Bayesian State Estimation Using Constrained Zonotopes

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Abstract: This paper proposes an approximate Bayesian recursive algorithm for the state estimation of a linear discrete time stochastic state space model. The involved state and observation noises are assumed to be bounded and uniformly distributed. The support of a posterior probability density function (pdf) is approximated by a constrained zonotope of an adjustable complexity. The behaviour of the proposed algorithm is illustrated by simulations and compared with other methods.

1 INTRODUCTION

State estimation or filtering has many applications in contexts where either the states, or the observations, or both, are constrained to particular sets. The constrained filtering applications are used for example in problems of fault detection (Scott et al., 2016), robust model predictive control (Sharma et al., 2018), estimation in sensor networks (Ge et al., 2019) and in applications involving constrained dynamics in physical processes (Simon and Simon, 2010).

Deterministic state estimation uses so called set membership approaches. There, states are guaranteed to be contained in a bounded set as orthotopes, parallelotopes, zonotopes and and ellipsoids (Althoff and Rath, 2021).

These geometric considerations are important also in a Bayesian filtering involving constrained (typically uniformly distributed) stochastic state and observation noise processes (Combastel, 2016).

The main advantages of Bayesian filtering are (i) the possibility to take account of the distribution of the states within their constrained support sets (Shao et al., 2010), (ii) the quantification of uncertainty in the states (Särkkä, 2013), and (iii) the evaluation of the optimality, in the sense of minimum Bayes' risk, of sequential estimation and decision-making, including control design (Kárný et al., 2006). Moreover, it has been shown that the deterministic approach is a particular case of the Bayesian general framework (Samada et al., 2023).

In the author's previous work, a Bayesian state estimator was proposed that provides optimally approximated state estimates within the class of uniform distributions on orthotopic support (Pavelková and Jirsa, 2018) and within the class of uniform distributions on parallelotopic support (Jirsa et al., 2019).

This paper aims to enhance the above mentioned Bayesian state estimator (Pavelková and Jirsa, 2018) and (Jirsa et al., 2019) and get a more flexible approximation of the true distribution while preserving the feasibility of the resulting algorithm. We achieve this by the by considering the state estimates within a constrained zonotopic support.

The class of constrained zonotopes (CZ) has been proposed in (Scott et al., 2016) as a tool for set-based estimation. CZ can describe arbitrary convex polytope when the complexity of the representation is not limited. At the same time, this representation allows the computation of exact projections, intersections, and Minkowski sums using very simple identities. To keep the computations feasible, methods for computing an enclosure of one CZ by another one of lower complexity are provided (Raghuraman and Koeln, 2022).

Set membership estimators based on CZ are presented e.g. in (Rego et al., 2020b) or (Pan et al., 2022). In contrast, we will use CZ to design a Bayesian estimator.

Throughout, I is the identity matrix, $\mathbb{R}^n$ is the n-dimensional real space. Matrices are denoted by capital letters (e.g. A), vectors and scalars by lowercase letters (e.g. b). Vector inequalities, e.g. $\underline{x} < \overline{x}$ are

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meant entry-wise; $\underline{x}$ and $\overline{x}$ are lower and upper bounds on x, respectively. ℓ_x denotes the length of a (column) vector x, and $\mathbb{X}$ denotes the set of x. x_t is the value of a time-variant vector x, at a discrete time instant, $t \in \mathbb{T} \equiv \{1, 2, \ldots, \bar{t}\}$; $x(t) \equiv \{x_t, x_{t-1}, \ldots, x_1\}$. The symbol $f(\cdot|\cdot)$ denotes a conditional probability density function (pdf); no notational distinction is made between a random variable and its realisation. $\mathcal{U}_x(\underline{x}, \overline{x})$ is the uniform pdf of x with an orthotopic (box) support $[\underline{x}, \overline{x}]$ and $\mathcal{U}_x(\mathbb{X})$ denotes the uniform pdf of x on a bounded convex set $\mathbb{X}$.

2 ADDRESSED PROBLEM

In the Bayesian filtering framework (Kárný et al., 2006), a system of interest is described by the following probability density functions (pdfs):

$$\begin{aligned} &\text{- prior pdf } f(x_1) \\ &\text{- observation model } f(y_t|x_t),\ t \in \mathbb{T} \qquad (1) \\ &\text{- time evolution model } f(x_{t+1}|x_t, u_t),\ t \in \mathbb{T} \setminus \bar{t} \end{aligned}$$

where $y_t \in \mathbb{R}^{\ell_y}$ is an observable output, $u_t \in \mathbb{R}^{\ell_u}$ is an optional known (exogenous) system input and $x_t \in \mathbb{R}^{\ell_x}$ is an unobservable (hidden) system state.

Bayesian state estimation or filtering consists in the evolution of the posterior pdf $f(x_t|d(t))$ where $d(t)$ is a sequence of observed data records $d_t = (y_t, u_t)$, $t \in \mathbb{T}$. The evolution of $f(x_t|d(t)$ is described by a two-steps recursion that starts from the prior pdf $f(x_1)$ and ends with the data update at the final time $t = \bar{t}$:

- data update (Bayes' rule) processing the new data

$$f(x_t|d(t)) = \frac{f(y_t|x_t)\, f(x_t|d(t-1))}{\int\limits_{\mathbb{X}_t} f(y_t|x_t)\, f(x_t|d(t-1))\, \mathrm{d}x_t}, \quad (2)$$

- time update (marginalization) evolving the state at the next time instant

$$f(x_{t+1}|d(t)) = \int\limits_{\mathbb{X}_t} f(x_{t+1}|u_t, x_t)\, f(x_t|d(t))\, \mathrm{d}x_t. \quad (3)$$

We consider that the stochastic system (1) is represented by a linear state-space model

$$\begin{aligned} y_t &= Cx_t + v_t \qquad (4) \\ x_{t+1} &= Ax_t + Bu_t + w_{t+1} \end{aligned}$$

with uniform prior pdf

$$x_1 = \mathcal{U}_x(\underline{x}_1, \overline{x}_1) \quad (5)$$

where $x_t \in \mathbb{R}^{\ell_x}$, $y_t \in \mathbb{R}^{\ell_y}$, $u_t \in \mathbb{R}^{\ell_u}$. A, B, C are known model matrices of appropriate dimensions; v_t and w_t are additive random observational and modelling uncertainties, respectively. We assume that v_t and w_t are mutually independent white noise processes uniformly distributed on *known* orthotopic supports:

$$f(v_t) = \mathcal{U}_v(-\nu, \nu), \qquad f(w_t) = \mathcal{U}_w(-\omega, \omega), \quad (6)$$

where $\nu \in \mathbb{R}^{\ell_y}$, $\omega \in \mathbb{R}^{\ell_x}$.

We denote the linear state space model with uniform noises (4) – (6) as a *LSU model*.

The exact Bayesian state estimation of the LSU model, according to (2) and (3) results in a non-uniformly distributed posterior pdf on a geometrically complex support.

To get applicable state estimation algorithm, two points has to be addressed:

(i) In each data update step (2), the support of posterior pdf corresponds to the intersection of the support from previous step and a strip given by new data. To avoid computational complexity, the resulting polytope is approximated in each step by a tightly circumscribing orthotope in (Pavelková and Jirsa, 2018) or parallelotope in (Jirsa et al., 2019). These approximations project the support back to the initial class, i.e. the function is kept, the support is changed.

(ii) In first time update step (3), the sum of two independent uniformly distributed random quantities results in a trapezoidal pdf (Kotz and Dorp, 2004), and each subsequent step further increases functional complexity of the resulting pdf. To preserve the class of the function and computational complexity, the trapezoidal pdf is approximated by a uniform pdf by minimising Kullback-Leibler divergence of these pdfs (Pavelková and Jirsa, 2018). The result is a uniform pdf on the support of the trapezoid, i.e. the support is kept, the function is changed.

This paper will use the above mentioned approximation (ii) to keep the posterior pdf uniform and will propose a more flexible approximation of its support, using constrained zonotopes (Scott et al., 2016).

3 LSU-CZ FILTER

In this section, the constrained zonotopic (CZ) sets are introduced and applied within the approximate Bayesian state estimation (2) and (3) of LSU model (4)–(6). The resulting constrained zonotopic state estimator is called *LSU-CZ* filter.

3.1 Constrained Zonotopes

A zonotope is a centrally symmetric convex polytope. It can be be described as the Minkowski sum of a set of n_g line segments, $n_g \geq n$ in n-dimensional space. Zonotopes are often used to approximate complex polytopes as their complexity can be easily tuned and relevant set operations result in simple matrix calculations (Combastel, 2015). Nevertheless, centrally symmetric zonotopes are not suitable for tight approximation (circumscription) of generally asymmetric convex polytopes. Therefore, a *constrained zonotope* (CZ) $\mathbb{Z}$ was introduced in (Scott et al., 2016):

$$\mathbb{Z} = \{G\xi + c : \|\xi\|_\infty \leq 1, \mathcal{A}\xi = b\} \equiv \{G, c, \mathcal{A}, b\}, \quad (7)$$

where $G \in \mathbb{R}^{n \times n_g}$ is a generator matrix of rank n (with n_g generator columns vectors or generators), $c \in \mathbb{R}^n$ is a zonotope centre and $\xi \in \mathbb{R}^{n_g}$, $n_g \geq n$, $\mathcal{A} \in \mathbb{R}^{n_c \times n_g}$ and $b \in \mathbb{R}^{n_c}$, n_c is a number of constraints (constraining equations in $\mathbb{R}^{n_g}$). Note that for $n_g = n$ linearly independent generators with no constraints, $\mathbb{Z}$ is a parallelotope.

The following set operations are defined for $\mathbb{Z}, \mathbb{U} \subset \mathbb{R}^n$, $\mathbb{Y} \subset \mathbb{R}^k$, $R \in \mathbb{R}^{k \times n}$:

$$R\mathbb{Z} = \{Rz : z \in \mathbb{Z}\}, \quad (8)$$

$$\mathbb{Z} + \mathbb{U} = \{z + u : z \in \mathbb{Z}, u \in \mathbb{U}\}, \quad (9)$$

$$\mathbb{Z} \cap_R \mathbb{Y} = \{z \in \mathbb{Z} : Rz \in \mathbb{Y}\}, \quad (10)$$

where (8) is a *linear mapping* of $\mathbb{Z}$ by R, (9) is the *Minkowski sum* of sets $\mathbb{Z}$ and $\mathbb{U}$ and (10) is a *generalized intersection* of sets $\mathbb{Z}$ and $\mathbb{Y}$. Note that for $R = I$ and $k = n$, a standard set intersection is obtained.

Applying to the constrained zonotopes, the operations (8), (9) and (10) result in (Scott et al., 2016):

$$R\mathbb{Z} = \{RG_z, Rc_z, \mathcal{A}_z, b_z\} \quad (11)$$

$$\mathbb{Z} + \mathbb{U} = \left\{[G_z G_u], c_z + c_u, \begin{bmatrix} \mathcal{A}_z & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_u \end{bmatrix}, \begin{bmatrix} b_z \\ b_u \end{bmatrix}\right\}, \quad (12)$$

$$\mathbb{Z} \cap_R \mathbb{Y} = \left\{[G_z \mathbf{0}], c_z, \begin{bmatrix} \mathcal{A}_z & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_y \\ RG_z & -G_y \end{bmatrix}, \begin{bmatrix} b_z \\ b_y \\ c_y - Rc_z \end{bmatrix}\right\}, \quad (13)$$

where subscripts z, u and y refer to the respective sets, $\mathbf{0}$ is a zero matrix of appropriate dimensions. The *lifted zonotope*

$$z \in \{G, c, \mathcal{A}, b\} \Longleftrightarrow \begin{bmatrix} z \\ \mathbf{0} \end{bmatrix} = \left\{\begin{bmatrix} G \\ \mathcal{A} \end{bmatrix}, \begin{bmatrix} c \\ -b \end{bmatrix}\right\} \quad (14)$$

is an unconstrained zonotope, with n_c added coordinates fixed to zero.

The operations (12) and (13) on constrained zonotopes increase their complexity, i.e. number of generators n_g and constraints n_c. To keep the complexity within the given limits, reduction operations are proposed in (Scott et al., 2016). These operations either (i) preserve the set (rescaling, removing zero columns in a lifted zonotope (14), zero constraints or parallel generators (Raghuraman and Koeln, 2022)) or (ii) approximate the set by circumscription (reduction of least significant generators or constraints).

3.2 State Estimation on a CZ Support

As mentioned in Section 2, one iteration of the Bayesian filtering task, applied to the LSU model (4)–(6), consist of (i) the data update (2) that corresponds to the intersection of two sets followed by an approximation that pushes the support of posterior pdf back to the chosen class, and (ii) the time update (3) followed by an approximation of resulting non-uniform pdf by the uniform one (Pavelková and Jirsa, 2018), (Jirsa et al., 2019).

Here, we propose a more flexible approximation of the support of a posterior pdf $f(x_t|d(t))$ in the data update (2) using a CZ (7). The approximation within the time update step is maintained.

We denote the support of posterior pdf $f(x_t|d(t))$ (2) by the symbol $\mathbb{X}_t$ and the support of the state predictor $f(x_{t+1}|d(t))$ (3) by $\mathbb{X}_{t|t-1}$. The support of prior pdf is denoted as $\mathbb{X}_1$.

3.2.1 Data Update

The data update (2) processes $f(x_t|d(t-1))$ (starting from prior $f(x_1)$ in $t = 1$) together with $f(y_t|x_t)$ given by (4) and (6). The exact pdf is uniformly distributed on a support $\mathbb{X}_t$ that results from the intersection of a support $\mathbb{X}_{t|t-1}$ obtained during previous time update (or $\mathbb{X}_1$ in the first step) and a strip given by new data (Pavelková and Jirsa, 2018). For CZ support $\mathbb{X}_t$, using (13), holds

$$\mathbb{X}_t = \mathbb{X}_{t|t-1} \cap_C (y_t - \mathbb{V}_t), \quad (15)$$

where $\mathbb{V}_t$ is a support of $f(v_t)$.

An advantage of CZ is that the relevant intersection stays within CZ class after data update. Nevertheless, the number of generators n_g and constraints n_c (7) continually increases. To maintain n_g and n_c below the required limits, the complexity reduction operations (Scott et al., 2016) are applied as needed.

3.2.2 Time Update

The time update step (3) processes $f(x_t|d(t))$ from previous data update together with $f(x_{t+1}|x_t, u_t)$ given by (4) and (6). The exact pdf $f(x_{t+1}|d(t))$ is non-uniformly distributed on a CZ support $\mathbb{X}_{t+1|t}$. Using the set operations (11), (12) gives

$$\mathbb{X}_{t+1|t} = A\mathbb{X}_t + Bu_t + \mathbb{W}_t, \quad (16)$$

where $\mathbb{W}_t$ is a support of $f(w_t)$.

For the next step, the resulting non-uniform pdf is approximated according to (Pavelková and Jirsa, 2018) which results in uniform pdf with support (16).

3.3 Point Estimates

To use the state estimates for the prediction and control problems, we need a point estimate. In the case of standard, i.e. unconstrained zonotope, its centre can be chosen as the point estimate. When using constrained zonotope (7), the centre is not guaranteed to be placed inside this set. Below, two ways of a choice of the point estimate are proposed.

Method 1. The point estimate $\hat{x}_1$ corresponds to the centre of an interval hull of the posterior pdf (15) (Rego et al., 2020a).

Method 2. The point estimate $\hat{x}_2$ corresponds to the point inside the posterior pdf (15) that is closest to the centre of the relevant unconstrained zonotope. The solution is obtained by solving two linear programs (Raghuraman and Koeln, 2022).

3.4 Algorithmic Summary

The algorithmic sequence for LSUCZ filter is provided in Algorithm 1.

4 EXPERIMENTS

In this section, the proposed Algorithm 1 is compared with the previous orthotopic (Pavelková and Jirsa, 2018) and parallelotopic (Jirsa et al., 2019) variants.

4.1 Simulation Settings

The matrices of the state space model (4), (6) are set as

$$A = \begin{bmatrix} 1.0 & -0.5 & 0.2 \\ 0.5 & 0.1 & 0.0 \\ 0.3 & 0.0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 0.6 \\ 0.3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.0 & 0.0 & 0.5 \\ 0.0 & 1.0 & 0.5 \end{bmatrix},$$

Algorithm 1: State estimation with LSUCZ filter.

Initialization:
- set the initial time $t = 1$ and the final time $\bar{t} > 1$
- set prior value of $f(x_1)$ (5)
- set noise parameters ν and ω in (6)
- set required maximal number of generators n_g and constraints n_c for posterior (15)

Recursion: for $t = 1, \dots, \bar{t} - 1$ **do**

I. *Data update:* process d_t into $f(x_t|d(t))$ via (15)

II. *Point estimate:* compute $\hat{x}_i, i = \{1,2\}$ according to Method 1 or 2, Subsection 3.3

III. *Time update:* compute $f(x_{t+1}|d(t))$ according to (16)

end

Termination: set $t = \bar{t}$

I. *Data update:* process final datum, $d_{\bar{t}}$, into $f(x_{\bar{t}}|d(\bar{t}))$ via (15)

II. *Point estimate:* compute $\hat{x}_i, i = \{1,2\}$ according to Method 1 or 2, Subsection 3.3

$$\omega = 10^{-4} * \mathbf{1}_{\ell_x}, \quad \nu = 10^{-K} * \mathbf{1}_{\ell_y}, \mathbf{K} \in [\mathbf{0}; \mathbf{4}],$$

where $\mathbf{1}_l$ denotes a unit column vector of a lenght l. Input is randomly generated with standard Gaussian pdf $u_t \sim \mathcal{N}(0,1)$. Length of data sequences $\bar{t} = 600$.

We run and compare the following algorithms:

LSUCZ-1 - Algorithm 1, point estimate - Method 1

LSUCZ-2 - Algorithm 1, point estimate - Method 2

LSUO - orthotopic filter (Pavelková and Jirsa, 2018)

LSUP - parallelotopic filter (Jirsa et al., 2019)

The results are compared by evaluating root mean square error (RMSE) of the state estimates, mean absolute error (MAE) of the state estimates, standard deviation (STD) of the state estimates.

4.2 Results and Discussion

The values of RMSE for a fixed state noise bounds and various observation noise are summarized in Table 1 and visualised in Figure 1. The values of MAE for a fixed state noise bounds and various observation noise are summarized in Table 2 and visualised in Figure 2. Both criteria indicate that for lower noises, LSUO and LSUP perform slightly better than both LSUCZ variants. For higher noises, LSUCZ-1 outperforms all other methods and LSUO is the worst one.

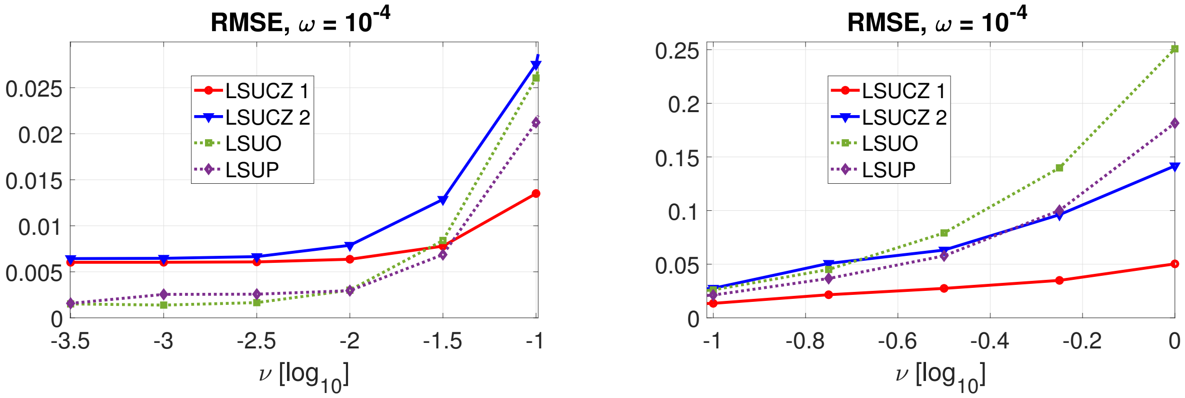

Figure 1: RMSE of state estimates for $\omega = 10^{-4} * \mathbf{1}_{\ell_x}$ and various ν, $\nu = 10^{-K} * \mathbf{1}_{\ell_y}$, $K \in [0;4]$.

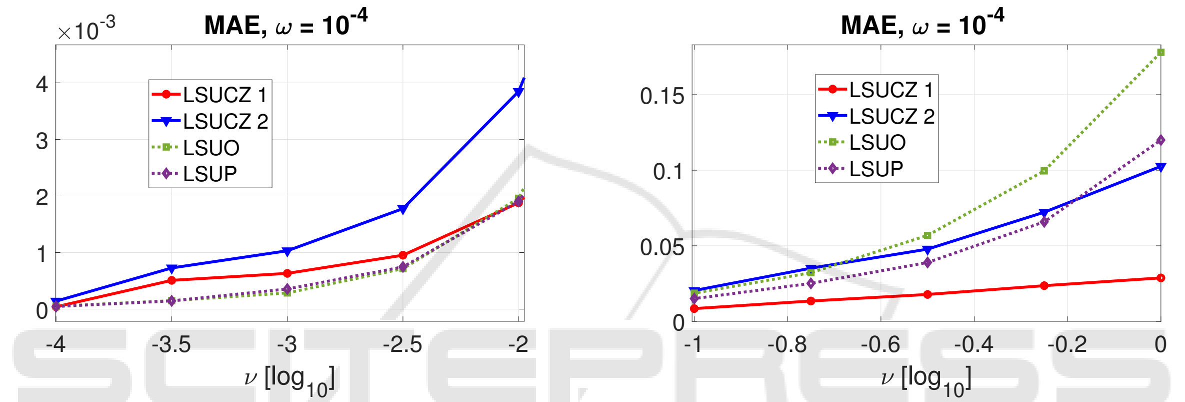

Figure 2: MAE of state estimates for $\omega = 10^{-4} * \mathbf{1}_{\ell_x}$ and various ν, $\nu = 10^{-K} * \mathbf{1}_{\ell_y}$, $K \in [0;4]$.

Table 1: RMSE$*10^{-3}$ of state estimates for state noise bounds $\omega = 10^{-4} * \mathbf{1}_{\ell_x}$ and various output noise bounds $\nu = 10^{-K} * \mathbf{1}_{\ell_y}$.

K	LSUCZ-1	LSUCZ-2	LSUO	LSUP
4	$< 10^{-3}$	$< 10^{-3}$	$< 10^{-3}$	$< 10^{-3}$
3.5	6.0	6.4	1.5	1.6
3	6.0	6.5	1.4	2.5
2.5	6.1	6.6	1.6	2.6
2	6.4	7.9	3.0	2.9
1.5	7.8	12.9	8.4	6.9
1	13.5	27.5	26.1	21.2
0.75	21.5	50.7	45.1	36.7
0.5	27.4	63.2	79.1	57.8
0.25	34.9	96.1	139.8	100.1
0	50.3	141.7	250.8	181.5

The values of STD for a fixed state noise bounds and various observation noise are visualised in Figure 3. While for lower noises, the results are comparable for all filters, for the highest state noise, the both LSUCZ variants perform better.

Table 2: MAE$*10^{-3}$ of state estimates for state noise bounds $\omega = 10^{-4} * \mathbf{1}_{\ell_x}$ and various output noise bounds $\nu = 10^{-K} * \mathbf{1}_{\ell_y}$.

K	LSUCZ-1	LSUCZ-2	LSUO	LSUP
4	$< 10^{-3}$	$< 10^{-3}$	$< 10^{-3}$	$< 10^{-3}$
3.5	0.5	0.7	0.1	0.1
3	0.6	1.0	0.3	0.4
2.5	0.9	1.8	0.7	0.7
2	1.9	3.8	2.0	1.9
1.5	3.9	8.8	5.9	4.8
1	8.4	20.4	18.7	15.0
0.75	13.4	35.2	32.2	25.1
0.5	17.8	47.8	56.9	39.0
0.25	23.6	72.2	99.6	65.9
0	28.7	102.6	178.0	120.0

5 CONCLUSION

This preliminary research suggests that constrained zonotopes are a promising class to deal with a constrained uncertainty in the context of a Bayesian state estimation. The proposed LSUCZ filter is more flex-

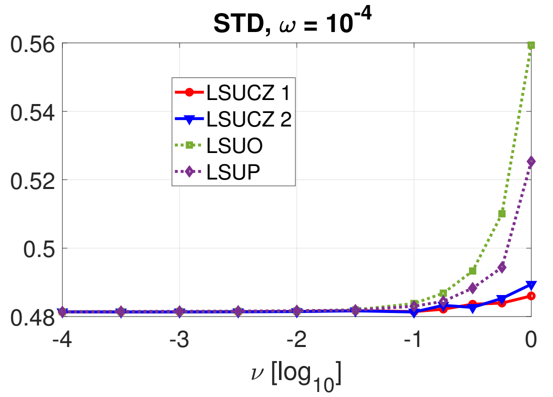

Figure 3: STD of state estimates for $\omega = 10^{-4} * \mathbf{1}_{\ell_x}$ and various ν, $\nu = 10^{-K} * \mathbf{1}_{\ell_y}$, $K \in [0;4]$.

ible compared to the previous LSUO and LSUP variants. It outperforms them from the point of view estimation errors for higher observation noises.

The further research will focus on the more detailed analysis including the posterior volumes and using the the proposed LSUCZ filter in the task of a Bayesian transfer learning schema (Kuklišová Pavelková et al., 2022).

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