to several external factors independent of the SC manager. If the actual ID is not properly taken into account in the design of the RP, a serious SC performance degradation will occur, especially in the case of highly perishable goods with uncertain deterioration rate and a long review period. A typical example is the food industry where the con-

[a] https://orcid.org/0000-0001-5617-8228
[b] https://orcid.org/0000-0003-4965-5262

Ietto, B. and Orsini, V.
Dynamic Modeling and Effective Inventory Management for Uncertain Perishable Supply Chains with non Synchronized Internal Dynamics.
DOI: 10.5220/0012269700003639

In *Proceedings of the 13th International Conference on Operations Research and Enterprise Systems (ICORES 2024)*, pages 25-34
ISBN: 978-989-758-681-1; ISSN: 2184-4372

sequent large waste and losses would result in a poor economic sustainability.

To the best of our knowledge, what is still missing about this issue is a general model of the inventory level dynamics in the case of perishability under uncertainty and a possibly unmanageable ID.

The main contribution of this paper is to fill this gap facing this topic at a formal level: here, we go into the details of the ID, and propose a general model of the dynamic SC encompassing other models proposed in the literature, as particular cases. Taking into account the possible non-synchronization of the OPs allows us to obtain a model that more carefully reflects the real dynamics of the SC and to define a more effective RP conciliating CS1-CS3. From this more general model we also derive a two-Evaluation Criteria of the BE: EC1) smoothness of the RP, EC2) "a priori" given bounds on the intervals over which the RP takes values.

In accordance with EC1 we define a cost functional adaptively penalizing excessive differences between two consecutive orders. As it concerns EC2, we suitably compute the time varying constraints to be imposed on the RP so as to guarantee the maximum amount of satisfied demand.

The actual computation of the RP is here set in the same general theoretical framework used in (Ietto and Orsini, 2022a; Ietto and Orsini, 2022b; Ietto and Orsini, 2023a; Ietto and Orsini, 2023b) where the OPs are assumed to be synchronized. The framework consists in a min-max MPC whose solution is parametrized using polynomial B-splines. The min-max formulation of the MPC allows us to deal with the uncertainties on the perishable rate and on the future customer demand. The B-splines parametrization allows us: 1) to reformulate the min-max MPC problem as a numerically simpler Constrained Robust Least Squares (CRLS) problem; 2) to easily impose hard constraints on the RP.

Compared to the latter references, we here provide more theoretical and implementation insights : more accurate description and prediction of the inventory dynamics, a more accurate estimate of the constraints on the control effort (i.e. the RP), deeper managerial insights. E. g. in the case of manageable ID, the man- ager can profitably use the information from our study for optimally scheduling the operations so as to minimize the wastage due to deterioration.

The paper is organized in the following way. Some mathematical preliminaries on B-splines and RLS are recalled in Section 2 The system model is described in Section 3. The min-max MPC problem is stated in Section 4. In Section 5 the min-max MPC problem is reformulated as a CRLS problem that can be efficiently solved using second order cone programming. Stability and feasibility are also discussed (see Remark 6). Numerical results and concluding remarks are reported in Sections 6 and 7 respectively.

2 PRELIMINARIES

2.1 Polynomial B-Splines Functions

Polynomial B-splines functions are universal optimal approximators of curves with complex shape. They are defined as a linear combination of B-splines basis functions and control points, (De-Boor, 1978):

$$s(t) = \sum_{i=1}^{\ell} c_i B_{i,d}(t), \quad t \in [\hat{t}_1, \hat{t}_{\ell+d+1}] \subseteq R, \quad (1)$$

where the c_i's are real numbers representing the control points of $s(t)$, the integer d is the degree of the B-spline, the $(\hat{t}_i)_{i=1}^{\ell+d+1}$ are the non decreasing knot points and the $B_{i,d}(t)$ are the uniformly bounded B-spline basis functions which can be computed by the Cox-de Boor recursion formula. An equivalent representation of $s(t)$ in (1) is

$$s(t) = \mathbf{B}_d(t)\mathbf{c}, \quad t \in [\hat{t}_1, \hat{t}_{\ell+d+1}] \subseteq R, \quad (2)$$

where $\mathbf{B}_d(t) = [B_{1,d}(t), \cdots, B_{\ell,d}(t)]$ and $\mathbf{c} = [c_1, \cdots, c_\ell]^T$.

Convex hull property. Any value assumed by $s(t)$, $\forall t \in [\hat{t}_j, \hat{t}_{j+1}]$, $j > d$, lies in the convex hull of its $d+1$ control points $c_{j-d}, \cdots, c_j$.

Smoothness property. Suppose that $\hat{t}_i < \hat{t}_{i+1} = \cdots = \hat{t}_{i+m} < \hat{t}_{i+m+1}$, with $1 \leq m \leq d+1$ then the B-spline function $s(t)$ has continuous derivative up to order $d-m$ at knot $\hat{t}_{i+1}$. This property implies that the spline smoothness can be changed using multiple knot points. It is common choice to set $m = d+1$ multiple knot points for the initial and the last knot points and to evenly distribute the other ones. In this way (1) assumes the first and the final control points as initial and final values.

2.2 The RLS Problem

Let $Ax \approx b$, be a set of linear equations, where $A \in R^{r \times m}$ is the design matrix and $b \in R^r$, $r > m$, is the observations vector. Both A and b are subject to unknown but bounded errors: $\|\delta A\| \leq \beta$ and $\|\delta b\| \leq \xi$ (where the matrix norm is the spectral norm). The RLS estimate $x \in R^m$ is the value of x solving the following min-max problem, (Lobo et al., 1998)

$$\min_x \max_{\|\delta A\| \leq \beta, \, \|\delta b\| \leq \xi} \|(A + \delta A)x - (b + \delta b)\| \quad (3)$$

As

$$\max_{\|\delta A\|\leq\beta,\|\delta b\|\leq\xi} \|(A+\delta A)x-(b+\delta b)\|$$
$$= \quad \|Ax-b\|+\beta\|x\|+\xi \qquad (4)$$

Problem (3) is equivalent to minimize the following sum of euclidean norms

$$\min_x \|Ax-b\|+\beta\|x\|+\xi \qquad (5)$$

The CRLS problem also requires that x satisfies the following conditions

$$\underline{x} \leq x \leq \bar{x} \qquad (6)$$

Remark 1. Note that the term $\|\delta b\|$ in (3) only appears in (5) through its norm upper bound ξ, which is independent of x. △

3 THE DYNAMIC UNCERTAIN MODEL

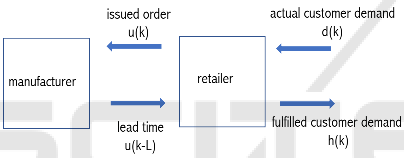

Figure 1: Single echelon SC model.

For ease of exposition and space limits we here consider the simple SC model shown in Figure 1. Inside each review period $[kT\ (k+1)T)$, $k \in Z^+$, the retailer performs four OPs: OP1 determines its on hand stock level; OP2 receives delivery from manufacturer; OP3 delivers the goods to meet demand; OP4 places a replenishment order according to a suitably defined RP. We assume

- **A1)** OP1 takes place at the beginning of the review period, OP2 and OP3 may not be simultaneous, OP2 is not subsequent to OP3, OP4 is executed last;
- **A2)** the manufacturer fully satisfies each issued non null replenishment order with a time delay $L=\ell T$, $\ell \in Z^+$;
- **A3)** the goods arrive at the retailer new and deteriorate while kept in stock;
- **A4)** inside each review period of length T, the uncertain perishability rate $\alpha_{T'}$ of the product is defined with respect to a sub-interval T', with $T=nT'$, $n\in Z^+$, $\alpha_{T'} \in [\alpha_{T'}^-\ \alpha_{T'}^+] \subset (0,1)$, then the corresponding decay factor over T' is $\rho_{T'} = 1-\alpha_{T'} \in [\rho_{T'}^-\ \rho_{T'}^+] \subset (0,1)$.

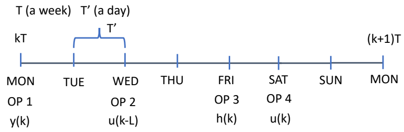

Figure 2: The operations performed by the retailer inside each review period: OP1(Monday), OP2 (Wednesday), OP3 (Friday), OP4 (Saturday).

A clarifying example is shown in Figure 2 where we suppose that T is a week and T' denotes a day of the week (namely $T=nT'$ with $n=7$).

The customer demand is modeled according to the following assumption

- **A5)** at any time instant k and limitedly to an M-steps prediction horizon $[k,k+M]$, the future customer demand $d(k+j)$, $j=0,\cdots,M$, fluctuates within a compact set D_k limited below and above by two known boundary trajectories: $d^-(k+j)$ and $d^+(k+j)$, $j=0,\cdots,M$. The set $\mathcal{D}$ containing the whole customer demand is given by the consecutive contiguous overlapping of all the sets $D_k, k\in Z^+$.

  Figure 3 shows a typical example of a customer demand $d(k+j)$, $j\geq 0$ over a fixed D_k. The dashed line is the forecasted demand $d(k+j|k)$, $j>0$. We assume it coincides with the central trajectory $\bar{d}(k+j)$ of D_k. Hence, the actual future values $d(k+j)$, $j>0$ can be expressed as

$$d(k+j) = d(k+j|k)+\delta d(k+j|k) \qquad (7)$$

where $d(k+j|k) = \bar{d}(k+j) = (d^+(k+j)+d^-(k+j))/2$ is the predicted demand and $\delta d(k+j|k)$ is the corresponding estimation error.

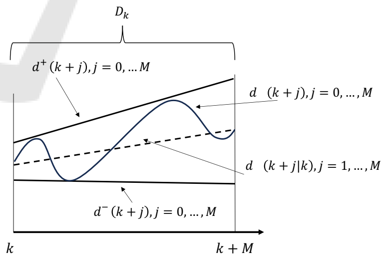

Figure 3: An example of customer demand $d(k+j)$, $j=0,\cdots,M$, over a fixed D_k. The dashed trajectory is the predicted demand $d(k+j|k)$, $j=1,\cdots,M$.

Remark 2. As $|\delta d(k+j|k)| \leq (d^+(k+j)+d^-(k+j))/2$, $j=1,\cdots,M$, then $\delta d(k+j|k)$ is the approximation error sequence with minimum maximum ℓ_2 norm over each prediction interval. This is useful in the CRLS formulation of the min-max MPC problem because allows reducing the maximum norm of the term that cannot be minimized with respect to x (i.e. the term ξ in (5)). For more details see Remark 4 reported in Section 5. △

The above considerations imply that the stock level dynamics is described by the following uncertain equation

$$y(k+1) = \rho_h(\rho_y y(k) + \rho_u u(k-L) - h(k)) \quad (8)$$

where:

- in accordance with A4), the quantities ρ_h, ρ_y and ρ_u in (8) are defined as

$$\rho_h \triangleq \rho_{T'}^{n_h} \in [(\rho_{T'}^-)^{n_h}, (\rho_{T'}^+)^{n_h}] \quad (9)$$

$$\rho_y \triangleq \rho_{T'}^{n_y} \in [(\rho_{T'}^-)^{n_y}, (\rho_{T'}^+)^{n_y}] \quad (10)$$

$$\rho_u \triangleq \rho_{T'}^{n_u} \in [(\rho_{T'}^-)^{n_u}, (\rho_{T'}^+)^{n_u}] \quad (11)$$

  for some positive integers n_h, n_y and n_u. For example, with reference to Figure 2 we have: $n_h = 3$, $n_y = 4$ and $n_u = 2$.
- $y(k)$ is the on hand stock level, i.e. the amount of remaining undamaged goods available at the beginning of the k review period; $u(k-L)$ is the replenishment order placed at time $k-L$ and realized at time k;
- $h(k)$ is the fulfilled customer demand and by A1) it is given by

$$h(k) = \min(\rho_y y(k) + \rho_u u(k-L), d(k)) \in [0, d(k)] \quad (12)$$

  where the quantity

$$\rho_y y(k) + \rho_u u(k-L) \quad (13)$$

  is the actual amount of goods available for sale while the corresponding wasted goods is

$$(1-\rho_y)y(k) + (1-\rho_u)u(k-L) \triangleq y_{lost}^1(k) \quad (14)$$

- the quantity

$$\rho_y y(k) + \rho_u u(k-L) - h(k) \triangleq y_{left}(k) \quad (15)$$

  is the amount of goods left in the stock just after OP1-OP3 have been performed. It follows that inside each review period, the total amount of wasted goods is

$$(1-\rho_h)y_{left}(k) + y_{lost}^1(k) \triangleq y_{lost}(k) \quad (16)$$

The model (8) generalizes the stock level balance equations with nonperishable goods ($\rho_y = \rho_u = \rho_h = 1$) as well as those ones with synchronized OP2 and OP3 ($\rho_u = 1$) and also those ones with synchronized OP1-OP3 at the beginning of the review period ($\rho_y = \rho_u = 1$).

For future developments we now rewrite equation (8) in a more convenient form where both $\rho_{T'}$ and $d(k)$ are made explicit.

By (12) an equivalent expression of $h(k)$ is

$$h(k) = d(k) - z(k) \quad (17)$$

for some $z(k) \in [0, d(k)]$ that represents the amount of possibly unsatisfied demand.

Definitions (9)-(11) and condition (17) imply

$$y(k+1) = \rho_{T'}^{n_h}(\rho_{T'}^{n_y} y(k) + \rho_{T'}^{n_u} u(k-L) - d(k) + z(k)) \quad (18)$$

4 PROBLEM FORMULATION

The control problem we consider is to determine an optimal RP matching the three antagonistic CSs defined in the introduction. Owing to the uncertainty on the future demand and the deterioration rate, we adopt a min-max MPC approach which requires to repeatedly solve a min-max constrained optimization problem over a future N steps control horizon $H_k = [k \;\; k+N-1]$ (for some $N < M$) and, according to the receding horizon control, to only apply the first sample of the computed optimal control sequence $u(k+j|k)$, $j = 0, 1, \cdots, N-1$.

At each fixed $k \in Z^+$ and over the corresponding control horizon H_k, the min-max MPC is formally defined as follows:

$$\min_{u(k+j|k)} \max_{\rho_{T'} \in [\rho_{T'}^-, \rho_{T'}^+]} J_k \quad (19)$$

$$\text{subject to: } (7), (18), \quad (20)$$

$$0 \le u_k^- \le u(k+j|k) \le u_k^+ < \infty \quad (21)$$

where

$$J_k = \sum_{i=1}^{N} e^T(k+L+i|k) q_i(k) e(k+L+i|k) + \lambda(k) \Delta u^2(k|k) \quad (22)$$

$$e(k+L+i|k) \triangleq d^+(k+L+i) - y(k+L+i|k) \quad (23)$$

$$\Delta u(k|k) \triangleq u(k|k) - u(k-1) \quad (24)$$

By (7) and (18) we have

$$\begin{aligned} y(k+L+i|k) &= \rho_{T'}^{(n_h+n_y)(L+i)} y(k) \\ &+ \sum_{\ell=0}^{L-1} \rho_{T'}^{(n_h+n_y)(L+i-\ell)-n_y+n_u} u(k+\ell-L) \\ &+ \sum_{\ell=0}^{i-1} \rho_{T'}^{(n_h+n_y)(i-\ell)-n_y+n_u} u(k+\ell|k) \\ &- \rho_{T'}^{(n_h+n_y)(L+i)-n_y} \big(d(k) - z(k)\big) \\ &- \sum_{\ell=1}^{L+i-1} \rho_{T'}^{(n_h+n_y)(L+i-\ell)-n_y} \\ &\times \big(\bar{d}(k+\ell) + \delta d(k+\ell|k) - z(k+\ell|k)\big) \end{aligned} \quad (25)$$

The following considerations are in order:

- by A5 and (23), it can be seen that $M \geq N+L$;
- in the light of CS1 and CS2, requiring the on hand stock level to track the maximum predicted customer demand has a double benefit: 1) the fulfillment of CS1 without incurring the inconvenience of the overstocking usually caused by a constant reference level conservatively fixed in advance; 2) the implicit guarantee of a safety stock to reduce the risk of lost sales and backorders, (Boulaksil, 2016).
- the hard constraints (21) need to guarantee the internal stability of the SC. Moreover, forcing the control effort to fluctuate within a predefined amplitude range allows us to contain the BE. How to compute u_k^- and u_k^+ is explained in Section 4.1;
- the term $\lambda(k)\Delta u^2(k|k)$ has been introduced in order to meet EC1: penalizing large deviations on the control variables smoothes the control effort, thus reducing its variability and the unavoidable costs related to sharp order quantity changes;
- following (G.F. Franklin, 1990), the weights $q_i(k)$, $i=1,\cdots,N$, and $\lambda(k)$, have been chosen inversely proportional to the square of the interval where the relative physical variables are allowed to vary.

4.1 Computing the Bounds U_k^- and U_k^+

The constraints u_k^- and u_k^+ on $u(k+j|k)$ have to be properly determined on the basis of the two following conflicting criteria: 1) the amplitude A_k of the interval $[u_k^- \;\; u_k^+] \triangleq \mathcal{C}_k$ should be large enough to allow the RP to follow demand fluctuations, 2) too large A_k should be avoided in the light of EC2 (the second criterion assessment of the BE).

Owing to the uncertainty on the future demand $d(k)$ and on the decay factor $\rho_{T'}$, we estimate u_k^- and u_k^+ with reference to two possible, limit situations compatible with (18). Consider the following scenario:

- the retailer is able to fully satisfy a customer demand that, over each prediction horizon, is a constant signal with value $\tilde{d}_k \in [d_k^-, d_k^+]$. This implies $z(k)=0$ in (18). The limits d_k^- and d_k^+ are the minimum and maximum values respectively of the customer demand over $[k \;\; k+M]$.
- each control horizon H_k is long enough to allow the output (the on hand stock level), to practically attain the steady-state value $\tilde{y}_k$ under the forcing action of a constant signal $\tilde{u}_k$.

The problem we now consider is: for any given constant demand $\tilde{d}_k \in [d_k^- \;\; d_k^+]$ it is required to find the interval $\mathcal{C}_k$ where the corresponding constant control input $\tilde{u}_k$ takes values, such that the resulting constant steady state value $\tilde{y}_k$ satisfies $\tilde{y}_k \geq \tilde{d}_k$, $\forall \rho_{T'} \in [\rho_{T'}^-, \rho_{T'}^+]$.

Using classical z-transform methods and applying the final value theorem (Kuo, 2007) we have

$$\tilde{y}_k = \left.\frac{\rho_{T'}^{n_h+n_u}}{z^L(z-\rho_{T'}^{n_h+n_y})}\right|_{z=1} \tilde{u}_k - \left.\frac{\rho_{T'}^{n_h}}{(z-\rho_{T'}^{n_h+n_y})}\right|_{z=1} \tilde{d}_k \quad (26)$$

If $\rho_{T'}$ were exactly known, then, choosing $\tilde{u}_k = \frac{1-(\rho_{T'})^{(n_h+n_y)}+(\rho_{T'})^{n_h}}{(\rho_{T'})^{(n_h+n_u)}}\tilde{d}_k$, equation (26) would readily imply $\tilde{y}_k = \tilde{d}_k$. As $\rho_{T'}$ is uncertain, the minimum $\tilde{u}_k$ guaranteeing $\tilde{y}_k \geq \tilde{d}_k$, $\forall \rho_{T'} \in [\rho_{T'}^- \rho_{T'}^+]$ is $\tilde{u}_k = \frac{1-(\rho_{T'}^-)^{(n_h+n_y)}+(\rho_{T'}^-)^{n_h}}{(\rho_{T'}^-)^{(n_h+n_u)}}\tilde{d}_k$.

Considering the two limit situations $\tilde{d}_k = d_k^-$ and $\tilde{d}_k = d_k^+$ we obtain

$$\mathcal{C}_k \triangleq [u_k^-, u_k^+] = \tilde{\rho}_{T'}\,[d_k^-, d_k^+] \quad (27)$$

$$\tilde{\rho}_{T'} \triangleq \frac{1-(\rho_{T'}^-)^{(n_h+n_y)}+(\rho_{T'}^-)^{n_h}}{(\rho_{T'}^-)^{(n_h+n_u)}} \quad (28)$$

The amplitude A_k of $\mathcal{C}_k$ is

$$A_k = \tilde{\rho}_{T'}(d_k^+ - d_k^-) \quad (29)$$

Conditions (27)-(29) give the "a priori" estimate of the BE corresponding to EC2. Condition (28) generalizes to any possible ID the analogous amplification factor $1/\rho^-$ estimated in (Ietto and Orsini, 2023b). As $1/\rho^-$ corresponds to $1/(\rho_{T'}^-)^n$, the two amplification factors coincide when OP1-OP4 are synchronized at beginning of the review period. In fact, in this case we would have $n_h = n$, $n_u = n_y = 0$ and, by (28): $\tilde{\rho}_{T'} = 1/(\rho_{T'}^-)^n = 1/\rho^-$.

5 A MORE CONVENIENT FORMULATION OF THE MIN-MAX MPC

In this section we reformulate the min-max MPC as a CRLS estimation problem. The purpose is to drastically reduce the numerical complexity of the algorithm solving the min-max MPC. For any fixed k, the functional J_k, defined in (19), is minimized assuming that the control sequence $u(j|k)$, $j=k,\cdots,k+N-1$, is given by the sampled version (with sampling period coinciding with the review period) of a B-spline function. Adapting the notation in (2) to specify that it is relative to the k-th fixed time instant we have

$$u(j|k) \triangleq \mathbf{B}_d(j)\mathbf{c}_k, \quad j \in [\hat{k}_1, \hat{k}_{\ell+d+1}] \quad (30)$$

with

1. $\mathbf{B}_d(j) \stackrel{\triangle}{=} [B_{1,d}(j), \cdots, B_{\ell,d}(j)]$
2. $\mathbf{c}_k \stackrel{\triangle}{=} [c_{k,1}, \cdots, c_{k,\ell}]^T$,
3. $\hat{k}_1 = \cdots = \hat{k}_{d+1} = k$ and $\hat{k}_{\ell+1} = \cdots = \hat{k}_{\ell+d+1} = k+N-1$
4. the remaining $\ell - d - 1$ knot points are evenly distributed over $(k, k+N-1)$

Remark 3. Point 3 and the smoothness property of B splines (recalled in Section 2.1) imply that the first sample $u(k|k)$ of the B spline $u(j|k)$ coincides with the first control point $c_{k,1}$ of the vector $\mathbf{c}_k$. $\triangle$

The parameter vector $\mathbf{c}_k$ defining $u(j|k)$, $j = k, \cdots, k+N-1$, is computed as the solution of the CRLS estimation problem defined beneath.

As $\rho_{T'} \in [\rho^-_{T'}\ \ \rho^+_{T'}]$, an equivalent representation of $\rho_{T'}$ is

$$\rho_{T'} = \bar{\rho}_{T'} + \delta\rho_{T'}, \qquad \bar{\rho}_{T'} \stackrel{\triangle}{=} (\rho^-_{T'} + \rho^+_{T'})/2 \tag{31}$$

where $\bar{\rho}_{T'}$ is the nominal value and $\delta\rho_{T'}$ is the perturbation with respect to $\bar{\rho}_{T'}$ satisfying $|\delta\rho_{T'}| \leq (\rho^+_{T'} - \rho^-_{T'})/2$.

From (31) it follows that

$$\rho^k_{T'} = (\bar{\rho}_{T'} + \delta\rho_{T'})^k = \bar{\rho}^k_{T'} + \Delta\rho_{T',k} \tag{32}$$

where $\Delta\rho_{T',k} \stackrel{\triangle}{=} (\bar{\rho}_{T'} + \delta\rho_{T'})^k - \bar{\rho}^k_{T'}$ is the sum of all terms containing $\delta\rho_{T'}$ in the explicit expression of $(\bar{\rho}_{T'} + \delta\rho_{T'})^k$.

Exploiting (32), one has that the term $\rho^{(n_h+n_y)(L+i)}_{T'} y(k)$ of (25) can be rewritten as

$$\begin{aligned} &\rho^{(n_h+n_y)(L+i)}_{T'} y(k) \\ =\ &\big(\bar{\rho}^{(n_h+n_y)(L+i)}_{T'} + \Delta\rho_{T',(n_h+n_y)(L+i)}\big) y(k) \end{aligned} \tag{33}$$

Analogously, the following terms of (25) can be rewritten as

$$\begin{aligned} &\sum_{\ell=0}^{L-1} \rho^{(n_h+n_y)(L+i-\ell)-n_y+n_u}_{T'} u(k+\ell-L) \\ &= \sum_{\ell=0}^{L-1} \Big(\bar{\rho}^{(n_h+n_y)(L+i-\ell)-n_y+n_u}_{T'} \\ &\quad + \Delta\rho_{T',(n_h+n_y)(L+i-\ell)-n_y+n_u}\Big) u(k+\ell-L) \end{aligned} \tag{34}$$

$$\begin{aligned} &\sum_{\ell=0}^{i-1} \rho^{(n_h+n_y)(i-\ell)-n_y+n_u}_{T'} u(k+\ell|k) \\ &= \sum_{\ell=0}^{i-1} \Big(\bar{\rho}^{(n_h+n_y)(i-\ell)-n_y+n_u}_{T'} + \Delta\rho_{T',(n_h+n_y)(i-\ell)-n_y+n_u}\Big) \\ &\times B_d(k+\ell)\mathbf{c}_k \end{aligned} \tag{35}$$

$$\begin{aligned} &\rho^{(n_h+n_y)(L+i)-n_y}_{T'} \big(d(k) - z(k)\big) \\ &= \Big(\bar{\rho}^{(n_h+n_y)(L+i)-n_y}_{T'} + \Delta\rho_{T',(n_h+n_y)(L+i)-n_y}\Big) \\ &\times \big(d(k) - z(k)\big) \end{aligned} \tag{36}$$

$$\begin{aligned} &\sum_{\ell=1}^{L+i-1} \rho^{(n_h+n_y)(L+i-\ell)-n_y}_{T'} \bar{d}(k+\ell) \\ &= \sum_{\ell=1}^{L+i-1} \Big(\bar{\rho}^{(n_h+n_y)(L+i-\ell)-n_y}_{T'} + \Delta\rho_{T',(n_h+n_y)(L+i-\ell)-n_y}\Big) \\ &\times \bar{d}(k+\ell) \end{aligned} \tag{37}$$

$$\begin{aligned} &\sum_{\ell=1}^{L+i-1} \rho^{(n_h+n_y)(L+i-\ell)-n_y}_{T'} \big(\delta d(k+\ell|k) - z(k+\ell|k)\big) \\ &= \sum_{\ell=1}^{L+i-1} \Big(\bar{\rho}^{(n_h+n_y)(L+i-\ell)-n_y}_{T'} + \Delta\rho_{T',(n_h+n_y)(L+i-\ell)-n_y}\Big) \\ &\times \big(\delta d(k+\ell|k) - z(k+\ell|k)\big) \end{aligned} \tag{38}$$

Equations (33)-(38) allow us: 1) to separate the terms depending on the optimal control sequence $u(k+\ell|k) = B_d(k+\ell)\mathbf{c}_k$ from the independent ones; 2) to separate, in either groups of terms, the known quantities from the unknown ones. Using (33)-(38), an equivalent representation of the predicted tracking error given by (23), formally similar to that given in (3), is

$$e(k+L+i|k) = (b_{k,i} + \delta b_{k,i}) - (A_{k,i} + \delta A_{k,i})\mathbf{c}_k \tag{39}$$

where

$$\begin{aligned} b_{k,i} &= d^+(k+L+i) - \bar{\rho}^{(n_h+n_y)(L+i)}_{T'} y(k) \\ &- \sum_{\ell=0}^{L-1} \bar{\rho}^{(n_h+n_y)(L+i-\ell)-n_y+n_u}_{T'} u(k+\ell-L) \\ &+ \bar{\rho}^{(n_h+n_y)(L+i)-n_y}_{T'} \big(d(k) - z(k)\big) \\ &+ \sum_{\ell=1}^{L+i-1} \bar{\rho}^{(n_h+n_y)(L+i-\ell)-n_y}_{T'} \bar{d}(k+\ell) \end{aligned} \tag{40}$$

$$\begin{aligned} \delta b_{k,i} &= -\Delta\rho_{T',(n_h+n_y)(L+i)} y(k) \\ &- \sum_{\ell=0}^{L-1} \Delta\rho_{T',(n_h+n_y)(L+i-\ell)-n_y+n_u} u(k+\ell-L) \\ &+ \Delta\rho_{T',(n_h+n_y)(L+i)-n_y} \big(d(k) - z(k)\big) \\ &+ \sum_{\ell=1}^{L+i-1} \Delta\rho_{T',(n_h+n_y)(L+i-\ell)-n_y} \bar{d}(k+\ell) \\ &+ \sum_{\ell=1}^{L+i-1} \Big(\bar{\rho}^{(n_h+n_y)(L+i-\ell)-n_y}_{T'} \\ &+ \Delta\rho_{T',(n_h+n_y)(L+i-\ell)-n_y}\Big) \\ &\times (\delta d(k+\ell|k) - z(k+\ell|k)) \end{aligned} \tag{41}$$

$$A_{k,i} = \sum_{\ell=0}^{i-1} \bar{\rho}_{T'}^{(n_h+n_y)(i-\ell)-n_y+n_u} B_d(k+\ell) \quad (42)$$

$$\delta A_{k,i} = \sum_{\ell=0}^{i-1} \Delta\rho_{T',(n_h+n_y)(i-\ell)-n_y+n_u} B_d(k+\ell) \quad (43)$$

By Remark 3 also the term $\Delta u(k|k) = u(k|k) - u(k-1) = c_{k,1} - u(k-1)$ in (22) can be rewritten as

$$\Delta u(k|k) = (b_{u_k} + \delta b_{u_k}) - (A_{u_k} + \delta A_{u_k})\mathbf{c}_k \quad (44)$$

where $b_{u_k} = -u(k-1)$, $\delta b_{u_k} = 0$, $A_{u_k} = -\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ and δA_{u_k} is a null row vector.

Defining the following augmented column vectors $\underline{e}_k$, $\underline{b}_k$, $\underline{\delta b}_k$ of $N+1$ elements

$$\underline{e}_k = \begin{bmatrix} q_1^{1/2}(k)e(k+L+1|k) \\ \vdots \\ q_N^{1/2}(k)e(k+L+N|k) \\ \lambda^{1/2}(k)\Delta u(k|k) \end{bmatrix}, \quad (45)$$

$$\underline{b}_k = \begin{bmatrix} q_1^{1/2}(k)b_{k,1} \\ \vdots \\ q_N^{1/2}(k)b_{k,N} \\ \lambda^{1/2}(k)b_{u_k} \end{bmatrix} \quad \underline{\delta b}_k = \begin{bmatrix} q_1^{1/2}(k)\delta b_{k,1} \\ \vdots \\ q_N^{1/2}(k)\delta b_{k,N} \\ \lambda^{1/2}(k)\delta b_{u_k} \end{bmatrix} \quad (46)$$

and the extended matrices $\underline{A}_k$ and $\underline{\delta A}_k$ of dimensions $((N+1)\times\ell)$

$$\underline{A}_k = \begin{bmatrix} q_1^{1/2}(k)A_{k,1} \\ \vdots \\ q_N^{1/2}(k)A_{k,N} \\ \lambda^{1/2}(k)A_{u_k} \end{bmatrix} \quad \underline{\delta A}_k = \begin{bmatrix} q_1^{1/2}(k)\delta A_{k,1} \\ \vdots \\ q_N^{1/2}(k)\delta A_{k,N} \\ \lambda^{1/2}(k)\delta A_{u_k} \end{bmatrix} \quad (47)$$

allow us to reformulate the min-max MPC (19)-(21) as the following CRLS estimation problem:

$$\min_{\mathbf{c}_k} \max_{\|\underline{\delta A}_k\|\le\beta_k \, \|\underline{\delta b}_k\|\le\xi_k} \|(\underline{b}_k + \underline{\delta b}_k) - (\underline{A}_k + \underline{\delta A}_k)\mathbf{c}_k\|^2 \quad (48)$$

$$\text{subject to} \qquad u_k^- \le c_{k,i} \le u_k^+, \quad i = 1,\cdots,\ell. \quad (49)$$

Constraints (49) derive from (30) and the convex hull property of B splines.

By (3) (that is equivalent to (5)) and considering that

$$\arg\min_x \sum_i \|g_i(x)\| \equiv \arg\min_x (\sum_i \|g_i(x)\|)^2$$

it is seen that (48) define a problem of the kind (3). Hence, according to Section 2.2, at any k, the solution $\mathbf{c}_k$ of the CRLS estimation problem (48)-(49) can be determined solving

$$\min_{\mathbf{c}_k} \|\underline{b}_k - \underline{A}_k\,\mathbf{c}_k\| + \beta_k\|\mathbf{c}_k\| + \xi_k \quad (50)$$

where the components of $\mathbf{c}_k$ must satisfy (49).

Remark 4. The maximum euclidean norm of the term $\underline{\delta b}_k$ given by (45) corresponds to the term ξ_k of (50). It is the analogous of the term ξ in (3). As ξ_k is independent of the control sequence $u(k+j|k), j = 0\cdots N-1$ (i.e. the term x of (3)) its maximum norm can be minimized with respect to $\Delta\rho_{T'}$ and $\delta d(k+\ell|k)$ assuming nominal $\rho_{T'}$ and predicted customer demand given by the respective central values $\bar{\rho}_{T'}$ and $\bar{d}(k+j), j = 1\cdots M$. $\triangle$

Remark 5. As for the numerical calculation of β_k and ξ_k, the following considerations hold: 1) as the term ξ_k of (50) is independent of $\mathbf{c}_k$, it cannot be minimized. Hence it can be removed from the objective function; 2) only the upper bound β_k on $\|\underline{\delta A}_k\|$ needs to be determined at each k. The way the B-spline basis functions are defined by the Cox de Boor formula implies that $\mathbf{B}_d(\tau) = \mathbf{B}_d(\tau+N), \forall\tau \in H_k, k \in Z^+$. Hence by (43) and (47) one has that $\beta_k \triangleq \beta, \forall k = 0,1,\cdots$ and moreover β is determined putting $\rho_{T'} = \rho_{T'}^+$. $\triangle$

Remark 6. Feasibility and internal asymptotic stability of the proposed min-max MPC control strategy are a direct consequence of the SC dynamics and of our approach. Feasibility of (21) derives from: (30), the consistency of (49) w.r.t. (48) and the convex hull property of B-splines. The uniform boundedness of all physical variables of the controlled SC derives from: $\rho_{T'} \in (0,1)$, the assumed uniform boundedness of $d(k), k \in Z^+$ and constraints (21). Both stability and feasibility are independent of the length of the prediction horizon.

6 ILLUSTRATIVE EXAMPLE

We consider a single echelon SC where the review period is $T = nT' = 2$(weeks) with $n = 14$ and $T' = 1$(day). The model parameters are: $L = 2$ (lead time), $\alpha_{T'} \in [\alpha_{T'}^- \ \ \alpha_{T'}^+] = [0.05 \ \ 0.1]$ (deterioration rate over T'), $\rho_{T'} \in [\rho_{T'}^- \ \ \rho_{T'}^+] = [0.9 \ \ 0.95]$ (decay factor over T') and $y(0) = 0$ (initial stock level).We suppose that equation (18) is characterized by: $n_h = 8, n_y = 6, n_u = 4$, so that $\rho_h = \rho_{T'}^{n_h} \in [0.4305 \ \ 0.6634]$, $\rho_y = \rho_{T'}^{n_y} \in [0.5314 \ \ 0.7351]$ and $\rho_u = \rho_{T'}^{n_u} \in [0.6561 \ \ 0.8145]$. At each $k \in Z^+$, the min-max MPC is solved parametrizing $u(j|k)$ in (30) as a B-spline function of degree $d = 1$ with $\ell = 3$ control points.

This allowed us to verify that good results can be obtained by simply imposing a C_0 continuity and few control points. By (31) we have $\bar{\rho}_{T'} = 0.925$ and, according to Section 4, we choose $q_i(k) = \frac{1}{(0.005\cdot d^+(k+L+i))^2}e^{(i-1)}$ and $\lambda(k) = \frac{1}{(0.005\cdot u(k-1))^2}$ (weights in (22)). By Remark 5, we obtain that the parameter $\beta_k \stackrel{\triangle}{=} \beta$ in (50) is given by $\beta = 0.6363$. We also assumed an "a priori" empirical knowledge on the future customer demand limited to an $M = 8$ steps prediction horizon. Hence, the length N of each control horizon H_k has been chosen as $N = M - L = 6$.

Figure 4 shows the compact set $\mathcal{D}$ enclosing the whole actual customer demand.

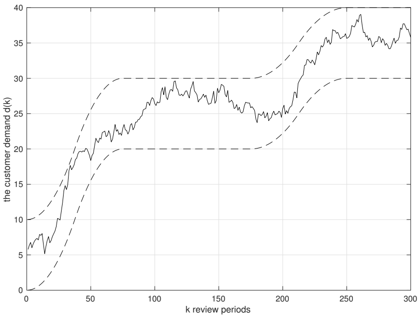

Figure 4: The actual end customer demand $d(k)$ (solid line). The dashed lines represents the compact set $\mathcal{D}$ given by the consecutive contiguous overlapping of all the "a priori" given sets D_k's.

The dynamic equation (18) has been implemented assuming an actual $\rho_{T'} = 0.9$ and the simulation has been stopped at time $k = 280$.

The generated RP is shown in figure 5. This figure shows a control signal satisfying EC1 and EC2: the RP has a smooth waveform and belongs to the interval defined by (27)- (28). The actual customer demand $d(k)$ and the fulfilled one $h(k)$ are reported in figure 6. This figure evidences the effectiveness of the proposed RP: the customer demand is not satisfied only for $k \leq L = 2$, as a consequence of lead time and null initial stock.

We have also carried out a second simulation to show the effects of neglecting the available information on the actual ID: we have calculated the RP under the erroneous hypothesis that that all the OPs are synchronized at the beginning of the review period. As a consequence, the min-max MPC has been implemented assuming $n_h = n = 14$ and $n_u = n_y = 0$. Although the amount of fulfilled customer demand is the same, a serious performance degradation of the SC is observed in terms of increased wasted and stocked goods (see Table 1).

A performance degradation is also observed in the EC2 measure of the BE: applying (28) we obtain $\tilde{\rho}_{T'} = 3.7360$ (actual ID) and a larger value $\tilde{\rho}_{T'} = 1/(\rho^{-}_{T'})^n = 4.3712$ (synchronized ID). Comparing figures 5 and 7 shows that the interval containing the RP is tighter in the min-max MPC strategy based on the actual ID and CS1 is satisfied with a smaller control effort.

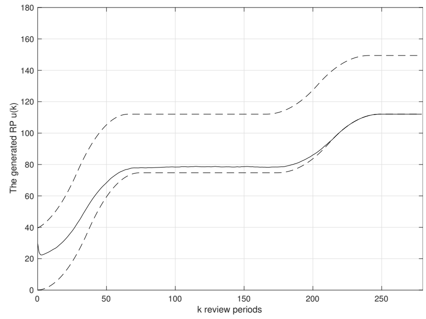

Figure 5: The generated RP (solid line) and the boundaries trajectories u_k^- and u_k^+ (dashed lines) computed by (27)- (28) with $n_h = 8$ and $n_u = 4$ and $n_y = 6$.

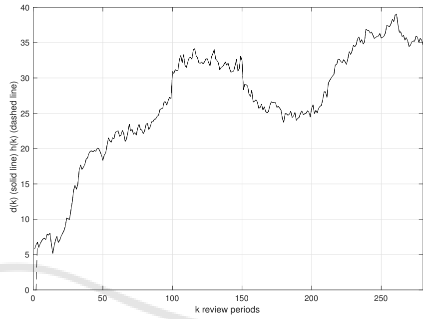

Figure 6: The actual customer demand $d(k)$ (solid line) and the fulfilled customer demand $h(k)$ (dashed line).

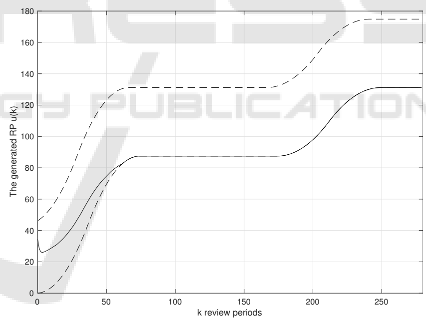

Figure 7: The generated RP (solid line) and the boundaries trajectories u_k^- and u_k^+ (dashed lines) computed by (27)- (28) with $n_h = n = 14$ and $n_u = n_y = 0$.

7 CONCLUSIONS

The rationale of our contribution is based on two considerations: 1) the simplistic assumption of an exactly known deterioration rate is never verified in practice, 2) the sequence of OPs inside each review period can be imposed by external factors that can not be controlled by the SC manager. To deal with these problems we introduced the notion of ID and defined a more realistic and general dynamic model encompassing many other SC models proposed in the literature. Endowing the min-max MPC with the informa-

Table 1: Quantitative performance evaluation.

	wasted goods $\sum_{k=0}^{280} y_{lost}(k)$	*stocked goods* $\sum_{k=0}^{280} y(k)$
MPC ($n_h = 8, n_u = 4, n_y = 6$)	14848	4999
MPC ($n_h = 14, n_u = n_y = 0$)	17782	6318

tion carried by the actual ID yields a more effective RP. In particular, the numerical simulations show a significant improvement in terms of reduced wasted and stocked goods.

Our study also reveals the following managerial insights with both academic and practical relevance:

- As stability and feasibility of the MPC control law are guaranteed independently of the length of the prediction horizon, the demand prediction problem is greatly facilitated;
- The worst-case approach provides the manager with the security of an effective inventory control despite the uncertainties;
- In the case of manageable IDs, our study provides the manager with the information necessary to define the best organizational policy: equations (14),(16) show that the waste of goods can be minimized synchronizing OP1, OP2 and OP3 at the beginning of each review period.

REFERENCES

Boulaksil, Y. (2016). Safety stock placement in supply chains with demand forecast updates. *Operations Research Perspectives*, 3:27–31.

Chaudary, V., Kulshrestha, R., and Routroy, S. (2018). State of the art literature review on inventory models for perishable products. *Journal of Advances in Management Research*, 1:306–346.

Cholodowicz, E. and Orlowski, P. (2023). Switching robust neural network control of perishable inventory with fixed shelf life products under time-varying uncertain demand. *J. of Computational Science*, 70.

De-Boor, C. (1978). *A practical guide to splines*. Springer Verlag, New York, 2nd edition.

Gaggero, M. and Tonelli, F. (2015). Optimal control of distribution chains for perishable goods. *IFAC Papers On Line*, 48:1049–1054.

G.F. Franklin, J.D. Powell, M. W. (1990). *Digital Control of Dynamic Systems*. Addison-Wesley Publishing Company, N.Y, 2nd edition.

Hipolito, T., Nabais, J., Benitez, R., Botto, M., and Negenborn, R. (2022). A centralised model predictive control framework for logistics management of coordinated supply chain of perishable goods. *International Journal of Systems Science: Operation & Logistics*, 9:1–21.

Ietto, B. and Orsini, V. (2022a). Effective inventory control in supply chain with large uncertain decay factor. In *30th Mediterranean Conference on Control and Automation*. IEEE.

Ietto, B. and Orsini, V. (2022b). Resilient robust model predictive control of inventory systems for perishable good under uncertain forecast information. In *2022 International Conference on Cyber-physical Social Intelligence (Best paper finalists award)*. IEEE.

Ietto, B. and Orsini, V. (2023a). Managing inventory level and bullwhip effect in multi stage supply chains with perishable goods: A new distributed model predictive control approach. In *12th International Conference on Operations Research and Enterprise Systems*. SCITEPRESS.

Ietto, B. and Orsini, V. (2023b). Optimal control of inventory level for perishable goods with uncertain decay factor and uncertain forecast information: a new robust mpc approach. *International Journal of Systems Science: Operations & Logistics*, 10:1–13.

Ignaciuk, P. (2014). Discrete inventory control in systems with perishable goods- a time delay system perspective. *IET Control Theory and Applications*, 8:11–21.

Ignaciuk, P. (2015). Discrete time control of production-inventory systems with deteriorating stock and unreliable supplies. *IEEE Transactions on Systems Man and Cybernetics*, 45:338–348.

Ivanov, D., Sethi, S., Dolgui, A., and Sokolov, B. (2018). A survey on control theory applications to operational systems, supply chain management, and industry 4.0. *Annual Reviews in Control*, 46:134–147.

Kuo, B. (2007). *Digital Control Systems*. Oxford University Press, Oxford, 2nd edition.

Lejarza, F. and Baldea, M. (2020). Closed-loop real-time supply chain management for perishable products. *IFAC PapersOnLine*, 53:11458–14463.

Leśniewsky, P. and Bartoszewicz, A. (2020). Optimal model reference sliding mode control of perishable inventory systems. *IEEE Trans. Autom. Science and Engineering*, 17:1647–1656.

Li, R., Lan, H., and Mawhinney, J. (2010). A review on deteriorating inventory study. *Journal of Service Science and Management*, 3:117–129.

Lobo, M., Vandenberghe, L., Boyd, S., and Lébret, H. (1998). Second-order cone programming. *Linear Algebra and its Applications*, 284:193–218.

Pan, X. and Li, S. (2015). Optimal control of a stochastic-inventory system under deteriorating items and environments constraints. *Int. Journal of Production Research*, 53:2937–2950.

Rossiter, J. and Bishop, R. (2004). *Model Based Predictive*

Control. A Practical Approach. CRC PRESS, Boca Raton, 1st edition.

Sarimveis, H., Patrinos, P., Tarantilis, C., and Kiranoudis, C. (2008). Dynamic modeling and control of supply chain system: A review. *Computers and Operation Research*, 35:353–356.

Taparia, R., Janardhanan, S., and Gupta, R. (2020). Inventory control for nonperishable and perishable goods based on model predictive control. *International Journal of Systems Science: Operations & Logistics*, 7:361–373.