Preliminary Results on Controllability of Serial Robot-Manipulators in Singular Configurations

Mir Mamunuzzaman and Jörg Mareczek
Faculty of Electrical and Industrial Engineering, Landshut University of Applied Sciences, 84036 Landshut, Germany

Keywords: Singularity, Non-Linear Controllability, Jacobian, Robot Motion Planning.

Abstract: We develop a standard system representation and analyse controllability properties for velocity kinematics of robot-manipulators located on singularities. These are positions where the Jacobian loses rank. Since its column vectors span the set of admissible workspace velocity directions, it is still a widespread misunderstanding that some directions would be locked on singularities and thus had to be bypassed as far as possible. We will show that this does not generally hold: On some types of singularities, the kinematic shows local redundancy, which can be used to generate paths crossing the singularity in any desired workspace velocity direction. To further analyse controllability properties, we develop an SVD-based method to represent the Jacobian-based velocity kinematics in standard system description of control theory without the need for inverse kinematics (IK). In many cases, IK do not offer a unique solution on singularities and, therefore, cannot be used. Furthermore, we present a modification of the SVD-based method for which the analytical calculation effort is feasible. The resulting system description has the advantage of being a simple decoupled set of single-integrators where the system states are divided into one set describing admissible workspace motions and a second set describing possible internal motions, also called nullspace motion. Based on this standard system representation, we determine local controllability and local accessibility for two different types of singularities. Finally, we illustrate our methods by means of a simple 3-DoF SCARA-type manipulator.

1 INTRODUCTION

Ongoing industrial transformation includes more networked, automated and intelligent manufacturing lines, (Kuo et al., 2017), (Mazzanti et al., 2019), (Nahavandi, 2019). This leads to an increasing degree of autonomy in industrial automation planning. One emerging technology towards this direction is computer-aided manufacturing (CAM), which aims to connect CAD directly with a CNC tooling machine. Beyond that, more and more industrial robots (in the following denoted as *manipulators*) are used for tooling. They offer the advantage of high dexterity and a large workspace. Prominent examples are additive manufacturing with large-scale 3D-printing, cladding or buildup welding, cover manipulation or human-robot collaboration.

These new manufacturing processes are characterized by paths that are either very long or non-predictable and are changed very often. This can also be observed in many other fields of applications like telesurgery, emergency manipulation inside nuclear reactor cells and explosive ordnance disposal.

Clearly, these types of paths cannot be implemented anymore by classical manual teaching or programming; the corresponding work effort would be excessive. In contrast to that, modern path planning requires the calculation of adequate joint paths (jointspace) from given, desired paths of the end-effector (workspace); the corresponding mapping from workspace to jointspace is denoted as *inverse kinematics* (IK). Problems arise from the fact that a method to solve IK analytically is only known for a very restricted type of kinematics fulfilling Pieper's requirements. Moreover, IK for six or more joints suffer from a high computational complexity.

In addition to joint positions, path-tracking control also requires joint velocities as feed-forward control input. These velocities are calculated by means of the inverse of the Jacobian matrix (in the following denoted as *Jacobian*) of the direct kinematics.

Unfortunately, the rank of the Jacobian typically depends on joint positions, and there exist positions where it loses rank, so-called *singularities*. It is well known that approaching such a singularity turns into excessively large joint speeds unless the commanded

Mamunuzzaman, M. and Mareczek, J.
Preliminary Results on Controllability of Serial Robot-Manipulators in Singular Configurations.
DOI: 10.5220/0012238000003543
In *Proceedings of the 20th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2023) - Volume 1*, pages 327-334
ISBN: 978-989-758-670-5; ISSN: 2184-2809

end-effector speed points into a special set of admissible directions. Although this problem has been known from the very beginning of robotics, singularities still suffer from little research interest. In fact, it is the general view of many roboticists that singularities impose an unsolvable problem and are thus to be bypassed as far as possible. This is also the basic idea behind all existing control methods, for example, *damped least-square control* (DLSC), (Wampler, 1986) or related methods like *Non-Redundancy Singularity Avoidance* (NRSA), (Huang et al., 2016), *Singularity Avoidance System* (SAS), (Raineri and Bianco, 2017), *Partially Decoupling and Linearizing Control*, (PDLC), (Mareczek, 2020). These approaches all lead to restrictions in tracking performance within close vicinities of singularities (called singular regions); these restrictions are not tolerable in many applications.

However, singularities may also offer advantages: Here, a manipulator can show local redundancy. Furthermore, maximum forces and torques exerted by the end-effector on the environment depend only on break limits of mechanics rather than on actuator limitations. For these reasons, a path planning method together with an appropriate control which does not have to avoid singularities is desirable. In fact, it is not necessary to avoid all kinds of singularities: As we will show by an example, there always exists a smooth path crossing one type of singularity in any predefined workspace direction. This is possible by taking advantage of kinematic redundancy available locally on that type of singularity.

Questions concerning admissible directions into which the system states may evolve can be characterized by its controllability properties. Therefore, this paper focuses on a standard system description of control theory for Jacobian-based velocity kinematics. Based on that, we present preliminary results towards the characterization of singularities with respect to controllability properties.

Actually, no such classification of singularities can be found in literature. Instead, common ways to classify singularities are: *internal or boundary type*, see for example (Spong et al., 2020). In (Kieffer, 1994) *isolated and bifurcation type* of singularities are defined by a differential geometric approach characterising the curve geometry of achievable paths in workspace. This approach uses the solution for IK, entailing its above-mentioned problems. Interestingly, the author suggests to "make use of kinematic singularities, rather than simply negotiate past them".

In (Nielsen et al., 1991), controllability has been investigated with respect to the standard rigid body model from mechanics. Hence, this system description is very complicated. Today's drive systems (motors with gears) are fast enough to operate in velocity rather than torque mode. Therefore, velocity kinematics are primarily used for motion control. However, the authors also concluded: "One may consider [other possibilities for controller design] instead of just giving up and saying that control in certain directions is impossible at singularities".

From the point of view of nonlinear control theory, the Jacobian-based velocity kinematics represent a nonlinear driftless system with the Jacobian column vectors as control vector fields. However, the latter depends on the time-integral of the control input variables. IK can solve this problem, but not on types of singularities where IK don't offer a unique solution. Therefore, we present a method to transform the Jacobian-based velocity kinematics into a standard system description without IK, thus being applicable and valid on singularities, also. Further advantages are: Decoupling of dynamics for admissible workspace motion and possible internal dynamics (nullspace motion). Thus, the nature of variable structure shows clearly by the variable number of system states describing admissible workspace motion. Moreover, this leads to a natural and easy way to investigate and classify internal dynamics. Last but not least, the new system appears as a decoupled set of single integrators, which strongly simplifies further system analysis and proofs.

We organize our paper as follows: First, we develop a method based on SVD to rewrite the velocity kinematics in the standard description of control theory without the need for IK. Then, in section 3, we present decoupling of the dynamics. In Section 4, we prove that for this sample kinematic, there always exists a smooth path crossing on type of singularities. Finally, in section 5, we apply the proposed method to a simple SCARA-type kinematic and conclude with some preliminary results on controllability.

2 PROBLEM FORMULATION

In this section, we outline the control problem of robot-manipulators on singularities. To do this, let's first review the kinematics of serial robotic manipulators. The reader is referred to standard textbooks for more details on robot kinematics.

We start by recalling the distinction between redundant and non-redundant robot-manipulators. There are various ways to define redundancy (Conkur and Buckingham, 1997). In this work, we solely take kinematic redundancy into account. The workspace for robot-manipulators typically describes the cartesian position (e.g., x, y, z) and/or orientation (e.g., Euler-angles α, β, γ) of the robot's end-effector. For robot-manipulators, the variables that define the configuration comprise the jointspace. When the workspace and jointspace dimension are the same[1], a robot-manipulator is said to be non-redundant. On the other hand, the manipulator is considered redundant when the workspace dimension is greater than the jointspace dimension.

Consider a non-redundant serial-link robot manipulator with n rigid links connected by revolute or prismatic joints. Defining n joint variables by $\mathbf{q} = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \end{pmatrix}^{\mathbf{T}} \in \mathcal{Q} \subset \mathbb{R}^n$, which is also referred to a configuration, and n workspace coordinates by $\boldsymbol{\eta} = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_n \end{pmatrix}^{\mathbf{T}} \in \mathcal{P} \subset \mathbb{R}^n$, the forward kinematics are expressed as

$$\boldsymbol{\eta} = \mathbf{f}(\mathbf{q}), \tag{1}$$

with a non-linear, continuous and differentiable direct mapping $\mathbf{f}: \mathcal{Q} \mapsto \mathcal{P}$. The IK problem is to compute the inverse mapping, $\mathbf{f}^{-1}$, that gives us joint variables for a given fixed end-effector's position and orientation. There is a countable set of solutions possible:

$$\mathbf{f}^{-1} : (\boldsymbol{\eta}, \kappa) \mapsto \mathbf{q},\ \kappa \in \{1, \cdots, k\} \tag{2}$$

with k as the number of possible configurations. Moreover, the solutions of IK may be infinite (hence an uncountable set of solutions), or no solution at all may exist.

Differentiating equation (1) with respect to time yields

$$\dot{\boldsymbol{\eta}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f}(\mathbf{q}(t)) = \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}}\,\dot{\mathbf{q}} = \tilde{J}(\mathbf{q})\,\dot{\mathbf{q}} = \sum_{i=1}^{n} \mathbf{j}_i(\mathbf{q})\,\dot{q}_i, \tag{3}$$

where $\tilde{J}(\mathbf{q}) = \begin{bmatrix} \mathbf{j}_1(\mathbf{q}) & \mathbf{j}_2(\mathbf{q}) & \cdots & \mathbf{j}_n(\mathbf{q}) \end{bmatrix} \in \mathbb{R}^{n\times n}$ is the configuration dependant Jacobian of $\mathbf{f}(\mathbf{q})$.

In this work, we define a configuration as *singular* when the Jacobian loses one or more ranks, hence if $\det(\tilde{J}) = 0$. This leads to the severe restriction that the robot's end-effector can no longer move in certain directions. To investigate this in more detail, we split the space of motion of the end-effector into two subspaces: *regular space*, which contains admissible directions, and *singular space*, which contains locked directions. The singular space must be an orthonormal complement of the range of $\tilde{J}$, which establishes the regular space.

However, it is still possible to move in the direction of singular space in some cases, as we will show later. To find out under which conditions this is possible, we consider velocity kinematics (3) as a control system and investigate its controllability properties on the singular subspace. First, let's recall the standard general nonlinear system representation:

$$\dot{\mathbf{x}} = \mathbf{g}_0(\mathbf{x}) + \sum_{i=1}^{n} \mathbf{g}_i(\mathbf{x})\,u_i, \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^m$ contains the system-states, $\mathbf{u} = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}^{\mathbf{T}} \in \mathbb{R}^n$ are the control-input variables, and $\mathbf{g}_i,\ i \in \{0, \cdots, n\}$ are smooth vector fields defining the unforced ($\mathbf{g}_0$: drift) and forced ($\mathbf{g}_1, \mathbf{g}_2, \cdots$: control) motions. Comparing (3) with standard model (4) yields $\mathbf{g}_0 = \mathbf{0}$ (driftless system), joint angle velocities as control-inputs $\mathbf{u} = \dot{\mathbf{q}}$, workspace variables as system-states $\mathbf{x} = \boldsymbol{\eta}$, and columns of $\tilde{J}(\mathbf{q})$ as control vector fields. The latter, however, explicitly depend on joint variables $\mathbf{q}$ rather than on system-states $\boldsymbol{\eta}$. One approach to eliminate the dependency of $\mathbf{q}$ is by substituting the IK (2):

$$\dot{\boldsymbol{\eta}} = \tilde{J}(\mathbf{f}^{-1}(\boldsymbol{\eta}, \kappa))\,\dot{\mathbf{q}}.$$

The IK, nonetheless, are only known for a few basic kinematics. Furthermore, most kinematics do not have a unique solution for the IK on a singularity. As a result, this strategy does not apply to most manipulators. Instead, we prefer a system model, see section 3, that does not require the knowledge of IK.

Additionally, the primary instrument for analysing controllability of a non-linear system, Lie brackets from differential geometry, cannot be used since necessary smoothness assumptions are not met on singularities. For system (3), we can see this issue mathematically. The Lie brackets of any two vector fields can be expressed by

$$[\mathbf{j}_1, \mathbf{j}_2] = \frac{\partial \mathbf{j}_2}{\partial \boldsymbol{\eta}}\,\mathbf{j}_1 - \frac{\partial \mathbf{j}_1}{\partial \boldsymbol{\eta}}\,\mathbf{j}_2.$$

It is apparent that, on singularities, $\frac{\partial \mathbf{j}_i}{\partial \boldsymbol{\eta}}$ is not defined in all workspace directions $\partial\boldsymbol{\eta}$. Therefore, the differential geometric approach to controllability analysis on singularities cannot be applied.

3 SYSTEM REPRESENTATION

As starting point for a state-space formulation of the system on singularities, in the system description, in addition to the dynamics for the workspace variables $\boldsymbol{\eta}$, we also take into account the dynamics for joint space variables $\mathbf{q}$:

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \tilde{J}(\mathbf{q})\,\mathbf{u} \\ \dot{\mathbf{q}} &= \mathbf{u} \end{aligned} \tag{5}$$

This seems to increase the system's total number of state variables to $2n$. The number of state variables,

[1]We assume the workspace and the jointspace both to be expressed in minimal coordinates.

however, remains n since the forward kinematics constrains the joint space variables to the workspace variables.

It has already been stated that the Jacobian loses ranks on singular configurations. We define a set that includes all singular configurations by

$$\mathbb{Q} = \{\mathbf{q}\,|\,\det(\tilde{J}(\mathbf{q})) = 0\}. \tag{6}$$

Elements of the above set are denoted as $\mathbf{q}^*$. Let, on $\mathbf{q}^* \in \mathbb{Q}$, the Jacobian loses r ranks, i.e., $\operatorname{rank}(\tilde{J}(\mathbf{q}^*)) = n-r$. This configuration is also called a *singular configuration of codimension r*.

To better understand possible motions on singularities, we decouple the system-states into two groups: One spanning the regular space, the other one spanning the singular space. This can be performed by singular value decomposition (SVD). An eigenvalue decomposition (EVD) could also be applied as alternative to SVD in case of a quadratic Jacobian. However, unlike EVD, SVD can also be used to address the more general case of redundant manipulators (i.e., non-square Jacobian).

Let us consider a singularity of codimension r. Here, the SVD of the manipulator Jacobian $\tilde{J}(\mathbf{q}^*)$ can be decomposed into

$$\tilde{J}(\mathbf{q}^*) = \tilde{U}(\mathbf{q}^*)\,\tilde{\Sigma}(\mathbf{q}^*)\,\tilde{V}^{\mathbf{T}}(\mathbf{q}^*), \tag{7}$$

where columns $\mathbf{u}_i$ of $\tilde{U}$ and $\mathbf{v}_i$ of $\tilde{V}$ for $i \in \{1, \cdots, n\}$ form an orthonormal basis of $\mathbb{R}^n$, and singular values $\sigma_i^*,\ i \in \{1, \cdots, n\}$ with the r smallest singular values equal to zero form the diagonal matrix $\tilde{\Sigma}(\mathbf{q}^*) \in \mathbb{R}^{n\times n}$. Without loss of generality we may assume that all vanishing singular values are gathered at the end, i.e., $\tilde{\Sigma}(\mathbf{q}^*) = \operatorname{diag}\{\sigma_1^*, \cdots, \sigma_{n-r}^*, 0, \cdots, 0\}$. Denoting the left and right singular matrix on a singular configuration with $\tilde{U}^* = \tilde{U}(\mathbf{q}^*)$ and $\tilde{V}^* = \tilde{V}(\mathbf{q}^*)$ and the diagonal matrix by $\tilde{\Sigma}^* = \tilde{\Sigma}(\mathbf{q}^*) = \operatorname{diag}\{\tilde{\Sigma}_{\mathrm{nz}}^*, 0, \cdots, 0\}$, the SVD of $\tilde{J}(\mathbf{q}^*)$ can be expressed as

$$\tilde{J}^* = \tilde{U}^*\,\tilde{\Sigma}^*\,\tilde{V}^{*\mathbf{T}}, \tag{8}$$

where $\tilde{\Sigma}_{\mathrm{nz}}^*$ is a diagonal matrix comprised of all non-zero singular values of the Jacobian. We now develop a state-space representation for the velocity kinematics using SVD of the Jacobian on singular configurations:

Proposition 3.1. *Assume velocity kinematics (5) are on a singular configuration. We introduce a virtual control* $\boldsymbol{\zeta} = \begin{pmatrix}\zeta_1 & \cdots & \zeta_n\end{pmatrix}^{\mathrm{T}}$. *Then by defining n new system states* $\mathbf{z} = \begin{pmatrix}z_1 & \cdots & z_n\end{pmatrix}^{\mathrm{T}}$, *it's always possible to rewrite system representation (5) in the following decoupled form:*

$$\begin{aligned}\dot{\mathbf{z}} &= \boldsymbol{\zeta}\\ \dot{\mathbf{q}} &= \mathbf{h}(\mathbf{q})\,\boldsymbol{\zeta},\end{aligned} \tag{9}$$

with $\mathbf{h}(\mathbf{q})$ *as a virtual control function possibly nonlinearly relates the virtual and the actual control variables.*

Proof. The assumption that velocity kinematics (5) are on a singular configuration implies rank deficiency of the Jacobian. Let us consider a singularity of codimension r. The left singular matrix $\tilde{U}^*$ can be decomposed as $\tilde{U}^* = \begin{bmatrix}\tilde{U}_{\mathrm{I}}^* & \tilde{U}_{\mathrm{S}}^*\end{bmatrix}$ with $\tilde{U}_{\mathrm{I}}^* = \begin{bmatrix}\mathbf{u}_1^* & \cdots & \mathbf{u}_{n-r}^*\end{bmatrix} \in \mathbb{R}^{n\times n-r}$ that spans the range space of $\tilde{J}^*$, i.e., $\mathcal{R}(\tilde{J}^*)$ and $\tilde{U}_{\mathrm{S}}^* = \begin{bmatrix}\mathbf{u}_{n-r+1}^* & \cdots & \mathbf{u}_n^*\end{bmatrix} \in \mathbb{R}^{n\times r}$ that spans the singular space of $\tilde{J}^*$. Similarly, the right singular matrix $\tilde{V}^*$ can also be decomposed as $\tilde{V}^* = \begin{bmatrix}\tilde{V}_{\mathrm{I}}^* & \tilde{V}_{\mathrm{S}}^*\end{bmatrix}$ with $\tilde{V}_{\mathrm{I}}^* = \begin{bmatrix}\mathbf{v}_1^* & \cdots & \mathbf{v}_{n-r}^*\end{bmatrix} \in \mathbb{R}^{n\times n-r}$, and $\tilde{V}_{\mathrm{S}}^* = \begin{bmatrix}\mathbf{v}_{n-r+1}^* & \cdots & \mathbf{v}_n^*\end{bmatrix} \in \mathbb{R}^{n\times r}$ that spans the null space of $\tilde{J}^*$, i.e., $\mathcal{N}(\tilde{J}^*)$.

We define $(n-r)$ new system states so that

$$(\dot{z}_1\ \cdots\ \dot{z}_{n-r})^{\mathrm{T}} = \tilde{U}_{\mathrm{I}}^{*\mathrm{T}}\,\dot{\boldsymbol{\eta}}. \tag{10}$$

Using velocity kinematics and performing SVD on the Jacobian, from (10), it follows:

$$(\dot{z}_1\ \cdots\ \dot{z}_{n-r})^{\mathrm{T}} = \tilde{U}_{\mathrm{I}}^{*\mathrm{T}}\,\tilde{U}^*\,\tilde{\Sigma}^*\,\tilde{V}^{*\mathbf{T}}\,\dot{\mathbf{q}} = \tilde{\Sigma}_{\mathrm{nz}}^*\,\tilde{V}_{\mathrm{I}}^{*\mathrm{T}}\,\dot{\mathbf{q}}. \tag{11}$$

Next, using the non-zero singular values of the Jacobian, we define a diagonal matrix as follows:

$$\tilde{S}^* = \tilde{S}(\mathbf{q}^*) = \operatorname{diag}\{\sigma_1^*, \cdots, \sigma_{n-r}^*, 1, \cdots, 1\}.$$

Note that the last r elements of the above diagonal matrix are set to a constant, non-zero value just to make it invertible. For convenience we chose number 1. The inverse, hence, exists by construction:

$$\tilde{S}^{*-1} = \operatorname{diag}\{\sigma_1^{*-1}, \cdots, \sigma_{n-r}^{*\ -1}, 1, \cdots, 1\}.$$

With $\tilde{S}^{*-1}$, we define virtual control variables $\boldsymbol{\zeta} = (\zeta_1 \cdots \zeta_n)^{\mathrm{T}}$ by

$$\dot{\mathbf{q}} = \mathbf{h}(\mathbf{q}^*)\,\boldsymbol{\zeta} = \tilde{V}^*\,\tilde{S}^{*-1}\,\boldsymbol{\zeta} \tag{12}$$

In this case, $(\zeta_1 \cdots \zeta_{n-r})^{\mathrm{T}}$ are to control in non-singular directions, while $(\zeta_{n-r+1} \cdots \zeta_n)^{\mathrm{T}}$ are to control potentially existing internal dynamics in the singular direction. Inserting (12) in (11) yields $\tilde{\Sigma}_{\mathrm{nz}}^*\,\tilde{V}_{\mathrm{I}}^{*\mathrm{T}}\,\tilde{V}^*\,\tilde{S}^{*-1}\,\boldsymbol{\zeta} = \begin{pmatrix}\zeta_1 & \cdots & \zeta_{n-r}\end{pmatrix}^{\mathrm{T}}$ and, thus, we obtain a set of single-integrator systems:

$$\dot{z}_i = \zeta_i,\ i \in \{1, \cdots, n-r\}. \tag{13}$$

Given (13), it is evident that the last r elements of the virtual control, i.e., $(\zeta_{n-r+1} \cdots \zeta_n)^{\mathrm{T}}$ are not included in the system description (as desired), but are included in (12) to determine $\dot{\mathbf{q}}$. If one resolves (12) for $\boldsymbol{\zeta}$, the results are:

$$\boldsymbol{\zeta} = \tilde{S}^*\,\tilde{V}^{*\mathrm{T}}\,\dot{\mathbf{q}} \Rightarrow \zeta_i = \begin{cases}\sigma_i^*\,\mathbf{v}_i^{*\mathrm{T}}\,\dot{\mathbf{q}}, & i \in \{1, \cdots, n-r\}\\ \mathbf{v}_i^{*\mathrm{T}}\,\dot{\mathbf{q}}, & i \in \{n-r+1, \cdots, n\}\end{cases}$$

Therefore, the last r rows define the dynamics in singular space. It completes the total dynamics of the system, including singular and non-singular directions. To get a standard form of the dynamics in singular space, a locally reversible coordinate transformation $\mathbf{q} \in \mathbb{R}^n \mapsto z_i \in \mathbb{R}$, for $i = \{n-r+1, \cdots, n\}$ is defined in such a way that it satisfies the following time derivative:

$$\dot{z}_i = \mathbf{v}_i^{*\mathrm{T}}(\mathbf{q})\,\dot{\mathbf{q}}. \tag{14}$$

This new set of variables $(z_{n-r} \cdots z_n)^{\mathrm{T}}$ contains the state variables describing the dynamics in singular directions. In summary, the complete state space representation thus follows from (13) and (14) to

$$\dot{\mathbf{z}} = \boldsymbol{\zeta}, \text{ with } \dot{\mathbf{q}} = \mathbf{h}(\mathbf{q})\boldsymbol{\zeta} = V^* S^{*-1}\boldsymbol{\zeta}. \qquad \square$$

Remark: In comparison to (5), state-space representation (9) has the advantage of totally decoupling workspace dynamics (described by $\mathbf{z}$) from joint space dynamics. Another significant advantage is that the workspace dynamics are separated into a non-singular range $(z_1 \cdots z_{n-r})^{\mathrm{T}}$ and a singular range $(z_{n-r+1} \cdots z_n)^{\mathrm{T}}$.

3.1 Alternative Decomposition of $\tilde{J}$

In the previous section, we observe that for rewriting the system dynamics, we need to compute $\tilde{\Sigma}^*$, $\tilde{U}^*$ and $\tilde{V}^*$ on singular configurations. With a large Jacobian, however, an analytical expression to perform SVD is highly complex. Hence, we propose an alternative and less complex method to SVD:

Proposition 3.2. *Assume velocity kinematics (5) are on a singular configuration. Consider virtual control* $\boldsymbol{\zeta} = (\zeta_1 \cdots \zeta_n)^{\mathrm{T}}$*, defined by:* $\dot{\mathbf{q}} = \tilde{K}^*\boldsymbol{\zeta} = \tilde{K}(\mathbf{q}^*)\boldsymbol{\zeta} = \begin{bmatrix}\mathbf{k}_1^* & \mathbf{k}_2^* & \cdots & \mathbf{k}_n^*\end{bmatrix}\boldsymbol{\zeta}$*, with matrix* $\tilde{K}^* \in \mathbb{R}^{n\times n}$*. Construct an orthonormal matrix*

$$\tilde{W}^* = \tilde{W}(\mathbf{q}^*) = [\mathbf{w}_1^* \cdots \mathbf{w}_{n-r}^*\ \mathbf{n}_1^* \cdots \mathbf{n}_r^*] \in \mathbb{R}^{n\times n}$$

such that its first $(n-r)$ *columns* $\{\mathbf{w}^* \cdots \mathbf{w}_{n-r}^*\}$ *span* $\mathcal{R}(\tilde{J}^*)$*, and the last r columns* $\{\mathbf{n}_1^* \cdots \mathbf{n}_r^*\}$ *span* $\mathcal{N}(\tilde{J}^{*\mathrm{T}})$*. If n new system-states* $\dot{\mathbf{z}} = (\dot{z}_1 \cdots \dot{z}_n)^{\mathrm{T}}$ *satisfy* $\dot{\mathbf{z}} = \tilde{W}^{*\mathrm{T}}\dot{\boldsymbol{\eta}}$*, then it is always possible to construct a* $\tilde{K}^*$ *that satisfies the decoupled state-space representation in (9) with* $\dot{\mathbf{q}} = \tilde{K}^*\boldsymbol{\zeta}$.

Proof. Consider (5) on a singular configuration of codimension r. According to the fundamental theorem of linear algebra, the following holds:

1. $\dim(\mathcal{R}(\tilde{J}^*)) = n-r$ and $\dim(\mathcal{N}(\tilde{J}^{*\mathrm{T}})) = r$
2. $\mathcal{N}(\tilde{J}^{*\mathrm{T}})$ is the orthogonal complement to $\mathcal{R}(\tilde{J}^*)$

By construction, $\tilde{W}^*$ is an orthogonal matrix, with its first $n-r$ columns describing non-singular directions and the last r columns describing singular directions.

Next, we consider n new system-states such that the first $(n-r)$ states $(z_1 \cdots z_{n-r})^{\mathrm{T}}$ describe the non-singular dynamics and the last r states $(z_{n-r+1} \cdots z_n)^{\mathrm{T}}$ describe the singular dynamics, and assume, they satisfy the following relation:

$$\dot{\mathbf{z}} = \tilde{W}^{*\mathrm{T}}\dot{\boldsymbol{\eta}} \overset{(3)}{=} \tilde{W}^{*\mathbf{T}}\tilde{J}^*\dot{\mathbf{q}}. \tag{15}$$

Since the last r columns span the space $\mathcal{N}(\tilde{J}^{*\mathrm{T}})$, it follows that all elements of the last r rows of $\tilde{W}^{*\mathbf{T}}\tilde{J}^*$ become zero, so that $\dot{z}_i = 0$, $i \in \{n-r+1, \cdots, n\}$. Next we define virtual control variables $\boldsymbol{\zeta}$ by $\dot{\mathbf{q}} = \begin{bmatrix}\mathbf{k}_1^* & \mathbf{k}_2^* & \cdots & \mathbf{k}_n^*\end{bmatrix}\boldsymbol{\zeta}$ with $\{\mathbf{k}_{n-r+1}^*, \cdots, \mathbf{k}_n^*\} \in \mathcal{N}(\tilde{J}^*)$. From (15) we further obtain $\dot{\mathbf{z}} = \tilde{W}^{*\mathbf{T}}\tilde{J}^*\tilde{K}^*\boldsymbol{\zeta}$.

We define a matrix $\tilde{E}_{a\times b} \in \mathbb{R}^{a\times b}$ which comprises an identity matrix $I_{\min\{a,b\}}$ and the rest of the elements are zeros. Since the last r rows of $\tilde{W}^{*\mathbf{T}}\tilde{J}^*$ vanish, we can reduce it to an $n-r$ dimension identity matrix i.e.,

$$\tilde{E}_{n-r\times n}\tilde{W}^{*\mathbf{T}}\tilde{J}^*\begin{bmatrix}\mathbf{k}_{n-r+1}^* & \cdots & \mathbf{k}_n^*\end{bmatrix} = \tilde{E}_{n-r\times n}.$$

In order to achieve a decoupled dynamic in $\begin{pmatrix}z_{n-r+1} & \cdots & z_r\end{pmatrix}$, one can arbitrarily choose $k_{ij} = 0$ for $i \in \{n-r+1, \cdots, n\}$ and $j \in \{1, \cdots, n-r\}$ in the above underdetermined system. The remaining $\mathbf{k}_i^* = \begin{pmatrix}k_{i1} & \cdots & k_{i(n-r)}\end{pmatrix}^{\mathrm{T}}$ results in a unique soluton:

$$\begin{bmatrix}\mathbf{k}_1^* & \cdots & \mathbf{k}_{n-r}^*\end{bmatrix} = \left(\tilde{E}_{n-r\times n}\tilde{W}^{*\mathrm{T}}\tilde{J}^*\tilde{E}_{n\times n-r}\right)^{-1}.$$

Finally, along with the last r columns of $\tilde{K}^*$, the following relation

$$\dot{\mathbf{z}} = \tilde{W}^{*\mathbf{T}}\tilde{J}^*\tilde{K}^*\boldsymbol{\zeta} = \operatorname{diag}\{1,\ldots,1,0,\ldots,0\}\boldsymbol{\zeta} \tag{16}$$

gives the desired state-space representation in single integrator form:

$$\begin{pmatrix}\dot{z}_1 & \cdots & \dot{z}_{n-r} & \dot{q}_{n-r+1} & \cdots & \dot{q}_n\end{pmatrix}^{\mathrm{T}} = \boldsymbol{\zeta}. \qquad \square$$

Remark: Expression (16) is similar to SVD $\tilde{U}^{*\mathrm{T}}\tilde{J}^*\tilde{V}^* = \tilde{\Sigma}^* = \operatorname{diag}\{\sigma_1^*, \cdots, \sigma_{n-r}^*, 0, \cdots, 0\}$ with two matrices $\tilde{W}^{*\mathbf{T}}$ and $\tilde{K}^*$ that are analogous to the left and right singular matrices $\tilde{U}^{*\mathrm{T}}$ and $\tilde{V}^*$. The difference is that a normalized matrix is considered while constructing $\tilde{W}^{*\mathbf{T}}$ and the orthonormality criterion in $\tilde{V}^*$ is lost while constructing $\tilde{K}^*$.

4 CONTROLLABILITY

In this section, we will investigate the controllability of serial robotic manipulators with the above formulated feed-forward control law (12) on different singular configurations. It is already stated in section

2 that differential geometric techniques, commonly used in the analysis of controllability and accessibility of nonlinear systems, are inapplicable due to the lack of smoothness around the singularity. Before going further, we first demonstrate that

Proposition 4.1. *The proposed control law (12) is continuous on an open-ball vicinity of singular configurations.*

Proof. Outside the singular configuration, the joint angle trajectory is always well-defined by $\dot{\mathbf{q}} = \tilde{J}^{-1}(\mathbf{f}^{-1}(\boldsymbol{\eta},\kappa))\dot{\boldsymbol{\eta}}$, since the inverse kinematics have a unique solution, and the Jacobian has full rank. On singularity, control law (12) gives the unique solution for the inverse of the velocity dynamics. This leaves only the case when we are in a vicinity of the singularity, i.e., $\mathbf{q} \to \mathbf{q}^*$. For continuity, it is sufficient to show that the following relation holds:

$$\lim_{\mathbf{q}\to\mathbf{q}^*} \tilde{V}(\mathbf{q})\tilde{S}^{-1}(\mathbf{q})\boldsymbol{\zeta} = \mathbf{q}^*. \tag{17}$$

If this does not hold, then there exists a $\boldsymbol{\varepsilon} \in \mathbb{R}^n \neq \mathbf{0}$ such that

$$\lim_{\mathbf{q}\to\mathbf{q}^*} \tilde{V}(\mathbf{q})\tilde{S}^{-1}(\mathbf{q})\boldsymbol{\zeta} - \mathbf{q}^* = \boldsymbol{\varepsilon}. \tag{18}$$

Then left multiplication with $(\tilde{S}^*\tilde{V}^{*\mathrm{T}})$ leads to

$$\lim_{\mathbf{q}\to\mathbf{q}^*} (\tilde{S}^*\tilde{V}^{*\mathrm{T}}\tilde{V}(\mathbf{q})\tilde{S}^{-1}(\mathbf{q})\boldsymbol{\zeta}) - \tilde{S}^*\tilde{V}^{*\mathrm{T}}\mathbf{q}^* = \tilde{S}^*\tilde{V}^{*\mathrm{T}}\boldsymbol{\varepsilon}.$$

The left side of this equation further details to:

$$\begin{aligned}
&\lim_{\mathbf{q}\to\mathbf{q}^*} (\tilde{S}^*\tilde{V}^{*\mathrm{T}}\tilde{V}(\mathbf{q})\tilde{S}^{-1}(\mathbf{q})\boldsymbol{\zeta}) - \tilde{S}^*\tilde{V}^{*\mathrm{T}}\mathbf{q}^* \\
&= \tilde{S}^*\tilde{V}^{*\mathrm{T}}\tilde{V}^* \lim_{\mathbf{q}\to\mathbf{q}^*} (\tilde{S}^{-1}(\mathbf{q})\boldsymbol{\zeta}) - \tilde{S}^*\tilde{V}^{*\mathrm{T}}\mathbf{q}^* \\
&= \tilde{S}^* \lim_{\mathbf{q}\to\mathbf{q}^*} (\tilde{S}^{-1}(\mathbf{q})\boldsymbol{\zeta}) - \tilde{S}^*\tilde{V}^{*\mathrm{T}}\mathbf{q}^* = \tilde{S}^*\tilde{S}^{*-1}\boldsymbol{\zeta}^* - \boldsymbol{\zeta}^* \\
&= \operatorname{diag}\{\sigma_1^*\sigma_1^{*-1},\cdots,\sigma_{n-r}^*{\sigma_{n-r}^*}^{-1},1,\cdots,1\}\boldsymbol{\zeta}^* - \boldsymbol{\zeta}^* = \mathbf{0}
\end{aligned}$$

Since the right-hand side of (18) certainly does not vanish, this contradicts the above assumption; therefore, (17) must be valid. Hence, by construction, the feed-forward control law is continuous. □

At this point, we discuss control strategies of robot-manipulators on singularities. The system representation is first formulated close to singular configurations, as shown in section 3. This provides us with $n-r$ control input variables for admissible motion in workspace and r control input variables for null space motion in singular space. Depending on the nature of the singularity, the null space motion can then be used to escape it.

Singularities of redundant manipulators, based on the presence of internal motion (also known as *self-motion*), are classified into two categories, see, e.g., (Chang and Khatib, 1995), (Oetomo and Ang, 2009). Internal motion changes joint configuration with constant end-effector pose. Type-1 singularities, where internal motion caused by null space motion changes the singular direction, and Type-2 singularities, where the null space motion doesn't provide any internal motion – moving in any direction is not possible.

Similarly, for non-redundant manipulators, based on the presence of internal motion, we classify singularities in Type-1 and Type-2 singularities since non-redundant manipulators show *local redundancy* on some singularities. To begin, it is crucial to determine which singularities provides internal motion.

In (6), a set $\mathbb{Q}$ was defined that contains all singular configurations. Consider, $\mathbb{P}$ to be an open subset of $\mathbb{Q}$ i.e., $\mathbb{P} \subseteq \mathbb{Q}$. Let's define a function on $\mathbb{P}$ as

$$\mathbb{P} \mapsto \mathbb{R} : \ \Phi(\mathbf{q}) = \det(\tilde{J}). \tag{19}$$

Then, $\mathbb{P}$ is time-invariant if $\dot{\Phi}(\mathbf{q}) = \mathbf{0}$ holds for all $\mathbf{q} \in \mathbb{P}$. In other words, internal motion is possible due to null space motion in the open subset $\mathbb{P}$. The following lemma gives us a sufficient condition for the existence of internal motion on singularities in (9):

Lemma 4.2. *A sufficient condition for existence of internal motion is* $\partial\Phi(\mathbf{q})/\partial\mathbf{q}^{\mathrm{T}}\tilde{K}(\mathbf{q})[n-r+1:n] = 0$ *for all* $\mathbf{q} \in \mathbb{P}$.

Proof. Differentiating (19) yields

$$\dot{\Phi} = \frac{\partial\Phi(\mathbf{q})}{\partial\mathbf{q}}^{\mathrm{T}}\dot{\mathbf{q}} = \frac{\partial\det(\tilde{J}(\mathbf{q}))}{\partial\mathbf{q}}^{\mathrm{T}}\tilde{K}^*\boldsymbol{\zeta}.$$

By definition of internal motion $\dot{\Phi}$ must vanish: $\partial\Phi(\mathbf{q})/\partial\mathbf{q}^{\mathrm{T}}\tilde{K}^*\boldsymbol{\zeta} = 0$. On the singular surface $\mathbb{P}$, $\zeta_1 = \cdots = \zeta_{n-r} = 0$ so that for all $\zeta_{n-r+1},\cdots,\zeta_n \in \mathbb{R}$ $\partial\Phi(\mathbf{q})/\partial\mathbf{q}^{\mathrm{T}}\begin{bmatrix}\mathbf{k}_{n-r+1}^* & \cdots & \mathbf{k}_n^*\end{bmatrix}\left(\zeta_{n-r+1}\cdots\zeta_n\right)^{\mathrm{T}} = 0$ $\iff \partial\Phi(\mathbf{q})/\partial\mathbf{q}^{\mathrm{T}}\begin{bmatrix}\mathbf{k}_{n-r+1}^* & \cdots & \mathbf{k}_n^*\end{bmatrix} = \mathbf{0}^{\mathrm{T}}$. □

5 NUMERICAL EXAMPLE

To illustrate the concepts introduced in section 3, a planar 3-DoF manipulator is considered; see Figure 1. For the end-effector (EE), position (x,y) and orientation α are considered as workspace coordinates $\boldsymbol{\eta} = (x,y,\alpha)^{\mathrm{T}}$ and three joint angles as joint variables $\mathbf{q} = (\theta_1,\theta_2,\theta_3)^{\mathrm{T}}$. To simplify the kinematics, we assume $l_1 = l_2 = l_3 = 1$.

The direct kinematics calculate to

$$\mathbf{f} : \mathbf{q} \mapsto \boldsymbol{\eta} = \begin{pmatrix} c_1+c_{12}+c_{123} \\ s_1+s_{12}+s_{123} \\ \theta_1+\theta_2+\theta_3 \end{pmatrix},$$

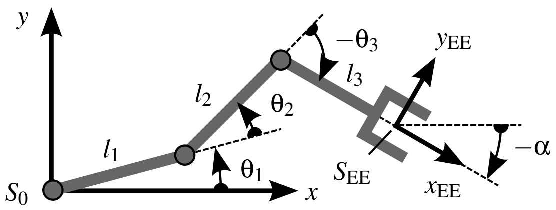

Figure 1: Planar 3-DoF Ellbow-Manipulator.

with the usual abbreviations: $c_1 = \cos\theta_1$, $c_{12} = \cos(\theta_1 + \theta_2), \cdots$. Then the Jacobian follows to

$$\tilde{J}(\mathbf{q}) = \begin{bmatrix} -s_1-s_{12}-s_{123} & -s_{12}-s_{123} & -s_{123} \\ c_1+c_{12}+c_{123} & c_{12}+c_{123} & c_{123} \\ 1 & 1 & 1 \end{bmatrix},$$

with singularities for $\theta_2 \in \{0, \pm\pi\}$. The IK solutions on these singular configurations might not be unique. For instance, when $\theta_2 = 0$, we have a unique IK solution for a fixed $\boldsymbol{\eta}$:

$$\begin{aligned} \theta_1(\boldsymbol{\eta}) &= \arctan2\,(x - c_\alpha, y - s_\alpha), \\ \theta_2(\boldsymbol{\eta}) &= 0, \\ \theta_3(\boldsymbol{\eta}) &= \alpha - \theta_1(\boldsymbol{\eta}). \end{aligned}$$

On the contrary, when $\theta_2 = \pi$, we have an infinite number of solutions represented by the straight line: $\theta_1 + \theta_3 = \alpha - \pi$. For the latter case, the singularity locus becomes a helix in the workspace $\mathcal{S} = \{(x, y, \alpha) \,|\, x = c_\alpha \wedge y = s_\alpha \wedge \alpha = \mathbb{R}\}$, with $s_\alpha = \sin(\alpha)$ and $c_\alpha = \cos(\alpha)$. Hence, on $\theta_2 = 0$, we have a unique IK solution but rank loss, while on $\theta_2 = \pi$, we have rank loss and an infinite number of solutions of IK.

5.1 System Representation

5.1.1 Case $\theta_2 = \pi$

With $\theta_1+\theta_3 = \alpha-\pi$ the Jacobian becomes

$$\tilde{J}^*(\theta_1, \alpha) = \begin{bmatrix} -s_\alpha & s_1-s_\alpha & -s_\alpha \\ c_\alpha & -c_1+c_\alpha & c_\alpha \\ 1 & 1 & 1 \end{bmatrix},$$

with Null space $\tilde{J}^*(\theta_1, \alpha)\begin{pmatrix}-1 & 0 & 1\end{pmatrix}^{\mathrm{T}} = \mathbf{0}$ and normal vector to the plane of admissible motions

$$\tilde{J}^{*\mathrm{T}}(\theta_1, \alpha)\begin{pmatrix}-c_1 & -s_1 & s_3\end{pmatrix}^{\mathrm{T}} = \mathbf{0}. \qquad (20)$$

A normalized orthogonal matrix is given by

$$\tilde{W}^*(\theta_1, \theta_3) = \frac{1}{b}\begin{bmatrix} bs_1 & c_1s_3 & -c_1 \\ -bc_1 & s_1s_3 & -s_1 \\ 0 & 1 & s_3 \end{bmatrix},$$

with $b = \sqrt{3-\cos(2\theta_3)}/\sqrt{2}$. Its first two columns span $\mathcal{R}(\tilde{J}^*)$ and the last column spans $\mathcal{N}(\tilde{J}^{*\mathrm{T}})$ as stated in Proposition 3.2. Inserting in the velocity kinematics yields

$$\tilde{W}^*\dot{\mathbf{z}} = \tilde{J}^*\dot{\mathbf{q}} \Leftrightarrow \dot{\mathbf{z}} = \tilde{W}^{*\mathrm{T}}\tilde{J}^*\dot{\mathbf{q}} = \begin{bmatrix} c_3 & 1+c_3 & c_3 \\ b & b & b \\ 0 & 0 & 0 \end{bmatrix}\dot{\mathbf{q}}. \quad (21)$$

Instead of the real control variables $\dot{\mathbf{q}}$ we introduce virtual control variables $\boldsymbol{\zeta}$ such that $\dot{\mathbf{q}} = \tilde{K}^*\boldsymbol{\zeta} = \begin{bmatrix}\mathbf{k}_1 & \mathbf{k}_2 & \mathbf{k}_3\end{bmatrix}\boldsymbol{\zeta}$ with $\mathbf{k}_3 = (-1\ 0\ 1)^{\mathrm{T}}$, null space of $\tilde{J}^*$ at $\theta_2 = \pi$. If one arbitrarily chooses $k_{13} = k_{23} = 0$, then

$$\tilde{K}^* = \begin{bmatrix} -1 & \frac{1}{b}(c_3+1) & -1 \\ 1 & \frac{1}{b}c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (22)$$

Inserting this in (21) we obtain

$$\dot{\mathbf{z}} = \tilde{W}^{*\mathrm{T}}\tilde{J}^*\dot{\mathbf{q}} = \tilde{W}^{*\mathrm{T}}\tilde{J}^*\tilde{K}^*\boldsymbol{\zeta} = \mathrm{diag}\{1, 1, 0\}\boldsymbol{\zeta}. \quad (23)$$

Hence, with the matrix $\tilde{K}^*$ in (22), the decoupled state-space representation for (5) on singularity $\theta_2 = \pi$ is as follows:

$$\begin{pmatrix}\dot{z}_1 & \dot{z}_2 & \dot{q}_3\end{pmatrix}^{\mathrm{T}} = \zeta_i,\ i \in \{1, 2, 3\}. \qquad (24)$$

5.1.2 Case $\theta_2 = 0$

With $\theta_1 + \theta_3 = \alpha$ and $\theta_2 = 0$, the Jacobian becomes

$$\tilde{J}^*(\theta_1, \alpha) = \begin{bmatrix} -2s_1-s_\alpha & -s_1-s_\alpha & -s_\alpha \\ 2c_1+c_\alpha & c_1+c_\alpha & c_\alpha \\ 1 & 1 & 1 \end{bmatrix}.$$

The null space of the Jacobian and a vector that is normal to the plane of admissible motions are $(1\ -2\,1)^{\mathrm{T}}$ and $(c_1\,s_1\,s_3)^{\mathrm{T}}$ respectively. Similar to the previous case, a normalized orthogonal matrix $\tilde{W}^*$ can be set up using the normal vector of the plane of motion by

$$\tilde{W}^*(\theta_1, \theta_3) = \frac{1}{b}\begin{bmatrix} bs_1 & c_1s_3 & c_1 \\ -bc_1 & s_1s_3 & s_1 \\ 0 & -1 & s_3 \end{bmatrix},$$

with $b = \sqrt{3-\cos(2\theta_3)}/\sqrt{2}$, whose first two columns span $\mathcal{R}(\tilde{J}^*)$ and the last column spans $\mathcal{N}(\tilde{J}^{*\mathrm{T}})$. Following all steps stated in proposition 3.2, we end up again with (24) and the virtual control law

$$\dot{\mathbf{q}} = \tilde{K}^*\boldsymbol{\zeta} = \begin{bmatrix} -1 & \frac{1}{b}(c_3+1) & 1 \\ 1 & \frac{1}{b}c_3 & -2 \\ 0 & 0 & 1 \end{bmatrix}\boldsymbol{\zeta}.$$

5.2 Controllability

Next, for this example, the function in (19) is defined as $\Phi(\mathbf{q}) = \det(\tilde{J}) = s_2$. The presence of a time-invariant subset implies that $\dot{\Phi} = {\partial\Phi(\mathbf{q})}/{\partial\mathbf{q}}^{\mathrm{T}}\dot{\mathbf{q}} = \begin{bmatrix}0 & c_2 & 0\end{bmatrix}\tilde{K}^*\boldsymbol{\zeta} = 0$. To stay on singular space, we set $\zeta_1 = \zeta_2 = 0$ and it follows $\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}k_{31} & k_{32} & k_{33}\end{pmatrix}^{\mathrm{T}} = 0$. This requires $k_{32} = 0$ and according to Lemma 4.2, internal motion exists, otherwise not.

For the case $\theta_2 = \pi$, it is evident that $k_{32} = 0$ and it also implies $\dot{\theta}_2 = 0$. Thus, a finitely extended journey in null space is possible, and identified as Type-1 singularity. On the other hand, for $\theta_2 = 0$ we have nonzero k_{32}. Hence, in this case, self-motion doesn't exist and is identified as Type-2 singularity.

Next, we investigate what type of controllability we have on Type-1 and Type-2 singularities for this example. Before moving forward, it is still unclear whether internal motion on Type-1 singularity permits coverage of the entire admissible direction in the workspace. For a pure null space motion, we have $\zeta_1 = \zeta_2 = 0$. Then, the corresponding motion in joint space is $\dot{\theta}_{1N} = -\zeta_3,\ \dot{\theta}_{2N} = 0,\ \dot{\theta}_{3N} = \zeta_3$. This is integrable over time so that $\theta_{1N} = -\zeta_3 t - \zeta_{30},\ \theta_{2N} = \zeta_{30},\ \theta_{3N} = \zeta_3 t + \zeta_{30}$. Inserting into normal vector (20) yields

$$\mathbf{n}(t) = \begin{pmatrix} -\cos(-\zeta_3 t - \zeta_{30}) \\ -\sin(-\zeta_3 t - \zeta_{30}) \\ \sin(\zeta_3 t + \zeta_{30}) \end{pmatrix} = \begin{pmatrix} -\cos(\phi) \\ \sin(\phi) \\ \sin(\phi) \end{pmatrix}.$$

It suffices to find at least two values for ϕ so that the corresponding $\mathbf{n}$ are linearly independent. This is given by $\phi \in \{0, \pi\}$: $\mathbf{n} \in \{\begin{pmatrix} -1 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^{\mathrm{T}}\}$. Since the two variants of $\mathbf{n}$ are linearly independent, this holds also for the corresponding vectors $\mathbf{n} \times \mathbf{j}_1$. This proves that along a null space motion, we can go in any direction.

We now have the necessary tools to specify controllability on singularities. On a Type-2 singularity, we can only move in the null space motion direction. As a result, we can pass the singularity, but cannot follow all the desired trajectories. In this situation, we have only *local accessibility*. On Type-1 singularities, on the other hand, we can move in any direction by changing the null space motion direction, and we can follow any desired trajectories. As a result, in this instance, we have *local controllability*. However, we cannot instantly follow trajectories since modifying the configuration with the internal motion takes some time to obtain the desired null space motion direction. As a result, on Type-2 singularities, we have local controllability but not *small-time local controllability*(STLC).

6 SUMMARY AND OUTLOOK

This work considers control issues in robot motion planning instead of avoiding singular configuration when the desired trajectory passes through singularities. We proposed a method for totally decoupling velocity kinematics representation for planner robot-manipulators on singularities. The proposed method is particularly attractive for analytical computation since it doesn't need SVD. An interesting outcome is the classification of singularities based on their controllability property. The limitation of this result is we can only classify singularities for the example we are considering here. Yet, the observations could lead to some interesting possibilities. We are confident that achieving *local controllability* is possible in various kinds of internal singularities. This line of research will be expanded in the future.

REFERENCES

Chang, K.-S. and Khatib, O. (1995). Manipulator control at kinematic singularities: A dynamically consistent strategy. *IEEE/RSJ International Conference on Intelligent Robots and Systems. Human Robot Interaction and Cooperative Robots*, pages 84–88.

Conkur, E. and Buckingham, R. (1997). Clarifying the definition of redundancy as used in robotics. *Robotica*, 15:583 – 586.

Huang, Y., Yong, Y. S., Chiba, R., Arai, T., Ueyama, T., and Ota, J. (2016). Kinematic control with singularity avoidance for teaching-playback robot manipulator system. *IEEE Transactions on Automation Science and Engineering*, 13:729–742.

Kieffer, J. (1994). Differential analysis of bifurcations and isolated singularities for robots and mechanisms. *IEEE Transactions on Robotics and Automation*, 10:1–10.

Kuo, C. J., Ting, K.-C., and Chen, Y.-C. (2017). State of product detection method applicable to industry 4.0 manufacturing models with small quantities and great variety: An example with springs. *International Conference on Applied System Innovation (ICASI)*, pages 1650–1653.

Mareczek, J. (2020). Local optimal tracking control for manipulators with restrictive joint velocity and acceleration limits. *IEEE/ASME International Conference on Advanced Intelligent Mechatronics (AIM)*, pages 230–237.

Mazzanti, M., Cristalli, C., Gagliardini, L., Carbonari, L., Lattanzi, L., and Massa, D. (2019). A novel trajectory generation algorithm for robot manipulators with online adaptation and singularity management. *IEEE 28th International Symposium on Industrial Electronics (ISIE)*, pages 1115–1120.

Nahavandi, S. (2019). Industry 5.0 - a human-centric solution. *Sustainability*, 11.

Nielsen, L., de Wit, C. C., and Hagender, P. (1991). Controllability issues of robots in singular configurations. *IEEE International Conference on Robotics and Automation*, pages 2210–2215.

Oetomo, D. and Ang, M. H. (2009). Singularity robust algorithm in serial manipulators. *Robotics and Computer-Integrated Manufacturing*, 25:122–134.

Raineri, C. G. and Bianco, M. L. (2017). An automatic system for the avoidance of wrist singularities in anthropomorphic manipulators. *13th IEEE Conference on Automation Science and Engineering (CASE)*, pages 1302–1309.

Spong, M. W., Hutchinson, S., and Vidyasager, M. (2020). *Robot Modeling and Control*. John Wiley & Sons, Ltd., secoind edition.

Wampler, C. (1986). Manipulator inverse kinematic solutions based on vector formulations and damped least-squares methods. *IEEE Transactions on Systems, Man, and Cybernetics*, 16:93–101.