Modelling Expressions of Physical Quantities

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Abstract: To express a quantity in conventional scientific notation, a number is paired with a unit of measurement, like $10\,\mathrm{m}\cdot\mathrm{s}^{-1}$. However, this notation can be ambiguous and may require people to understand the context in order to resolve interpretation difficulties. Also, the notation is intended to describe a certain type of scientific data and is ill-equipped to express other kinds of measurement results. This paper discusses an alternative formalism that is suitable for digital systems and overcomes many of the difficulties associated with conventional written notation. We present the proposal using modelling elements that are closely related to scientific concepts that underpin a wide range measurements. The alternative format for expressions is a triplet: a number and a pair of references to information stored centrally. The mathematical properties of data and, in a general sense, the property that is measured, can be captured in this extended format.

1 INTRODUCTION

Meaningful scientific communication needs a shared understanding of the elements of language used to describe data, including the intended meanings of unit and quantity names and symbols. The International System of Units (SI) is generally preferred for scientific work (BIPM, 2019), although some groups find it convenient to adopt other units, or modify SI unit notation to better suit their needs. Many customary units are also used outside scientific communities.

The preferred scientific notation for physical data does not always capture important information. So, effective exchange of information about physical quantities relies on skilled people who can access contextual information and recognise appropriate interpretations of data. Those skills are acquired during years of formal education; but digital systems would benefit from more direct logical representation of data. This paper discusses some of the difficulties with our current scientific notation and suggests a formalism better suited to digital systems.

To express a quantity in conventional notation, a number is paired with a unit of measurement, like $10\,\mathrm{m}\cdot\mathrm{s}^{-1}$ (BIPM, 2019). This notation is complementary to a form of scientific modelling in which mathematical expressions describe relationships between quantities without any reference to units of measurement (e.g., Newton's second law, $f = ma$). Terms in these so-called quantity equations represent abstract physical quantities and the rules governing permissible mathematical operations (known as the quantity calculus) are formulated in terms of the kind of quantity of each term. A quantity equation can be evaluated when 'concrete quantities' (i.e., a number and a unit) are provided for each term (Lodge, 1888).

Several important assumptions are made about the semantics of terms in quantity equations. The quantity concept encompasses the amount of a quantity that is attributed a value of zero. This is often said to be a natural zero (e.g., we may describe a length as being zero without concern for units). It is further assumed that a ratio of two terms for the same kind of quantity does not depend on units (e.g., a linear scale factor is a ratio of lengths that does not depend on any unit of length). These two assumptions allow the notion of a physical dimension to apply to terms (Fourier, 1878), where a dimension may be thought of as the class of units that can be used to express a particular quantity (Ellis, 1964). This, in turn, allows the analytical technique of dimensional analysis to be applied to quantity equations (Barenblatt, 1987).

However, different types of measurement can attribute the value of zero to different physical references. For example, Fahrenheit temperature and Celsius temperature adopt different reference temperatures as zero, so each can be considered a quantity in its own right, with a distinct dimension. There is, however, no benefit in doing so. Quantity equations

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cannot express the relationships between these quantities, or with thermodynamic temperature, nor can the different dimensions associated with Fahrenheit, Celsius, and thermodynamic temperatures be related to each other.

Nevertheless, when physical data is incompatible with quantity equations, it can still be used in what are called numerical value equations. These equations specify the form of expression for each term to ensure that appropriate numbers are used in calculations. For instance, a numerical value equation is given in the SI Brochure that relates Celsius temperature to temperature expressed in kelvin, $t/^{\circ}\mathrm{C} = T/\mathrm{K} - 273.15$ (BIPM, 2019). This equation uses t for a Celsius temperature (with zero at the ice point, which is equivalent to 273.15 K) and T for temperature expressed in kelvin. The stylistic flourish of dividing terms by appropriate units is a way of indicating the data required, and harks back to the quantity concept. However, numerical value equations may use any convenient notation.

Scientific notation for physical quantities and the idea of a formal quantity calculus supporting quantity equations took shape in the 19th century; however, since then, a broader classification of measurement scale types has been developed, which covers a range of different measurement methodologies (Stevens, 1946). The issues associated with current notation can be addressed by these ideas, which can be usefully incorporated in digital representations of physical data.

The remainder of this article is organised as follows. Section 2 reviews, in a series of subsections, the scientific ideas involved in current notation and explains some of the modelling difficulties. Then, subsection 2.5 explains more about the classification of types of measurement scales, which is needed in the proposed formalism but is not part of conventional notation. Section 3 describes a model that can overcome current notational difficulties to represent a broad range of measurement data. Section 4 discusses further how modelling elements relate to basic measurement concepts and section 5 summarises the main points made in the paper.

2 THE CURRENT APPROACH

2.1 The SI

The modern SI is based on the idea of a formal unit system proposed by Maxwell (Maxwell, 1873). These unit systems are built up from a small set of base quantities, with a corresponding set of quantity dimensions, and a set of base units. Relationships between quantities in the system are established by quantity equations involving the base quantities. These relationships can be associated with expressions that take the form of products of powers of base dimensions, which we call system dimensions (e.g., the SI system dimension for speed is LT^{-1}, the dimension for length divided by the dimension for time).[1]

Compound names for derived-quantity units are generated systematically by replacing terms in a system dimension with the corresponding base-unit name. This produces names like, $\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$ (corresponding to the system dimension MLT^{-2}, where M is the SI symbol for the mass dimension). These names are effectively mnemonics for the system dimensions.

The SI has seven base quantities: mass, length, time, electrical current, thermodynamic temperature, amount of substance, and luminous intensity; and seven corresponding units: kg, m, s, A, K, mol, and cd, respectively (BIPM, 2019). However, the relationships between some unit names and quantities is one-to-many, because system dimensions can sometimes be associated with several quantities. For instance, the $\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$ is an acceptable unit for both torque and energy (BIPM, 2019, section 2.3.4).

The SI also defines 22 special unit names and symbols (e.g., rad, °C, Ω, etc.). Like the base units, these special units are associated with specific quantities so, when the quantity is known, theses names can be used interchangeably with the corresponding compound names (e.g., the volt, V, may replace $\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3} \cdot \mathrm{A}^{-1}$).

2.2 Challenges Posed by SI Notation

A formal unit system based on quantities and dimensions is a simple, elegant, and logical idea. There would be no particular difficulty in representing this concept. However, the SI is a dynamic system that continues to evolve. While adopting a pragmatic approach over the years, it has adopted features that create modelling difficulties (Foster, 2009a; Mills, 2009; Foster, 2009b). Four main difficulties are apparent.

Quantities Cannot be Uniquely Identified. Each of the base SI units and the special units is associated with a kind of quantity. However, the

[1] The term ‘dimension’ is commonly used for a system dimension; but this is misleading, because these expressions do not designate actual dimensions—classes of similar scales for a particular kind of quantity. As explained by Emerson, “To say that two quantities are of the same dimension implies a relationship that has significance when in fact it has none” (Emerson, 2005).

same cannot be said for other SI unit names. In general, the kind of quantity cannot be determined from the unit symbol alone. For example, the SI has a special unit joule (J) for energy but the compound units $N \cdot m$ and $kg \cdot m^2 \cdot s^{-2}$ are legitimate alternatives for expressing both torque and energy (BIPM, 2019, section 2.3.4).

Unit Conversion is Sometimes Quantity-Dependent. The special unit names can complicate rules for permissible unit conversions. For example, the special name for the unit of activity, the becquerel (Bq), may always be replaced by the reciprocal second (s^{-1}), whereas data expressed in s^{-1} may not be expressed in Bq unless the expression is known to be a measure of activity, because s^{-1} is also a unit for angular frequency.

Quantities Representing Temperature are Unusual. Relationships between expressions in kelvin and degrees Celsius are unusually complicated. The kelvin is the SI base unit for thermodynamic temperature and the degree Celsius is a special unit name for temperature. The numerical value of a temperature difference is the same when expressed in either unit. However, Celsius temperature places zero at the ice point, while zero on the kelvin scale refers to absolute zero of thermodynamic temperature. Transformation of temperatures between kelvin and degrees Celsius must take the different zeros into account (Hall et al., 2023).

Dimensionless Quantities are not Plain Numbers. The class of *dimensionless quantities* contains a large number of quantities that can be expressed in the SI unit one. Only two special names for dimensionless quantities are defined (the radian, for plane angle, and the steradian, for solid angle), so most dimensionless quantities are expressed in terms of the unit one, which provides no information about the kind of quantity. In practice, dimensionless-quantity data is often treated as plain numbers, but these quantities have unique characteristics that are not generally comparable to each other. Notation showing a ratio of unit symbols is encouraged (e.g., mm/m, g/kg, etc.), as this can convey useful information. However, not all dimensionless quantities are simple ratios of the same kind of quantity (e.g., the dimensionless numbers that arise in fluid mechanics (Wikipedia, 2023)).

These types of problem are known. Some are brought to the attention of readers in the SI Brochure, so people can act to mitigate their impact.

2.3 Units Outside the SI

The SI has strict formatting rules and style conventions, which are intended to ensure that notation is used consistently. However, there are groups that deviate from the rules, and sometimes alternative symbols, or even alternative interpretations of standard symbols, are adopted, effectively introducing *ad hoc* notation.

For example, some specialists in humidity and hygrometry favour the symbol %rh to express the dimensionless quantity relative humidity. This breaks SI rules by annotating the percent symbol (%, representing 1/100) with ‘rh’ to indicate a kind of quantity. Another example is when symbols are introduced to identify particular chemical elements. For instance, 12 kg C may be intended to express a mass of the chemical element carbon; but the symbol C represents a coulomb—the special SI unit for electric charge.

Customary units are often organised into systems, like the British Imperial System of units (Encyclopaedia Britannica, 2019). However, these systems do not have a formal structure: no base quantities or units are defined. Most customary units are not accepted for use in the SI. However, authoritative conversion factors to SI equivalents are sometimes published (e.g., (Butcher et al., 2006)). When authoritative conversion factors are available, data can be disseminated in customary units while maintaining a strict relationship to SI units.

2.4 Kind of Quantity

The earliest description of quantity calculus is attributed to Lodge (Lodge, 1888; Copley, 1960). He explained that terms in a quantity equation are associated with ‘kinds of quantity’, and that, if equality is to be meaningful, both sides of an equation must represent quantities of the same kind. This requirement is reminiscent of dimensional homogeneity, but Lodge reminded readers that homogeneity is not a sufficient condition for quantities to be of the same kind, and gave examples where an understanding of the physics is needed to identify the kinds of quantity.

Although Lodge referred to kinds of quantity for terms when describing quantity calculus, he made no attempt to define this terminology. We assume the ordinary English sense of ‘kind’ is adequate: as being of the same class, sort, or variety. So, length is both the name of a quantity and of a kind of quantity, whereas breadth, height, thickness, radius, diameter, circumference, etc., are all names of quantities, but they are not quantity kinds. The ISO 80000 standard, which documents the International System of Quantities, also interprets 'kind of quantity' in this sense (ISO, 2013).

Kinds of quantity are used to establish acceptable computational steps. There is no restriction on multiplication of terms, but only terms associated with the same kind of quantity may be added or subtracted. Division is interpreted as the inverse of multiplication, and terms may be exponentiated as an alternative notation for multiplication and division. Dimensionless quantity terms are considered pure numbers in the calculus, allowing non-linear operations (e.g., trigonometric, logarithmic, and exponential functions) to be applied, and the results to be treated as pure numbers as well.

The kind of quantity of a sum or difference is the same as the terms involved, which seems self-evident: the cliché that apples cannot be added to oranges is even more compelling when quantities such as length and time are considered. However, the calculus has no way of determining the kind of quantity for a product or quotient. A product is simply understood as being proportional to its factors, while a quotient is proportional to factors in the numerator and inversely proportional to factors in the denominator (Lodge, 1888).

2.5 Types of Scale

During the first half of the 20th century, research disciplines outside the traditional physical sciences were challenged about the validity of quantitative data obtained without the underlying conceptual support of notions like a physical quantity. From this debate, a classification scheme for different types of measurement emerged (Stevens, 1946). This classification relates certain types of experimental procedures to the mathematical properties of data obtained.

Four fundamental types of measurement scale were identified: ratio, interval, ordinal, and nominal. All of these are important to metrology (White, 2010), although we focus on ratio and interval scales here.

Each scale type can be associated with a mathematical transformation that preserves certain properties of the data and does not change the type of scale. For a ratio scale, multiplication by any positive number transforms data to another ratio scale. This corresponds to the familiar process of unit conversion for quantities, which does not affect data ratios (e.g., aspect ratio or linear scale factor).

Affine functions transform one interval scale to another by applying a scale factor and an offset. The Fahrenheit and Celsius temperature scales are examples of interval scales. Conversion from one to the other requires an adjustment for the different reference points associated with zero and a rescaling of the data (the degree Fahrenheit is smaller than the degree Celsius). The arithmetic mean and standard deviation are not affected by affine transforms. For example, the following two processes yield the same result: 1) take the mean of a sample of data in degrees Fahrenheit and transform it to degrees Celsius; or 2) transform data in degrees Fahrenheit to degrees Celsius and take the mean.

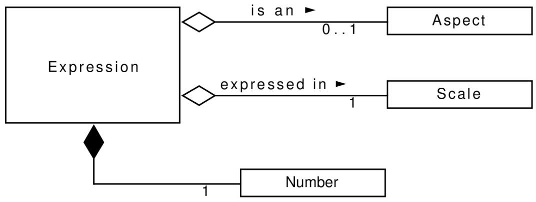

Figure 1: An expression is a triplet, consisting of a number with references to a scale and an aspect. To accommodate legacy data, the aspect reference is shown as optional; but if no aspect is specified, fewer legitimate forms of expression can be identified. This figure, and those following, are class diagrams in the unified modelling language (UML) version 2.5 (Object Management Group, 2015).

3 MODELLING QUANTITY EXPRESSIONS

This section describes a more detailed formalism for expressions of physical data that avoids the difficulties with unit notation identified above (Hall and Kuster, 2022; Hall, 2023). This can be thought of as extending the conventional formalism.

A datum is expressed as a triplet (figures 1 and 2): a number accompanied by references to an *aspect* and a *scale*. This corresponds to an English phrase like "9.8 is the acceleration expressed in $\mathrm{kg \cdot m \cdot s^{-2}}$ on a ratio scale" (acceleration is the aspect and the SI unit combined with the ratio scale-type is the scale).

An aspect may be regarded as a generalisation of the role played by a kind of quantity. This is explained further in subsection 3.2. A scale may be regarded as a generalisation of the role played by a unit. Scales are defined by associating a unit, and perhaps other appropriate references, with a scale type. For example, a scale can be defined for temperature by associating the unit degree Celsius, a reference to the ice point (at $0\,^{\circ}\mathrm{C}$), and the interval scale type.

In digital records, expressions hold a numeric value and unique identifiers for the aspect and scale (figure 2). The identifiers refer to digital objects that encapsulate information about what is expressed and how. These objects are stored in a central registry, which can be indexed by the identifiers. The purpose of using identifiers for the aspect and scale is to ensure that alternative legitimate expressions for data can be identified. This is achieved using a table of expression transformations in the central registry, indexed by aspect–scale identifier pairs.

```
<root xmlns:mlayer="http://mlayer.org/ns">
  <mlayer:Expr>
    <mlayer:Number x=12.3>
    <mlayer:Aspect id="AS2"/>
    <mlayer:Scale id="SC1"/>
  </mlayer:expr>
</root>
```

Figure 2: Digital records identify an aspect and a scale in expressions, as shown in this XML snippet. The succinct identifiers index more detailed information in a central register. Figures 5 and 6 show more details for the scale (kg) and the aspect (mass), respectively.

3.1 Scale

The different types of scale may be regarded as specialisations of a generic type (figures 3 and 4), which captures the attributes needed. For example, a `Unit` is associated with both a `RatioScale` and an `IntervalScale`, but an `IntervalScale` is also associated with a `Reference` that defines one point on the scale (figure 4).

In systems like the SI (here qualified as *formal* unit systems), each unit is associated with a product of powers of base dimensions (here called a *system dimension*). For instance, the system dimension LT^{-2} is associated with the unit for acceleration, $\mathrm{m \cdot s^{-2}}$. However, customary unit systems do not define base quantities or base units. So, units in these systems may be associated with ratio scales but the notion of system dimensions does not apply.

The UML class diagram in figure 3 shows relationships between system, ratio scale, unit and system dimension. Figure 5 shows an example of a scale object representing the ratio scale for the SI kilogram.

As explained in section 2.2, when Celsius temperature is considered a quantity, its relationship to thermodynamic temperature is lost (only numerical value equations relate expressions of temperature in different units). However, most specialists in temperature measurement would not think of Celsius temperature as being a different quantity to thermodynamic temperature. A better representation of scientific understanding is provided by explicit scales, such as the degree Celsius paired with the interval scale type, and the kelvin paired with the ratio scale type. These scales can be associated with an aspect for temperature. This use of aspect is more general than the quantity concept applied to Celsius temperature and thermodynamic temperature (which attribute different temperatures to zero).

3.2 Aspect

In the conventional expression of a quantity, the kind of quantity may be hard to determine unless it can be inferred directly from a special unit name, or a base unit name. The formalism proposed here overcomes this difficulty by making explicit reference to an aspect, which may be used to disambiguate data. For example, knowing that a datum expressed in s^{-1} on a ratio scale is associated with the aspect *frequency* establishes equivalence with an alternative expression in Hz. Aspects can address difficulties in the interpretation of SI units related to the concept of quantity (and hence kind of quantity), but they can also be used to denote measurable properties in a wider sense than kinds of quantity.

One difficulty arises because temperature differences are expressed in degrees Celsius on a ratio scale, but absolute temperatures are expressed in degrees Celsius on an interval scale. So, in degrees Celsius, the scale type can disambiguate between temperature data and temperature differences. However, both forms of expression can be converted to kelvin on a ratio scale, after which the quantity is ambiguous. Explicit reference to the kind of quantity cannot resolve this problem. As explained in section 2.4, the kind of quantity is always the most general sense of a quantity—the superordinate class. So, the kind of quantity of a temperature difference is the same as a temperature (this is reflected in the dimensional homogeneity requirement for a quantity equation like $\Delta T = T - T_0$, for the difference between temperatures T and T_0). Instead, distinct aspects for absolute temperature and temperature difference can be used to resolve this ambiguity (see figure 7).

Aspects are also useful for dimensionless quantities, many of which represent ratios of the same kind of quantity, such as aspect ratio and linear scale factor (both length ratios). Although all dimensionless quantities have identical system dimensions, many are arguably distinct kinds of quantity (Ellis, 1964). Some dimensionless quantities are best represented as particular quantity kinds—angle being a good example—while others may be best represented as quantity ratios, in which case the aspect is the ratio of the quantity kinds for the numerator and denominator. For instance, data for both a linear scale factor and an aspect ratio will be associated with a length-ratio aspect (similar to notation that shows unit ratios, like mm/m), although the nature of the lengths is different (lengths are perpendicular in an aspect ratio, but they are collinear in a linear scale factor).

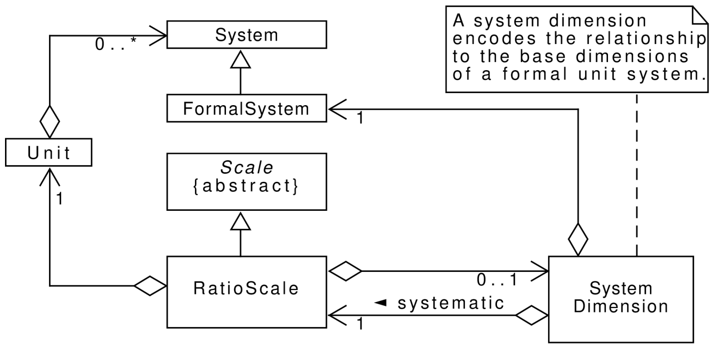

Figure 3: A `RatioScale` is associated with a unit, and may be associated with a system dimension (a product of powers of the base dimensions in a formal unit system—often encoded as a sequence of integers). Many ratio scales can be associated with a single system dimension, but just one has the unit with a name derived from the system dimension. This scale is designated 'systematic'. No system dimension is associated with scales for units that do not belong to a formal system.

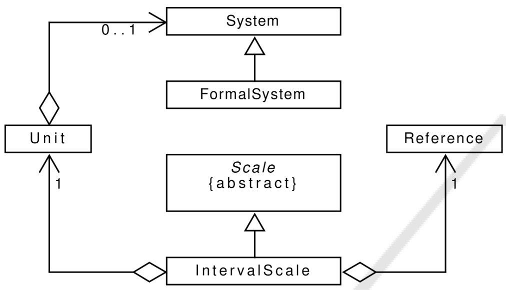

Figure 4: An `IntervalScale` is associated with a unit, which may be part of a unit system. In addition to the unit (which determines the size of scale divisions), the `Reference` establishes a physical reference to one point on the scale (often defining the zero point).

```
{
  "id": "SC1",
  "ml_name": "ra_si_kg",
  "type": "ratio",
  "unit_id": "UN1",
  "system_dimension_id": "DI1",
}
```

Figure 5: A JSON object representing a `RatioScale` associated with the SI kilogram (some details are elided). Unit and system dimension identifiers index further information.

The aspect may be used to distinguish between measurements of closely related but different properties. Physical properties of practical importance are sometimes difficult to measure when a large number of factors can influence measurement results. In such cases, standard methods may be developed. These allow measurements to be made in a reproducible manner, so results can be compared with similar data obtained according to the same standard. One example is viscosity, for which there may be as many as a hundred different standard methods, each defining a protocol suited to specific needs for information (White, 2010). Viscosity may be expressed in SI units but, if data is to be compared in any meaningful sense, the method of measurement has to be identifiable. Aspect can be used for this purpose.

```
{
  "id": "AS2",
  "ml_name": "as_mass",
  "name": "mass",
}
```

Figure 6: A JSON object representing the `Aspect` mass (some details are elided). References to external information about the physical concept mass would be provided but are not shown.

4 DISCUSSION

The purpose of the formalism described here is to capture information about data in sufficient detail to enable other legitimate forms of expression to be identified. In other words, it supports the abstract notion of a measurable magnitude that can be expressed in different but equivalent ways. Our modelling elements represent the various constructs and measurement concepts involved in the expression of data. The relationships between these elements do not seem to have been modelled before.

Many attempts to find satisfactory digital representations for units of measurement for programming languages and databases have been made over the years—see (McKeever et al., 2021) for a recent review. There are also a number of ontologies that define quantities and units—see (Aameri et al., 2020). Nevertheless, there is often confusion about fundamental concepts, like quantity, dimension, and quantity calculus. For example, a digital encoding resembling system dimensions is often used to represent quantity kinds and units. This can produce apparently nifty software; however, it does not accurately

$$\left.\begin{matrix}\langle T\rangle[\![{}^\circ\mathrm{C},\mathrm{I}]\!]\\ \langle \Delta T\rangle[\![{}^\circ\mathrm{C},\mathrm{R}]\!]\\ \langle T_{90}\rangle[\![{}^\circ\mathrm{C},\mathrm{I}]\!]\\ \langle \Delta T_{90}\rangle[\![{}^\circ\mathrm{C},\mathrm{R}]\!]\end{matrix}\right\}{}^\circ\mathrm{C}\quad \left.\begin{matrix}\langle T\rangle[\![\mathrm{K},\mathrm{R}]\!]\\ \langle \Delta T\rangle[\![\mathrm{K},\mathrm{R}]\!]\\ \langle T_{90}\rangle[\![\mathrm{K},\mathrm{R}]\!]\\ \langle \Delta T_{90}\rangle[\![\mathrm{K},\mathrm{R}]\!]\end{matrix}\right\}\mathrm{K}$$

$$\left.\begin{matrix}\langle T\rangle[\![{}^\circ\mathrm{F},\mathrm{I}]\!]\\ \langle \Delta T\rangle[\![{}^\circ\mathrm{F},\mathrm{R}]\!]\end{matrix}\right\}{}^\circ\mathrm{F}\quad \left.\begin{matrix}\langle T_{90}\rangle[\![{}^\circ\mathrm{F},\mathrm{I}]\!]\\ \langle \Delta T_{90}\rangle[\![{}^\circ\mathrm{F},\mathrm{R}]\!]\end{matrix}\right\}{}^\circ\mathrm{F}$$

Figure 7: Pairing of an aspect and a scale allows different types of temperature data to be expressed without ambiguity. Aspect–scale pairs are shown grouped with the corresponding conventional unit symbol. Angle brackets $\langle\cdot\rangle$ indicate an aspect and $[\![\cdot]\!]$ a scale. $\langle T\rangle$ is thermodynamic temperature, or temperature difference when prefixed by Δ. $\langle T_{90}\rangle$ is a defined scale that approximates thermodynamic temperature (Hall et al., 2023). The scale types are: R, for ratio scale; and I, for interval scale.

represent the underlying metrological concepts and, although extra dimensions can be added for internal use, it is vulnerable to the type of problems with ambiguity discussed above for SI system dimensions.

Figure 7 illustrates the proposed approach applied to temperature data. As mentioned in §2.2, the kelvin (K) and degree Celsius (°C) are both SI units for temperature, but the relationships between data expressed in these units can be complicated. Also shown is the degree Fahrenheit (°F), which is a commonly used customary unit in the United States. In addition, there is a widely used method-defined temperature scale called ITS-90 that approximates thermodynamic temperature. The figure shows that different types of data can be resolved by using explicit aspects and scales. This makes it possible to identify alternative expressions. For instance, the conversion of temperature data from °C to °F has multiple interpretations (such as the illegitimate conversion from $\langle T\rangle[\![{}^\circ\mathrm{C},\mathrm{I}]\!]$ to $\langle \Delta T\rangle[\![{}^\circ\mathrm{F},\mathrm{R}]\!]$), whereas an explicit conversion from $\langle T\rangle[\![{}^\circ\mathrm{C},\mathrm{I}]\!]$ to $\langle T\rangle[\![{}^\circ\mathrm{F},\mathrm{I}]\!]$) is unambiguous (and legitimate).

Using aspect–scale identifier pairs, a table of transformations that map between expressions can be maintained in a central register. Entries in this register are indexed by aspect–scale pairs for the initial and final expressions. The table holds numerical coefficients and mathematical functions to transform from one expression to another. This register-based approach accurately captures the nature of relationships between expressions without being limited to particular unit systems. The register can include mappings based on data published by authoritative bodies when the units involved do not belong to the same system.

Certain sets of mappings in this table are useful and interesting. For example, the general notion of dimension—a class of similar ratio scales for a quantity (Ellis, 1964)—corresponds to the set of mappings between the ratio scales paired with a given aspect. For example, units of length, like the inch, yard, etc. are not recognised by the SI. However, authoritative conversion factors from these units to the metre are published (Butcher et al., 2006), so ratio scales for these units will be included in the dimension set for length by virtue of the known transformations.

Another example is the dimension set for angle, which has mappings between ratio scales associated with an angle aspect. The radian is a special name for the unit of plane angle and the imperial unit for plane angle—the degree—is an accepted non-SI unit. Scales associated with these units are included in the dimension set for angle because there are recognised transformations.

There are other useful sets of related expressions that can be identified in the table. There is the set of mappings between all interval scales for a given aspect, which is analogous to the dimension set. For example, the interval-scale expressions for temperature in degrees Celsius and degrees Fahrenheit are related by affine transformations, which would be included in such a set. Similarly, and perhaps more useful, there are the sets of mappings between different types of scale for a given aspect. For instance, temperature expressed in kelvin on a ratio scale may be transformed to temperature in degrees Celsius on an interval scale.

The ideas presented here are being implemented as an online service with a RESTful interface to a cloud-based register. This project, in collaboration with the Measurement Information Infrastructure technical committee 141 (MII) of the NCSL International organisation.[2]), is at an early stage, so details may change. Governance of data in the register will be provided by the MII committee under the auspices of NCSL International.

5 CONCLUSIONS

A formalism that overcomes many of the difficulties associated with conventional notation for physical quantity data has been discussed. The scientific concepts that underpin expressions of quantities can be clearly related to elements of the formalism used to model data. Conventional notation for written expression of quantities is intended for skilled people and sometimes requires supplementary contextual information to be interpreted. This lack of expressive-

[2] Founded under the name 'National Conference of Standards Laboratories', this global non-profit organisation is now known as NCSL International (see, https://ncsli.org/page/AB

ness is addressed in the alternative formalism by including information about the mathematical structure of data (the scale type) and the nature, in a general sense, of the property that was measured (the aspect). The conventional couple notation, of a number with a unit symbol, is replaced by a triplet: a number and a pair of digital identifiers that refer to centrally-stored information concerning the measured aspect and the scale of measurement. A central register of transformations that map between alternative forms of expression allows sets of related aspect–scale expressions to be identified. One such category of sets corresponds to the general notion of a quantity dimension.

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