On the Categories of Coalgebras, Dialgebras and Powerset Theory over L-Fuzzy Approximation Spaces

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Abstract: This paper is to establish a relationship between powerset theories and the category of dialgebras over the category of L-fuzzy approximation space, where L is a residuated lattice. Also, we show that the category **FAS** of L-fuzzy approximation spaces is a category of F-coalgebras. Interestingly, we introduce a functor having both the left/right adjoint from the category **FAS** to the category **UAS** of upper approximation sets. Further, the category **J-Coal** of J-coalgebras and the category **(J,K)-Dial** of (J,K)-dialgebras are introduced over the category **FAS** and it is shown that they are topological categories.

1 INTRODUCTION

The notion of rough sets was introduced by Pawlak (Pawlak, 1982) in his seminal paper in the early eighties. This idea applies to data systems that contain inconsistencies. Rough set theory is a database mining or knowledge discovery approach for relational databases. Equivalence relations played an important part in the rough set theory presented by Pawlak. Several generalizations of rough sets have been made in the literature (Kondo, 2006; Qin et al., 2008; Yao, 1998) by using an arbitrary relation in the place of equivalence relation. By using the ideas of fuzzy set theory, Dubois and Prade (Dubois and Prade, 1990) introduced the concept of fuzzy rough set, in which fuzzy relations play the key role instead of crisp relations, which turned into a powerful tool in analyzing inconsistent and vague data. Further, the combinations of fuzzy sets and rough sets were studied using binary fuzzy relations and different fuzzy logic operations in (D'eer et al., 2015; Mi et al., 2008; Morsi and Yakout, 1998; Radzikowska and Kerre, 2002; Radzikowska and Kerre, 2005; Tiwari et al., 2018; Wang and Hu, 2015; Wu et al., 2013; Yao et al., 2014); and the topological properties of fuzzy rough sets were discussed in (Perfilieva et al., 2017; Srivastava and Tiwari, 2003; Tiwari et al., 2014; Tiwari and Srivastava, 2013; Wang, 2023).

Eilenberg and Mac Lane (Eilenberg and MacLane, 1945) developed the concept of category theory, which is well-known. A number of researchers further developed this theory (Freyd, 1964; Lawvere, 1963; Lawvere, 1966) and demonstrated it to help develop many aspects of theoretical computer science. Coalgebra is an abstract theory that emerged as a relatively new theory within or closely related to category theory. Coalgebras have the advantage of naturally dealing with nondeterminism and undefinedness concepts that are difficult or impossible to deal with algebraically. Categories of dialgebras were initially investigated as extended algebraic categories (Adámek, 1976; Trnková and Goralčík, 1969). In computer science, these structures have been used to specify data types (Hagino, 1987; Poll and Zwanenburg, 2001) Powerset structures are commonly used in algebra, logic, topology, and computer science. Almost all areas of mathematics and its applications, including computer science, use the primary example of a powerset structure $P(X) = \{A : A \subseteq X\}$ and the related extension of a mapping $f : X \to Y$ to the map $f_P^{\to} : P(X) \to P(Y)$. Classical set theory can be thought of as a subset of the fuzzy set theory; naturally, powerset objects associated with fuzzy sets were examined as generalizations of classical powerset objects. The first approach was made by Zadeh (Zadeh, 1965), who defined a new powerset object instead of $P(X)$ and introduced new powerset operators named Zadeh's forward and backward operator. Many papers (Rodabaugh, 1999a; Rodabaugh, 1999b) have been published about Zadeh's extension and generalizations. Rodabaugh investigated Zadeh's extension for lattice-valued fuzzy sets for the first time in (Rod-

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abaugh, 1999a). Rodabaugh's work laid a solid foundation for future research into powerset objects and operators. The relationship of algebraic theories to powerset theories and fuzzy topological theories have been studied in (Rodabaugh et al., 2007). In (Močkoř, 2020), it is proved that fuzzy soft sets also give rise to a powerset theory, which is defined by a monad. Further, in (Močkoř, 2023), two basic types of relational powerset theory have been introduced for semiring-valued fuzzy structures and examine the basic relationships between these theories. The relationships between powerset theories and F-transforms are investigated in (Močkoř, 2018). Recently, in (Perfilieva et al., 2017), it has been shown that F-transform is a realization of an abstract fuzzy rough set theory. So, it is obvious to think about the relationship between fuzzy approximation spaces and powerset theories. We use category and powerset theories to expand this study's idea of L-fuzzy approximation spaces. Specifically,

- we show that the category **FAS** is a category of F-coalgebras where F is a functor from the category **SET** to **SET**;
- The categories J-**Coal** and (J,K)-**Dial** are introduced over **FAS** and it is shown that they are topological categories;
- The relationship between powerset theories and the category of dialgebras is established over the category **EFAS** of L-fuzzy equivalence approximation space.

The paper is organized as follows. In Section 2, we recall some basic properties of residuated lattice, L-fuzzy sets and category theory. In Section 3, it is shown that the category **FAS** is a category of F-coalgebras. The categories **J-Coal** and **(J,K)-Dial** are introduced and it is shown that they are topological categories. Next, the relationship between powerset theories and the category of dialgebras is established. At end, we conclude our work.

2 PRELIMINARIES

This section recalls the ideas respective to the residuated lattice, L-fuzzy sets, MV-algebra, and category theory (cf., (Barr and Wells, 1990; Belohlavek, 2012; Blount and Tsinakis, 2003; Ćirić et al., 2007; Goguen, 1967; Mac Lane, 2013; Pei, 2004; She and Wang, 2009; Zadeh, 1975)). We begin with the following definition.

Definition 2.1. *A complete residuated lattice is a structure* $L = (L,\wedge,\vee,\otimes,\rightarrow,0,1)$ *such that* $(L,\wedge,\vee,0,1)$ *is a complete lattice with lower bound 0 and upper bound 1,* $(L,\otimes,1)$ *is a commutative monoid and* $(\rightarrow,\otimes)$ *is an adjoint pair, i.e.* $a\otimes b \leq c \iff a \leq b \rightarrow c$, $\forall a,b,c \in L$.

A complete residuated lattice L is said to be divisible if for $b,c \in L$ and $b \leq c$, $\exists\ d \in L$ such that $b = c \odot d$. A negation in L is a map $\neg : L \rightarrow L$ such that $\neg b = b \rightarrow 0$, $\forall\ b \in L$. If $\neg(\neg b) = b$, $\forall\ b \in L$, then L is called a complete regular residuated lattice. An MV-algebra is a complete regular residuated lattice satisfies divisibility property.

Proposition 2.1. *Let L be a complete residuated lattice. Then for* $a,b,c \in L$,

- $1\otimes a = a\otimes 1 = a$, $a\otimes 0 = 0\otimes a = 0$;
- $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$;
- $a\otimes(\vee_{i\in I}b_i) = \vee_{i\in I}(a\otimes b_i)$;
- $a \rightarrow (\wedge_{i\in I}b_i) = \wedge_{i\in I}(a \rightarrow b_i)$.

If L is a complete regular residuated lattice, Then

- $\neg a \rightarrow \neg b = b \rightarrow a$;
- $\neg \vee_{i\in I} a_i = \wedge_{i\in I}\neg a_i$.

Throughout this paper, an L-fuzzy set is identified with its membership function and takes values from a fixed complete residuated lattice L. For a nonempty set X, the collection of all L-fuzzy sets of X is denoted by L^X. Also, for all $a \in L$; $\mathbf{a}(x) = a$ denotes a constant L-fuzzy set. Further, for $A \in L^X$, the $core(A)$ is a set of all elements $x \in X$ such that $A(x) = 1$ and 1_x is the characteristic function of $\{x\}$ in X.

For $A_1,A_2 \in L^X$ and $x \in X$, new L-fuzzy sets are defined as follows: $(A_1\vee A_2)(x) = A_1(x)\vee A_2(x)$; $(A_1 \wedge A_2)(x) = A_1(x) \wedge A_2(x)$; $(A_1 \odot A_2)(x) = A_1(x)\odot A_2(x)$; $(A_1 \rightarrow A_2)(x) = A_1(x) \rightarrow A_2(x)$.

We recall the following concept of an L-fuzzy relation and Zadeh's L-fuzzy operators from (Mahato and Tiwari, 2020).

Definition 2.2. *An L-fuzzy relation R on X is an L-fuzzy set of* $X\times X$. *An L-fuzzy relation R is called reflexive if* $R(x,x) = 1$, *symmetric if* $R(x,y) = R(y,x)$ *and transitive if* $R(x,y)\otimes R(y,z) \leq R(x,z)$, *for all* $x,y,z \in X$.

A reflexive, symmetric and transitive L-fuzzy relation on X is called an L-fuzzy equivalence relation on X.

Definition 2.3. *Let* $\phi : X \rightarrow Y$ *be a map, then Zadeh's L-fuzzy forward operators* $\phi_Z^{\rightarrow} : L^X \rightarrow L^Y$ *is defined as* $\phi_Z^{\rightarrow}(A)(x') = \vee_{\phi(x)=x'}A(x)$, $\forall A \in L^X, \forall x' \in Y$.

Next, we recall the ideas related to category theory. For details, we refer to the work done in (Barr and Wells, 1990; Mac Lane, 2013). Some standard categories used in this paper are: category **SET** of sets as objects and maps as morphisms; and the category **CSLAT**($\vee$) of complete $\vee$-semilattices as objects with $\vee$-preserving maps as morphisms. Similarly, we can define the category **CSLAT**($\wedge$). We shall write **CSLAT** if there is no need to differentiate between $\vee$ and $\wedge$.

Definition 2.4. *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and M be a $\mathbf{D}$-object. Then the functor F has left adjoint if for some $\mathbf{C}$-object N, there exists a pair (N,τ), where $\tau : M \to F(N)$ is a $\mathbf{D}$-morphism such that for all $\mathbf{C}$-object N' and $\mathbf{D}$-morphism $f : M \to F(N')$, there exists a unique $\mathbf{C}$-morphism $g : N \to N'$ such that the diagram in Figure 1 commutes.*

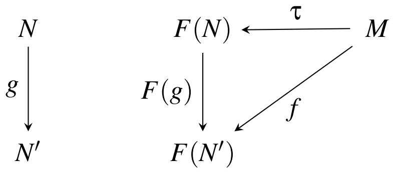

Figure 1: Diagram for Definition 2.4.

Definition 2.5. *Let $G : \mathbf{C} \to \mathbf{D}$ be a functor and M be a $\mathbf{D}$-object. Then the functor G has right adjoint if for some $\mathbf{C}$-object N, there exists a pair (N,η), where $\eta : G(N) \to M$ is a $\mathbf{D}$-morphism such that for all $\mathbf{C}$-object N' and $\mathbf{D}$-morphism $f : G(N') \to M$, there exists a unique $\mathbf{C}$-morphism $g : N' \to N$ such that the diagram in Figure 2 commutes.*

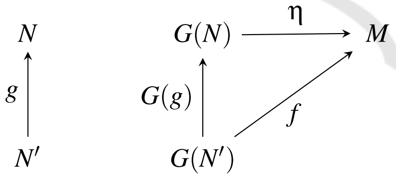

Figure 2: Diagram for Definition 2.5.

We close this section by recalling the concepts of F-coalgebras and (F,G)-dialgebras.

Definition 2.6. *For a functor $F : \mathbf{C} \to \mathbf{C}$, F-coalgebra is a pair (X,α), where $X \in obj(\mathbf{C})$ and $\alpha : X \to F(X)$ is a map of the coalgebra. A morphism between two F-coalgebras (X_1,α_1) and (X_2,α_2) is a map $f : X_1 \to X_2$ such that $F(f) \circ \alpha_1 \leq \alpha_2 \circ f$. The class of F-coalgebras alongwith their morphisms form a category.*

Definition 2.7. *For functors $F,G : \mathbf{C} \to \mathbf{D}$, a (F,G)-dialgebra is a pair (X,γ), where $X \in obj(\mathbf{C})$ and $\gamma : F(X) \to G(X)$ is a map . A morphism between two (F,G)-dialgebra (X_1,γ_1) and (X_2,γ_2) is a map $f : X_1 \to X_2$ such that $G(f) \circ \gamma_1 \leq \gamma_2 \circ F(f)$. The class of all (F,G)-dialgebras alongwith their morphisms form a category.*

3 COALGEBRAS AND DIALGEBRAS OVER *L*-FUZZY APPROXIMATION SPACES

In this section, we show that the category of L-fuzzy approximation spaces **FAS** is a category of F-coalgebra, where F is a functor from **SET** to **SET**. The concept of the category of upper approximation set **UAS** is introduced, and we show that there exists a functor G from the category **FAS** to the category **UAS**, which has both the left and right adjoint. Further, the category **J-Coal** of J-coalgebras and the category **(J,K)-Dial** of (J,K)-dialgebras are introduced over the category **FAS** and it is shown that they are topological categories.

Now, we recall the following from (Mahato and Tiwari, 2020).

Definition 3.1. *A pair (X,R) is called an L-fuzzy approximation space, where X is a nonempty set and R is an L-fuzzy relation on X. The operators $\overline{R},\underline{R} : L^X \to L^X$ are respectively called the L-fuzzy upper and L-fuzzy lower approximation operators of (X,R), where for $A \in L^X$*

$$\overline{R}(A)(x) = \vee_{y\in X}(R(x,y) \otimes A(y))$$
$$\underline{R}(x) = \wedge_{y\in X}(R(x,y) \to A(y)).$$

The pair $(\underline{R}(A),\overline{R}(A))$ is called an L-fuzzy rough set of an L-fuzzy set of $A \in L^X$ in (X,R). Further, let (X_1,R_1) and (X_2,R_2) be two L-fuzzy approximation spaces. Then a map $f : X_1 \to X_2$ is relation preserving if $R_1(x_1,y_1) \leq R_2(f(x_1), f(y_1))$, $\forall x_1,y_1 \in X_1$.

L-fuzzy approximation spaces along with relation preserving maps form a category, say, **FAS**.

Proposition 3.1. *Let (X,R) be an L-fuzzy approximation space. Then*
(1) If R is an L-fuzzy equivalence relation on X, then $\underline{R} \circ \underline{R} = \underline{R}$ and $\overline{R} \circ \overline{R} = \overline{R}$.
(2) If L is a regular residuated lattice, then $\neg\overline{R}(A) = \underline{R}(\neg A)$.

Now, we introduce a functor $F : \mathbf{SET} \to \mathbf{SET}$ such that for $X \in obj(\mathbf{SET})$, $F(X) = L^X$ and for a map f, $F(f) = f_Z^{\to}$. The following is towards the category **FAS**, which is a category of F-coalgebras. Let (X,R) be an L-fuzzy approximation space. Then the L-fuzzy relation $R : X \times X \to L$ may also be interpreted as a F-coalgebra map, $R : X \to L^X$ such that for all $x,y \in X$, $R(x)(y) = R(x,y)$.

Theorem 3.1. *The category **FAS** is a category of F-coalgebra.*

Proof: Let (X_1,R_1) and $(X_2,R_2) \in obj(\mathbf{FAS})$. Then $X_1, X_2 \in obj(\mathbf{SET})$ and the maps $R_1 : X_1 \to L^{X_1}$ and $R_2 : X_2 \to L^{X_2}$ are defined as $R_1(x_1)(y_1) = R_1(x_1,y_1)$ and $R_2(x_2)(y_2) = R_2(x_2,y_2)$. Thus (X_1,R_1) and (X_2,R_2) are F-coalgebras. Now, let $f : X_1 \to X_2$ be a map in **SET** such that f is relation preseving map and $F(f) = f_Z^{\to}$. Then the diagram in Figure 3 commutes, i.e., for $x_1 \in X_1$ and $y_2 \in X_2$,

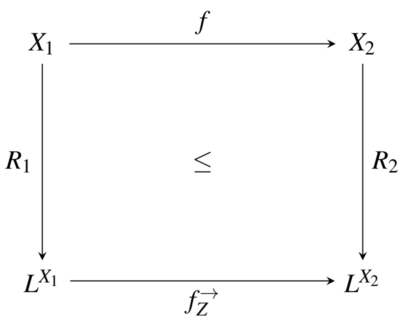

Figure 3: Diagram for Theorem 3.1.

$$\begin{aligned}(F(f)\circ R_1(x_1))(y_2) &= (f_Z^{\to}(R_1(x_1)))(y_2)\\ &= \vee_{f(y_1)=y_2} R_1(x_1)(y_1)\\ &= \vee_{f(y_1)=y_2} R_1(x_1,y_1)\\ &\leq \vee_{f(y_1)=y_2} R_2(f(x_1),f(y_1))\\ &\leq R_2(f(x_1),y_2)\\ &\leq (R_2\circ f)(x_1)(y_2).\end{aligned}$$

Thus $F(f)\circ R_1 \leq R_2 \circ f$ and f is a morphism of the category of F-coalgebras, whereby the category **FAS** is a category of F-coalgebras.

Now, we introduce the following.

Definition 3.2. *Upper approximation set $\overline{R}_X$ of an L-fuzzy approximation space (X,R) is defined as*

$$\overline{R}_X = \{\overline{R}(A) : A \in L^X\} \subseteq L^X.$$

Let (X_1,R_1) and (X_1,R_2) be two L-fuzzy approximation spaces and $\phi : X_1 \to X_2$ be their relation preserving map. Again, let $\overline{R_1}_{X_1}$ and $\overline{R_2}_{X_2}$ be corresponding upper approximation sets of (X_1,R_1) and (X_2,R_2) respectively. Then the morphism $\phi^{\uparrow} : \overline{R_1}_{X_1} \to \overline{R_2}_{X_2}$ is defined as $\phi^{\uparrow}(\overline{R}_1(A_1) = \overline{R}_2(\phi_z^{\to}(A_1))$, $\forall A_1 \in L^{X_1}$.

The class of all upper approximation sets alongwith their morphism form a category, say, **UAS**. Next are to introduce a functor G from the category **FAS** to the category **UAS** having left and right adjoint.

Proposition 3.2. *Let $G : \mathbf{FAS} \to \mathbf{UAS}$ be a map such that for all $(X,R) \in obj(\mathbf{FAS})$, $G(X,R) = \overline{R}_X$ and for every **FAS**-morphism $\phi : (X_1,R_1) \to (X_2,R_2)$, $G(\phi) = \phi^{\uparrow}$. Then G is a functor.*

Theorem 3.2. *The functor $G : \mathbf{FAS} \to \mathbf{UAS}$ has a left adjoint.*

Proof: To prove that G has left adjoint, it is sufficient to show that for each **UAS**-object $\overline{R}_X$ corresponding to an L-fuzzy approximation space (X,R), there exist a pair (M,τ), where M is **FAS**-object and $\tau : \overline{R}_X \to G(M)$ is an **UAS**-morphism alongwith for all **FAS**-object M' and **UAS**-morphism $g : \overline{R}_X \to G(M')$, there exists a unique **FAS**-morphism $f : M \to M'$ such that the diagram in Figure 4 commutes.

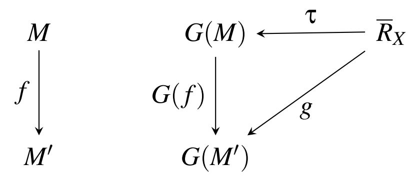

Figure 4: Diagram for Theorem 3.2.

Now, let $M = (X,R)$ be an **FAS**-object and $\tau : \overline{R}_X \to G(M) = \overline{R}_X$ be a map such that for all $\overline{R}(A) \in \overline{R}_X$, $\tau(\overline{R}(A)) = \overline{R}(A)$. To show that τ is an **UAS**-morphism, let us construct an identity map $I : (X,R) \to (X,R)$, then $\tau = G(I)$ is an identity morphism of **UAS**. Let $M' = (X',R')$ be an **FAS**-object and $g : \overline{R}_X \to G(M') = \overline{R'}_{X'}$. Then there exists an **FAS**-morphism $f' : (X,R) \to (X',R')$ such that for $\overline{R}(A) \in \overline{R}_X$, $g(\overline{R}(A)) = \overline{R'}(f'^{\to}_Z(A))$. Next, we define a map $f : M \to M'$ such that $f = f'$. Then for $\overline{R}(A) \in \overline{R}_X$,

$$\begin{aligned}(G(f)\circ\tau)(\overline{R}(A)) &= G(f)(\tau\overline{R}(A))\\ &= G(f)(\overline{R}(A))\\ &= G(f')(\overline{R}(A))\\ &= \overline{R'}(f'^{\to}_Z(A))\\ &= g(\overline{R}(A)).\end{aligned}$$

Thus $G(f)\circ\tau = g$. Further, the uniqueness of f is trivial. Hence G has a left adjoint.

Theorem 3.3. *The functor $G : \mathbf{FAS} \to \mathbf{UAS}$ has a right adjoint.*

Proof: To prove that G has a right adjoint, it is sufficient to show that for each **UAS**-object $\overline{R}_X$ corresponding to an L-fuzzy approximation space (X,R), there exists a pair (M,η), where M is an **FAS**-object and $\eta : G(M) \to \overline{R}_X$ is an **UAS**-morphism alongwith for all **FAS**-object M' and **UAS**-morphism $g : G(M') \to \overline{R}_X$, there exists an **FAS**-morphism $f : M' \to M$ such that the diagram in Figure 5 commutes.

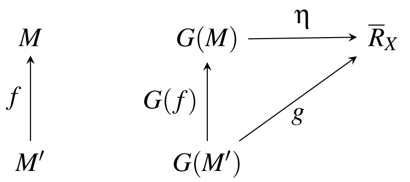

Figure 5: Diagram for Theorem 3.3.

Let $M = (X,R)$ be an **FAS**-object and $\eta : G(M) = \overline{R}_X \to \overline{R}_X$ be a map such that for all $\overline{R}(A) \in \overline{R}_X$, $\eta(\overline{R}(A)) = \overline{R}(A)$. To show that η is an **UAS**-morphism, let us construct an identity map $I : (X,R) \to (X,R)$, then $\eta = G(I)$ is an identity morphism of **UAS**. Let $M' = (X',R')$ be an **FAS**-object and $g : G(M') = \overline{R'}_{X'} \to \overline{R}_X$. Then there exists an **FAS**-morphism $f' : (X',R') \to (X,R)$ such that for $\overline{R'}(A') \in \overline{R'}_{X'}$, $g(\overline{R'}(A')) = \overline{R}(f'^{\to}_Z(A'))$. Next, we define a map $f : M' \to M$ such that $f = f'$. Then for $\overline{R'}(A') \in \overline{R'}_{X'}$,

$$\begin{aligned}(\eta \circ G(f))(\overline{R'}(A')) &= \eta(G(f)(\overline{R'}(A'))) \\ &= \eta(\overline{R}(f'^{\to}_Z(A'))) \\ &= \overline{R}(f'^{\to}_Z(A')) \\ &= g(\overline{R'}(A')).\end{aligned}$$

Thus $\eta \circ G(f) = g$. Further, the uniqueness of f is trivial. Hence G has a right adjoint.

Let $\overline{R}_X$ be an upper approximation set and $S : \overline{R}_X \times \overline{R}_X \to L$ be the L-fuzzy relation on $\overline{R}_X$. Then the pair $(\overline{R}_X, S)$ is called upper approximation space, where $S(\overline{R}(A), \overline{R}(B)) = \vee_{x \in X}\{\overline{R}(A)(x) \otimes \overline{R}(B)(x)\}$, for $\overline{R}(A), \overline{R}(B) \in \overline{R}_X$. Now, we have the following.

Proposition 3.3. *Let (X_1,R_1) and (X_2,R_2) be two L-fuzzy approximation spaces and $\phi : X_1 \to X_2$ be their relation preserving map. Then $(\overline{R_1}_{X_1}, S_1)$ and $(\overline{R_2}_{X_2}, S_2)$ are the object of **FAS**. Again let $\phi^{\uparrow} : (\overline{R_1}_{X_1}, S_1) \to (\overline{R_2}_{X_2}, S_2)$ be a morphism such that $\phi^{\uparrow}(\overline{R}_1(A_1)) = \overline{R}_2(\phi^{\to}_Z(A_1))$. Then $\phi^{\uparrow}$ is a relation preserving map, i.e., $\phi^{\uparrow} \in$ **FAS**-morphism.*

Proof: By the definition, $\overline{R}_X$ is a set and S is an L-fuzzy relation on $\overline{R}_X$. Thus $(\overline{R}_X, S)$ is an L-fuzzy approximation space. Now $\overline{R}_1(A_1), \overline{R}_1(B_1) \in \overline{R}_X$,

$$\begin{aligned}S_2(\phi^{\uparrow}(\overline{R}_1(A_1)), \phi^{\uparrow}(\overline{R}_1(B_1))) &= \vee_{x_2 \in X_2}\phi^{\uparrow}\overline{R}_1(A_1)(x_2) \otimes \phi^{\uparrow}(\overline{R}_1(B_1)(x_2) \\ &= \vee_{x_2 \in X_2}\{\phi^{\uparrow}\overline{R}_1(A_1)(x_2) \otimes \phi^{\uparrow}(\overline{R}_1(B_1)(x_2)\} \\ &= \vee_{x_2 \in X_2}\overline{R}_2(\phi^{\to}_z(A_1))(x_2) \otimes \overline{R}_2(\phi^{\to}_z(B_1))(x_2) \\ &\geq \vee_{x_1 \in X_1}\overline{R}_1(A_1)(x_1) \otimes \overline{R}_1(B_1)(x_1) \\ &\geq S_1(\overline{R}_1(A_1), \overline{R}_1(B_1)).\end{aligned}$$

Thus $\phi^{\uparrow}$ is a relation preserving map. This completes the proof.

The following is towards to a category of J-coalgebras based on the category **FAS**. Let J : **FAS** $\to$ **FAS** be a map such that $J(X,R) = (\overline{R}_X, S)$ and $J(\phi) = \phi^{\uparrow}$. Then J is also a functor. Before stating the next theorem, let define a morphism $\alpha^R_X : (X,R) \to (\overline{R}_X, S)$, such that $\alpha^R_X : X \to \overline{R}_X$ is a map and $\alpha^R_X(x) = \overline{R}(1_x)$.

Theorem 3.4. *Let J : **FAS** $\to$ **FAS** be a functor, $(X,R) \in obj($**FAS**$)$, and ϕ be an **FAS**-morphism. Then the set of the pairs $((X,R), \alpha^R_X)$ with their morphisms ϕ form a category of J-coalgebras.*

Proof: Let $(X_1,R_1), (X_2,R_2) \in obj($**FAS**$)$ and $\phi : (X_1,R_1) \to (X_2,R_2) \in$ **FAS**-morphism such that $J(X_1,R_1) = (\overline{R_1}_{X_1}, S_1)$, $J(X_2,R_2) = (\overline{R_2}_{X_2}, S_2)$ and $J(\phi) = \phi^{\uparrow}$. Then by the definition, it is clear that the pairs $((X_1,R_1), \alpha^{R_1}_{X_1})$ and $((X_2,R_2), \alpha^{R_2}_{X_2})$ are J-coalgebras. Now, we show that ϕ is a morphism of the category of J-coalgebras, i.e., the diagram in Figure 6 commutes.

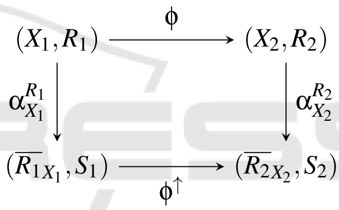

Figure 6: Diagram for Theorem 3.4.

For $x_1 \in X_1$ and $x_2 \in X_2$,

$$\begin{aligned}(J(\phi) \circ \alpha^{R_1}_{X_1}(x_1))(x_2) &= J(\phi)(\alpha^{R_1}_{X_1}(x_1)(x_2)) \\ &= \phi^{\uparrow} \circ \overline{R_1}(1_{x_1})(x_2) \\ &= \overline{R_2} \circ \phi^{\to}_z(1_{x_1})(x_2) \\ &= \overline{R_2}(1_{\phi(x_1)})(x_2) \\ &= \alpha^{R_2}_{X_2}(\phi(x_1))(x_2) \\ &= (\alpha^{R_2}_{X_2} \circ \phi)(x_1)(x_2).\end{aligned}$$

Thus $J(\phi) \circ \alpha^{R_1}_{X_1} = \alpha^{R_2}_{X_2} \circ \phi$. Hence ϕ is a morphism of the category of J-coalgebra.

We denote the above category of J-coalgebras as **J-Coal**. Before stating the next theorem, we recall the definition of topological category from (Močkoř, 2019).

Definition 3.3. *A concrete category **C** with a forgetful functor U : **C** $\to$ **SET** is a topological category if let a system of objects $A_i \in obj($**C**$)$ and $X \in$ **SET**. Then for any system of maps $g_i : X \to U(A_i)$, there exists an initial lift, which is to say*

(i) an object $A \in obj(\mathbf{C})$, such that $U(A) = X$,
(ii) a system of K-morphism $f_i : A \to A_i$, such that $U(f_i) = g_i$,
(iii) for each $B \in obj(\mathbf{C})$, a map $w : U(B) \to X$, a system of $\mathbf{C}$-morphism $t_i : B \to A_i$ such that $g_i \circ w = U(t_i)$, there exists the unique K-morphism $h : B \to A$ such that $U(h) = w$ and $f_i \circ h = t_i$.

Theorem 3.5. *The category **J-Coal** is a topological category.*

Proof: Let U : **J-Coal** $\to$ **SET** be a functor. For $((X,R),\alpha_X^R) \in obj -$ **J-Coal**, $U((X,R),\alpha_X^R) = X$. Again, let f be a morphism of **J-Coal**. Then $U(f) = f$. Now, let $((X_i,R_i),\alpha_{X_i}^{R_i})$ be a system of **J-Coal**-objects and $f_i : X \to X_i$ be maps, where $X \in$ **SET** is set. Then define an L-fuzzy relation $R : X \times X \to L$ such that for $x,y \in X$, $R(x,y) = \wedge_{i\in I} R_i(f_i(x), f_i(y))$ and $\alpha_X^R(x) = \overline{R}(1_x)$. Thus $f_i : ((X,R),\alpha_X^R) \to ((X_i,R_i),\alpha_{X_i}^{R_i})$ are the **J-Coal**-morphisms, where $U(f_i) = f_i$. Again, let $((Y,S),\eta)$ be an object of **J-Coal**. Then $U((Y,S),\eta) = Y$ and for a map $w : Y \to X$ and a morphism $t_i : ((Y,S),\eta) \to ((X_i,R_i),\alpha_{X_i}^{R_i})$, the diagram in Figure 7 commutes, i.e., $f_i \circ w = U(t_i) = t_i$. Now, we have to prove $h : ((Y,S),\eta) \to ((X,R),\alpha_X^R)$ is a **J-Coal**-morphism such that $U(h) = w$.

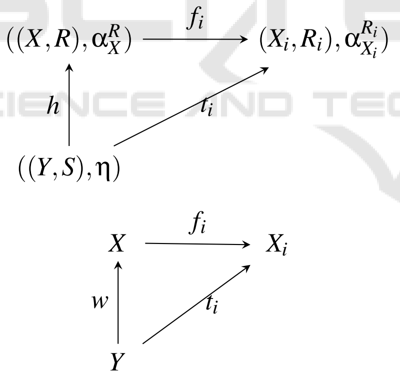

Figure 7: Diagram for Theorem 3.5.

$$\begin{aligned} F(h)\circ\eta(y)(x) &= F(h)\circ\overline{S}(1_y)(x) \\ &= \overline{R}(h_Z^{\rightarrow}(1_y)(x) \\ &= \overline{R}(1_{h(y)})(x) \\ &= \alpha_X^R(h(y)(x) \\ &= (\alpha_X^R \circ h)(y)(x). \end{aligned}$$

Thus $F(h)\circ\eta = \alpha_X^R \circ h$. Since $U(h) = w$ implies $h = w$. Thus h is unique morphism. This completes the proof.

Similar to the concept upper approximation space, we next introduce the concept of lower approximation space.

Definition 3.4. *Lower approximation space of an L-fuzzy approximation space (X,R) is defined as a pair $(\underline{R_X},T)$, where $\underline{R_X} = \{\underline{R}(A) : A \in L^X\} \subseteq L^X$ and $T : \underline{R_X} \times \underline{R_X} \to L$ is an L-fuzzy relation on $\underline{R_X}$, such that for $\underline{R}(A),\underline{R}(B) \in \underline{R_X}$,*

$$T(\underline{R}(A),\underline{R}(B)) = \wedge_{x\in X}\{\underline{R}(A)(x) \to \neg\underline{R}(B)(x)\}.$$

Now, we have the following.

Proposition 3.4. *Let L be a regular complete residuated lattice, (X_1,R_1) and (X_2,R_2) be two L-fuzzy approximation spaces, and $\phi : X_1 \to X_2$ be their relation preserving map. Then $(\underline{R_1}_{X_1},T_1)$ and $(\underline{R_2}_{X_2},T_2)$ are the object of **FAS**. Again let $\phi^{\downarrow} : (\underline{R_1}_{X_1},T_1) \to (\underline{R_2}_{X_2},T_2)$ be a morphism such that $\phi^{\downarrow}(\underline{R_1}(A_1)) = \underline{R_2}(\neg\phi_Z^{\rightarrow}(\neg A_1))$. Then $\phi^{\downarrow}$ is a relation preserving map, i.e., $\phi^{\downarrow} \in$ **FAS**-morphism.*

Proof: By the definition, $\underline{R_X}$ is a set and T is an L-fuzzy relation on $\underline{R_X}$. Thus $(\underline{R_X},T)$ is an L-fuzzy approximation space. Now $\underline{R_1}(A_1),\underline{R_1}(B_1) \in \underline{R_X}$,

$$\begin{aligned} T_2(\phi^{\downarrow}(\underline{R_1}(A_1)), \quad &= \wedge_{x_2\in X_2}\{\phi^{\downarrow}\underline{R_1}(A_1) \\ \phi^{\downarrow}(\underline{R_1}(B_1))) \quad &\quad (x_2) \to \neg\phi^{\downarrow}(\underline{R_1}(B_1)(x_2))\} \\ &= \wedge_{x_2\in X_2}\{\phi^{\downarrow}\underline{R_1}(A_1)(x_2) \\ &\quad \to \neg\phi^{\downarrow}(\underline{R_1}(B_1)(x_2)\} \\ &= \wedge_{x_2\in X_2}\{\underline{R_2}(\neg\phi_z^{\rightarrow}(\neg A_1)(x_2) \to \\ &\quad \neg\underline{R_2}(\neg\phi_z^{\rightarrow}(\neg B_1)(x_2)\} \\ &\geq \wedge_{x_1\in X_1}\{\underline{R_1}(A_1)(x_1) \to \\ &\quad \neg\underline{R_2}(\neg\phi_z^{\rightarrow}(\neg B_1)(x_2) \\ &\geq \wedge_{x_1\in X_1}\{\underline{R_1}(A_1)(x_1) \to \\ &\quad \neg\underline{R_1}(B_1)(x_1)\} \\ &\geq T_1(\underline{R_1}(A_1),\underline{R_1}(B_1)). \end{aligned}$$

Thus $\phi^{\downarrow}$ is a relation preserving map. This completes the proof.

Now, we introduce a functor K from the category **FAS** to the category **FAS**.

Proposition 3.5. *Let K : **FAS** $\to$ **FAS** be a map such that for all $(X,R) \in obj($**FAS**$)$, $K(X,R) = (\underline{R}_X,T)$ and for every **FAS**-morphism $\phi : (X_1,R_1) \to (X_2,R_2)$, $K(\phi) : (\underline{R_1}_{X_1},T_1) \to (\underline{R_2}_{X_2},T_2)$ is a map such that $K(\phi) = \phi^{\downarrow}$. Then K is a functor.*

Theorem 3.6. *Let L be a complete regular residuated lattice, J,K : **FAS** $\to$ **FAS** be two functors such that for all $(X,R) \in obj($**FAS**$)$, and $\phi \in$ **FAS**-morphism such that $J(X,R) = (\overline{R}_X,S)$, $K(X,R) = (\underline{R}_X,T)$, $J(\phi) = \phi^{\uparrow}$, and $K(\phi) = \phi^{\downarrow}$. Again, let γ :*

$J(X,R) \to K(X,R)$ be a map such that $\gamma(\overline{R}(A)) = \underline{R}(\neg A)$, for all $\overline{R}(A) \in \overline{R}_X$. Then the set of all pair $((X,R),\gamma)$ with their morphisms ϕ form a category of (J,K)-dialgebras.

Proof: Let J,K : **FAS** $\to$ **FAS** be two functors. Now, for all $(X,R) \in$ **FAS**, $J(X,R) = (\overline{R}_X, S)$, $K(X,R) = (\underline{R}_X, T)$ and $\gamma : J(X,R) \to K(X,R)$ is a map such that $\gamma(\overline{R}(A)) = \underline{R}(\neg A)$. Also, by the definition of dialgebra, the pair $((X,R),\gamma)$ is an (J,K)-dialgebra. Again, let $\phi : (X_1,R_1) \to (X_2,R_2)$ be a morphism of the category **FAS**. Then we show that ϕ is a morphism of the category of (J,K)-dialgebras, i.e., the diagram in Figure 8 commutes.

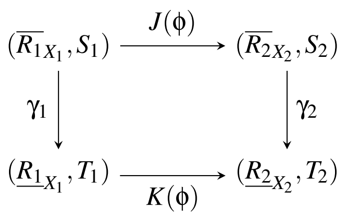

Figure 8: Diagram for Theorem 3.6.

$$\begin{aligned} K(\phi)\circ\gamma_1(\overline{R_1}(A_1)) &= K(\phi)(\underline{R_1}(\neg A_1)) \\ &= \underline{R_2}(\neg\phi_Z^{\to}(\neg\neg A_1)) \\ &= \underline{R_2}(\neg\phi_Z^{\to}(A_1)) \\ &= \gamma_2(\overline{R_2}(\phi_Z^{\to}(A_1)) \\ &= \gamma_2 J(\phi)(\overline{R_1}(A_1)). \end{aligned}$$

Thus $K(\phi)\circ\gamma_{X_1} = \gamma_{X_2}\circ J(\phi)$. Hence ϕ is a homomorphism of the category of (J,K)-dialgebras.

We shall denote this category by **(J,K)-Dial**.

Theorem 3.7. *The category **(J,K)-Dial** is a topological category.*

Proof: Let V :**(J,K)-Dial**$\to$ **SET** be a functor. Then for $((X,R),\gamma) \in obj -$ **(J,K)-Dial**, $V((X,R),\gamma) = X$. Again, let f be a morphism of **(J,K)-Dial**. Then $V(f) = f$. Now, let $((X_i,R_i),\gamma_i)$ be a system of **(J,K)-Dial**-objects and $f_i : X \to X_i$ be maps, where $X \in$ **SET** is set. Then define an L-fuzzy relation $R : X\times X \to L$ such that for $x,y \in X$, $R(x,y) = \wedge_{i\in I} R_i(f_i(x), f_i(y))$ and $\gamma(\overline{R}(A)) = \underline{R}(\neg A)$, for all $\overline{R}(A) \in J(X,R)$. Since f_i are relation preserving map then $f_i : ((X,R),\gamma) \to (X_i,R_i),\gamma_i)$ are the **(J,K)-Dial**-morphisms, where $V(f_i) = f_i$. Now, let $((Y,S),\delta)$ be an object of **(J,K)-Dial**. Then $V((Y,S),\delta) = Y$ and there exist a map $w : Y \to X$ and a morphism $t_i : ((Y,S),\eta) \to ((X_i,R_i),\gamma_i)$ such that the diagram in Figure 9 commutes, i.e., $f_i \circ w = V(t_i) = t_i$. Now, we prove $h : ((Y,S),\eta) \to ((X,R),\gamma)$ is a **(J,K)-Dial**-morphism such that $V(h) = w$ implies $h = w$. Firstly, we show that $h : (Y,S) \to (X,R)$ is a relation preserving map.

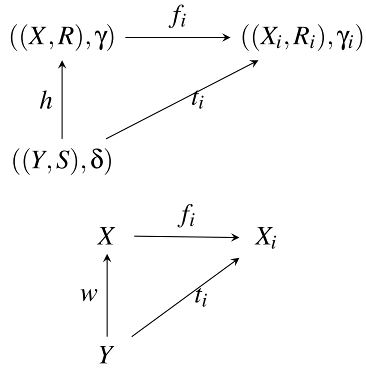

Figure 9: Diagram for Theorem 3.7.

For $y_1, y_2 \in Y$,

$$\begin{aligned} R(h(y_1),h(y_2)) &= \wedge_{i\in I} R_i(f_i(h(y_1)), f_i(h(y_2))) \\ &= \wedge_{i\in I} R_i(f_i(w(y_1)), f_i(w(y_2))) \\ &= \wedge_{i\in I} R_i(t_i(y_1), t_i(y_2)) \\ &\geq \wedge_{i\in I} S(y_1,y_2) \\ &\geq S(y_1,y_2). \end{aligned}$$

Thus h is a relation preserving map. Now, for $B \in L^Y$,

$$\begin{aligned} K(h)\circ\delta(\overline{S}(B)) &= K(h)(\underline{S}(\neg B)) \\ &= \underline{R}(\neg h_Z^{\to}(\neg\neg B)) \\ &= \underline{R}(\neg h_Z^{\to}((B)) \\ &= \gamma(\overline{R}(h_Z^{\to}((B)) \\ &= \gamma(J(h)(\overline{S}(B))). \end{aligned}$$

Thus $K(h)\circ\delta = \gamma\circ J(h)$. Also, $U(h) = w$ implies $h = w$, whereby h is unique morphism. This completes the proof.

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4 POWERSETS VS DIALGEBRAS

In this section, the relationship between powerset theories and the category of dialgebras is established. We recall the following definition of powerset theory from (Močkoř, 2018).

Definition 4.1. *Let **C** be a category with a forgetful functor $|.| : \boldsymbol{C} \to \boldsymbol{SET}$. Then $(T,\to,\eta)$ is called a **CSLAT**-powerset theory in **C**, if*
(1) $T : obj(\boldsymbol{C}) \to obj(\boldsymbol{CSLAT})$ is a map.
*(2) For each **C**-morphism $f : X \to Y$, there exist $f_T{}^{\to}$: $T(X) \to T(Y)$ in **CSLAT**.*

(3) η *determines for each* $X \in \boldsymbol{C}$ *a mapping* $\eta_X : |X| \to T(X)$ *in* ***SET****.*
(4) For each $f : X \to Y$ *in* $\boldsymbol{C}$, $f_T{}^{\to} \circ \eta_X = \eta_Y \circ |f|$.

In general T may not be a functor. If T is a functor then it is called a functor type powerset theory.

Definition 4.2. *A structure* $(T, \to, \eta)$ *is called* $L^{\vee}$*-powerset in a category* $\boldsymbol{C}$*, if*
(1) $(T, \to, \eta)$ *is a* ***CSLAT***$(\vee)$*-powerset theory in the category* $\boldsymbol{C}$*.*
(2) For each $X \in obj(\boldsymbol{C})$*,*
(a) there exist $\vee$*-preserving embedding* $i_X : T(X) \hookrightarrow L^X$*,*
(b) for each $x \in |X|$*,* $core(i_X(\eta_X)(x)) \neq \emptyset$*,*
(c) there exist an external operation $\star : L \times T(X) \to T(X)$*, for* $a \in L$ *and* $A \in T(X)$*,* $i_X(a \star A) = a \otimes i_X(A)$ *and* $f_T^{\to}(a \star A) = a \star f_T^{\to}(A)$*.*

Definition 4.3. *Let* (X, R) *be an L-fuzzy approximation space and* R *be an L-fuzzy equivalence relation. Then the L-fuzzy approximation space is called L-fuzzy equivalence approximation space.*

L-fuzzy equivalence approximation spaces with their relation preserving maps form a category, say, **EFAS**.

Example 4.1. *Let* $(X_1, R_1), (X_2, R_2) \in obj(\boldsymbol{EFAS})$ *and* $T(X_i, R_i) = \overline{R}_{iX_i}$*,* $i = 1, 2$*. Then* $f_T{}^{\to} : T(X_1, R_1) \to T(X_2, R_2)$ *such that for all* $\overline{R_1}(A) \in \overline{R_1}_{X_1}$*,* $f_T{}^{\to}(\overline{R_1}(A_1)) = \overline{R_2}(f_Z{}^{\to}(A_1))$*. Also, for each* $(X, R) \in \boldsymbol{EFAS}$*,* $\eta_X : X \to \overline{R_X}$ *such that* $\eta_X(x) = \overline{R}(1_x)$ *and* $i_X : \overline{R}_X \to L^X$ *is* $\vee$*-preserving map such that* $i_X(\overline{R}(A)) = \overline{R}(A) \in L^X$*. Since* R *is reflexive then* $core(i_X(\overline{R}(1_x))) \neq \emptyset$*. For* $x \in X_1$ *and* $y' \in X_2$*,*

$$
\begin{aligned}
f_T^{\to} \circ \eta_{X_1}(x)(y') &= f_T^{\to}(\overline{R_1}(1_x))(y') \\
&= \overline{R_2}(f_Z^{\to}(1_x))(y') \\
&= \vee_{z' \in X_2}\{R_2(y', z') \otimes f_Z^{\to}(1_x)(z')\} \\
&= \vee_{z' \in X_2}\{R_2(y', z') \otimes (1_{f(x)})(z')\} \\
&= \overline{R_2}(1_{f(x)})(y') \\
&= \eta_{X_2}(f(x))(y') \\
&= (\eta_{X_2} \circ f)(x)(y').
\end{aligned}
$$

Thus $f^{\to}{}_T \circ \eta_{X_1} = \eta_{X_2} \circ f$*.*
Again, for $a \in L$ *and* $\overline{R}(A) \in \overline{R}_X$*, we define external operation* $a \star \overline{R}(A) = a \otimes \overline{R}(A)$*. Then* $i_{X_1}(a \star \overline{R_1}(A_1)) = i_X(a \otimes \overline{R_1}(A_1)) = a \otimes \overline{R_1}(A_1) = \overline{R_1}(a \otimes A_1) \in T(X_1)$*. Also,* $f_T^{\to}(a \otimes \overline{R_1}(A_1)) = f_T^{\to}(\overline{R_1}(a \otimes A_1)) = \overline{R_2}(f_Z{}^{\to}(a \otimes A_1)) = a \otimes \overline{R_2}(f_Z{}^{\to}(A_1)) = a \otimes f_T^{\to}(\overline{R_1}(A_1))$*. Hence* $(T, \to, \eta)$ *is* $L^{\vee}$*-powerset theory of* ***EFAS****.*

Definition 4.4. *The structure* $(P, \to, \zeta)$ *is called* $L^{\wedge}$*-powerset theory in the category* $\boldsymbol{C}$*, if*
(1) $(P, \to, \zeta)$ *is a* ***CSLAT***$(\wedge)$*-powerset theory in category* $\boldsymbol{C}$*.*
(2) For each $X \in \boldsymbol{C}$*,*
(a) there exist $\wedge$*-preserving embedding* $j_X : P(X) \hookrightarrow L^X$*,*
(b) for each $x \in |X|$*, such that* $core(\neg j_X(\zeta(x))) \neq \emptyset$*,*
(c) there exist an external operation $+ : L \times P(X) \to P(X)$*, for* $a \in L$ *and* $A \in P(X)$*,* $j_X(a + A) = \neg a \to j_X(A)$ *and* $f_P^{\to}(a + A) = a + f_P^{\to}(A)$*.*

Example 4.2. *Let* L *be a complete regular residuated lattice and* $(X_1, R_1), (X_2, R_2) \in obj(\boldsymbol{EFAS})$ *such that* $P(X_i, R_i) = \underline{R_i}_{X_i}$*, for* $i = 1, 2$*. Also,* $f_P{}^{\to} : P(X_1, R_1) \to P(X_2, R_2)$ *is defined as* $f_P{}^{\to}(\underline{R_1}(A_1)) = \underline{R_2}(\neg f_Z{}^{\to}(\neg A_1))$*. For each* $(X, R) \in \boldsymbol{EFAS}$*,* $\zeta_X : X \to P(X, R)$ *is defined as* $\zeta_X(x) = \underline{R}(\neg 1_x)$ *and* $j_X(\underline{R}(A)) = \underline{R}(A)$*. Then for* $x, y \in X$*,* $\neg j_X(\zeta_X(x))(y) = \neg \underline{R}(\neg 1_x)(y) = R(x, y)$*. Since* R *is reflexive relation then* $core(\neg j_X(\zeta_X(x)) \neq \emptyset$*. Now for* $x \in X_1$ *and* $y' \in X_2$*,*

$$
\begin{aligned}
f_P{}^{\to} \circ \zeta_{X_1}(x)(y') &= f_P{}^{\to}(\underline{R_1}(\neg 1_x))(y') \\
&= \underline{R_2}(\neg f_Z{}^{\to}(\neg\neg 1_x))(y') \\
&= \underline{R_2}(\neg f_Z{}^{\to}(1_x))(y') \\
&= \wedge_{z' \in X_2}\{R_2(y', z') \to \\
&\quad \neg f_Z{}^{\to}(1_x)(z')\} \\
&= \wedge_{z'}\{R_2(y', z') \to \\
&\quad \neg \vee_{z : f(z) = z'} (1_x)(z)\} \\
&= \wedge_{z'}\{R_2(y', z') \to \\
&\quad \wedge_{z : f(z) = z'}(\neg 1_x)(z)\} \\
&= \wedge_{z'} \wedge_{z : f(z) = z'} \{R_2(y', z') \to \\
&\quad (\neg 1_x)(z)\} \\
&= \wedge_{z'} R_2(y', z') \to (\neg 1_{f(x)})(z') \\
&= \underline{R_2}(\neg 1_{f(x)})(y') \\
&= \zeta_{X_2}(f(x))(y').
\end{aligned}
$$

Thus $f_P{}^{\to} \circ \zeta_{X_1} = \zeta_{X_2} \circ f$*.*
Now, for $a \in L$ *and* $\underline{R}(A) \in \underline{R}_X$*, we define external operation* $a + \underline{R}(A) = \neg a \to \underline{R}(A)$*. Then* $j_X(a + \underline{R}(A)) = j_X(\underline{R}(a + A)) = \underline{R}(\neg a \to A) = \neg a \to \underline{R}(A) = \neg a \to j_X(\underline{R}(A))$*. For* $\underline{R_1}(A_1) \in \underline{R_1}_{X_1}$*,* $x' \in X_2$*,*

$$
\begin{aligned}
f_P^{\to}(a + \underline{R_1}(A_1))(x') &= f_P^{\to}(\underline{R_1}(a + A_1))(x') \\
&= \underline{R_2}(\neg f_Z^{\to}(\neg(a + A_1)))(x') \\
&= \wedge_{y' \in X_2} R_2(x', y') \to \\
&\quad \neg f_Z^{\to}(\neg(a + A_1)))(y') \\
&= \wedge_{y' \in X_2}\{R_2(x', y') \to \\
&\quad \wedge_{f(y) = y'}(a + A_1)(y)\} \\
&= \wedge_{y' \in X_2}\{R_2(x', y') \to \\
&\quad (\neg a \to \wedge_{f(y) = y'}(A_1)(y))\} \\
&= \neg a \to \wedge_{y' \in X_2}\{R_2(x', y') \to \\
&\quad \neg f_Z^{\to}(\neg A_1)(y')\} \\
&= a + \underline{R_2}(\neg f_Z^{\to}(\neg A_1))(x') \\
&= a + f_P^{\to}(\underline{R_1}(A_1))(x').
\end{aligned}
$$

Thus $f_P^{\rightarrow}(a+\underline{R}(A)) = a + f_P^{\rightarrow}(\underline{R}(A))$. *Hence* $(P,\rightarrow,\zeta)$ *is* $L^{\wedge}$*-powerset theory.*

Let $(T,\rightarrow,\eta)$ be a $L^{\vee}$-powerset theory of a category **C**. Then define a map $\overline{T} : L^{|X|} \rightarrow T(X)$, such that $\overline{T}(A) = \vee_{x\in|X|}\{\eta_X(x) \star A(x)\}$. Again let $(P,\rightarrow,\zeta)$ be $L^{\wedge}$-powerset theory in **C**. Then define a map $\underline{P} : L^{|X|} \rightarrow T(X)$ such that $\underline{P} = \wedge_{x\in|X|}\{\neg\zeta_X(x) + A(x)\}$. Now, we have the following.

Theorem 4.1. *Let* $(T,\rightarrow,\eta)$ *be an* $L^{\vee}$*-powerset theory of* ***EFAS****. Then* $(T,\rightarrow,\eta)$ *is also* $L^{\wedge}$*-powerset theory of* ***EFAS*** *such that* $\overline{R} = \overline{T}$ *and* $\underline{R} = \underline{T}$*, provided* L *is a complete* MV*-algebra.*

Proof: Let $f : (X_1,R_1) \rightarrow (X_2,R_2)$ be a morphism in the category **EFAS**. We define T : **EFAS** $\rightarrow$ **CSLAT**$(\vee)$ such that $T(X_1,R_1) = L^{X_1}$ and $|(X_1,R_1)| = X_1$. Also, define $f_T^{\rightarrow} : T(X_1,R_1) \rightarrow T(X_2,R_2)$ such that $f_T^{\rightarrow} = f_Z^{\rightarrow}$. It is clear that $T(X,R)$ is a complete $\vee$-semilattice and $f_T^{\rightarrow}$ is $\vee$-preserving map. Next, we define $\eta_X : X \rightarrow T(X,R)$ such that for $x,y \in (X,R)$, $\eta_X(x)(y) = R(x,y)$. Then we show that the following diagram commute.

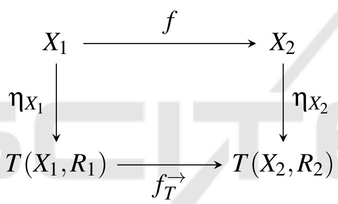

Figure 10: Diagram for Theorem 4.1.

$$\begin{aligned} f_T^{\rightarrow}(\eta_{X_1}(x))(y') &= f_Z^{\rightarrow}\eta_{X_1}(x)(y') \\ &= \vee_{y:f(y)=y'}\eta_{X_1}(x)(y) \\ &= \vee_{y:f(y)=y'}R_1(x,y) \\ &= \vee_{y:f(y)=y'}R_2(f(x),f(y)) \\ &= \vee_{y:f(y)=y'}R_2(y',f(x)) \\ &= R_2(y',f(x)) \\ &= \eta_{X_2}(f(x))(y). \end{aligned}$$

Thus $f_T^{\rightarrow} \circ \eta_{X_1} = \eta_{X_2} \circ f$. Hence $(T,\rightarrow,\eta)$ is **CSLAT**$(\vee)$ powerset theory. For $a \in L$, $x \in X$ and $A \in L^X$, $(a \star A)(x) = a \otimes A(x)$. For $A \in L^X, x \in X$, we have

$$\begin{aligned} \overline{T}(A)(x) &= \vee_{y\in X}(\eta_X(y) \star A(y))(x) \\ &= \vee_{y\in X}\eta_X(y)(x) \otimes A(y)(x) \\ &= \vee_{y\in X}R(x,y) \otimes A(y)(x) \\ &= \overline{R}(A)(x) \end{aligned}$$

Hence $\overline{T} = \overline{R}$.

Let L be the complete MV-algebra. For arbitrary morphism $f : (X_1,R_1) \rightarrow (X_2,R_2)$, $T(X,R) = L^X$ is also complete $\wedge$-semilattice. Since any complete MV-algebra is completely distributive and $f_T^{\rightarrow} = f_Z^{\rightarrow}$ is $\wedge$-preserving. Thus $(T,\rightarrow,\eta)$ can also consider as **CSLAT**$(\wedge)$ powerset theory. To prove T is also $L^{\wedge}$-powerset theory, we need to change only definition of the external operation $+$ as follows. For $A \in L^X$, $a \in L$ and $x \in X$, $(a+A)(x) = \neg a \rightarrow A(x)$. Then for $A \in L^X$ and $x \in X$,

$$\begin{aligned} \underline{T}(A)(x) &= (\wedge_{y\in X}\neg\eta_X(y) + A(y))(x) \\ &= \wedge_{y\in X}\{\neg\neg\eta_X(y)(x) \rightarrow A(y)\} \\ &= \wedge_{y\in X}\{\eta_X(y)(x) \rightarrow A(y)\} \\ &= \wedge_{y\in X}\{R(x,y) \rightarrow A(y)\} \\ &= \underline{R}(A)(x). \end{aligned}$$

Thus $\underline{T} = \underline{R}$. This completes the proof.

Proposition 4.1. *Let* $(T,\rightarrow,\eta)$ *be a* $L^{\vee}$*-powerset theory of a category* ***C****. Then for any* $A \in T(X)$ *can be written as* $A = \vee_{y\in X}\{i_X(A)(y) \star \eta_X(y)\}$*, provided* $i_X(\eta_X(x)) = 1_x$.

Proof: Let $A \in T(X)$. Then $i_X(A) \in L^X$. Thus $i_X(A)$ can be written as $i_X(A) = \vee_{x\in X}\{i_X(A)(x) \star 1_x\}$. Now,

$$\begin{aligned} i_X(A) &= \vee_{x\in X}\{i_X(A)(x) \star 1_x\} \\ &= \vee_{x\in X}\{i_X(A)(x) \star i_X(\eta_X(x))\} \\ &= \vee_{x\in X}i_X\{i_X(A)(x) \star \eta_X(x)\} \\ &= i_X\{\vee_{x\in X}\{i_X(A)(x) \star \eta_X(x)\}\}. \end{aligned}$$

Since i_X is one-one, whereby $A = \vee_{x\in X}\{i_X(A)(x) \star \eta_X(x)\}$.

Theorem 4.2. *Let* $(T_1,\rightarrow,\eta_1)$ *and* $(T_2,\rightarrow,\eta_2)$ *be two functor type* $L^{\vee}$ *and* $L_{\wedge}$*-powerset theories of a category* ***C****, respectively. Again, for each* $X \in obj(\boldsymbol{C})$, $\phi_X : T_1(X) \rightarrow T_2(X)$ *is a map such that* $\phi_X \circ \eta_{1X} = \eta_{2X}$*. Then the set of pair* (X,ϕ_X) *with their morphisms as the morphisms in* ***C*** *form a category of* (T_1,T_2)*-dialgebras, provided* $A = \vee_{x\in X}\{i_{1X}(A)(x) \star \eta_{1X}(x)\}$*, for all* $A \in T_1(X)$.

Proof: Let $(T_1,\rightarrow,\eta_1)$ and $(T_2,\rightarrow,\eta_2)$ be two functor type $L^{\vee}$ and $L_{\wedge}$-powerset theories of a category **C**, respectively. Then for a **C**-morphism $f : X \rightarrow Y$, $f_{T_1}^{\rightarrow} \circ \eta_{1X} = \eta_{1Y} \circ |f|$ and $f_{T_2}^{\rightarrow} \circ \eta_{2X} = \eta_{2Y} \circ |f|$. Now, T_1, T_2 : **C** $\rightarrow$ **CSLAT** are two functors such that $T_1(f) = f_{T_1}^{\rightarrow}$ and $T_2(f) = f_{T_2}^{\rightarrow}$. For each $X \in$ **C**, define a map $\phi_X : T_1(X) \rightarrow T_2(X)$ such that for all $A \in T_1(X)$, $\phi_X(A) = \phi_X(\vee_{x\in X}(i_{1X}(A)(x) \star \eta_{1_X}(x))) = \wedge_{x\in X}(\neg i_{1X}(A)(x) + \phi_X(\eta_{1_X}(x)))$. Then by the definition of dialgebra, the pair (X,ϕ_X) is an (T_1,T_2)-dialgebra. Now, we show that the map $f : X \rightarrow Y$ is an (T_1,T_2)-morphism, i.e., the diagram in Figure 11 commutes. Now, for all $A \in T_1(X)$,

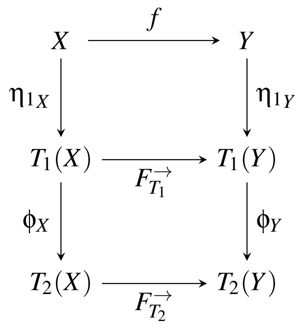

Figure 11: Diagram for Theorem 4.2.

$$\begin{aligned}
f_{T_2}^{\rightarrow}\circ\phi_X(A) &= f_{T_2}^{\rightarrow}\circ\phi_X(\vee_{x\in X}(i_{1X}(A)(x)\star\eta_{1_X}(x)))\\
&= f_{T_2}^{\rightarrow}(\wedge_{x\in X}(\neg i_{1_X}(A)(x)+\phi_X(\eta_{1_X})(x)))\\
&= f_{T_2}^{\rightarrow}(\wedge_{x\in X}(\neg i_{1_X}(A)(x)+\eta_{2_X}(x)))\\
&= \wedge_{x\in X}\{\neg i_{1_X}(A)(x)+f_{T_2}^{\rightarrow}(\eta_{2_X}(x))\}\\
&= \wedge_{x\in X}\{\neg i_{1_X}(A)(y)+(\eta_{2_Y}\circ|f|)(x)\}\\
&= \wedge_{x\in X}\{\neg i_{1_X}(A)(x)+\phi_Y\circ\eta_{1_Y}\circ|f|(x)\}\\
&= \phi_Y\{\vee_{x\in X}(i_{1_X}(A)(x)\star f_{T_1}^{\rightarrow}\circ\eta_{1_X}(x))\}\\
&= \phi_Y\circ f_{T_1}^{\rightarrow}\{\vee_{x\in X}(i_{1_X}(A)(x)\star\eta_{1_X}(x))\}\\
&= \phi_Y\circ f_{T_1}^{\rightarrow}(A).
\end{aligned}$$

Thus $f_{T_2}^{\rightarrow}\circ\phi_{X_1}=\phi_Y\circ f_{T_1}^{\rightarrow}$. Hence f is a morphism of the category of (T_1,T_2)-Dialgebras.

Example 4.3. *Let $(T,\rightarrow,\eta)$ and $(P,\rightarrow,\zeta)$ be the $L^{\vee}$, $L^{\wedge}$-powerset theories of **EFAS** as defined in Example 4.1 and Example 4.2, respectively. Then T,P : **EFAS** $\rightarrow$ **CSLAT** are functor type powerset theories. Since R is an equivalence L-fuzzy relation, $\overline{R}(A)\in\overline{R}_X$ can be written as $\overline{R}(A)=\vee_{x\in X}(i_X(\overline{R}(A))(x)\otimes\eta_X(x))$. For $(X,R)\in$ **EFAS**, $\phi_X:\overline{R}_X\rightarrow\underline{R}_X$ is defined as follows.*

$$\begin{aligned}
\phi_X(\overline{R}(A)) &= \phi_X\{\vee_{x\in X}(i_X(\overline{R}(A))(x)\otimes\eta_X(x))\}\\
&= \wedge_{x\in X}(\neg i_X(\overline{R}(A))(x)+\phi_X(\eta_X(x)))\\
&= \wedge_{x\in X}(\neg\overline{R}(A)(x)+\zeta_X(x))\\
&= \wedge_{x\in X}(\underline{R}(\neg A)(x)+\underline{R}(\neg 1_x)(y))\\
&= \wedge_{x\in X}(\neg\underline{R}(\neg A)(x)\rightarrow\neg R(x,y))\\
&= \wedge_{x\in X}(R(x,y)\rightarrow\underline{R}(\neg A)(x))\\
&= \underline{R}(\underline{R}(\neg A))(x)\\
&= \underline{R}(\neg A)(x).
\end{aligned}$$

*Thus the pair $((X,R),\phi_X)$ is a (T,P)-dialgebra. Now, let $f:(X_1,R_1)\rightarrow(X_2,R_2)$ be a **EFAS**-morphism. Then for $\overline{R_1}(A_1)\in\overline{R_1}_{X_1}$,*

$$\begin{aligned}
f_P^{\rightarrow}(\phi_{X_1}(\overline{R_1}(A_1))) &= f_P^{\rightarrow}(\underline{R_1}(\neg A_1))\\
&= \underline{R_2}(\neg f_Z^{\rightarrow}(\neg\neg A_1))\\
&= \underline{R_2}(\neg f_Z^{\rightarrow}(A_1))\\
&= \phi_{X_2}(\overline{R_2}(f_Z^{\rightarrow}(A_1)))\\
&= \phi_{X_2}(f_T^{\rightarrow}(\overline{R_1}(A_1))).
\end{aligned}$$

Thus $f_P^{\rightarrow}\circ\phi_{X_1}=\phi_{X_2}\circ f_T^{\rightarrow}$. So, f is a morphism of the category of (T,P)-dialgebras.

5 CONCLUSIONS

In this paper, we have studied L-fuzzy approximation space from the categorical point of view. It is shown that the category **FAS** is the category of F-coalgebras. Also, we have introduced two categories **J-Coal** and **(J,K)-Dial**, which are topological categories. In fuzzy set theory and applications, fuzzy approximation operators and powerset theories in fuzzy structures are extensively used concepts. Despite the fact that these concepts appear to be independent in terms of methodology, there are significant connections between them. Interestingly, it is shown that there is a special type of powerset theories in which maps defined by these powerset theories are fuzzy approximation operators. Furthermore, we have established a bijective correspondence between powerset theories and the category of dialgebras. Because of the close link of L-fuzzy approximation operators with powerset theories, in the future, it will be interesting to derive or discuss the results in fuzzy rough set theory via the properties of powerset theories.

REFERENCES

Adámek, J. (1976). Limits and colimits in generalized algebraic categories. *Czechoslovak Mathematical Journal*, 26(1):55–64.

Barr, M. and Wells, C. (1990). *Category theory for computing science*, volume 1. Prentice Hall New York.

Belohlavek, R. (2012). *Fuzzy relational systems: foundations and principles*, volume 20. Springer Science & Business Media.

Blount, K. and Tsinakis, C. (2003). The structure of residuated lattices. *International Journal of Algebra and Computation*, 13(04):437–461.

Ćirić, M., Ignjatović, J., and Bogdanović, S. (2007). Fuzzy equivalence relations and their equivalence classes. *Fuzzy Sets and Systems*, 158(12):1295–1313.

D'eer, L., Verbiest, N., Cornelis, C., and Godo, L. (2015). A comprehensive study of implicator–conjunctor-based and noise-tolerant fuzzy rough sets: definitions, properties and robustness analysis. *Fuzzy Sets and Systems*, 275:1–38.

Dubois, D. and Prade, H. (1990). Rough fuzzy sets and fuzzy rough sets. *International Journal of General System*, 17(2-3):191–209.

Eilenberg, S. and MacLane, S. (1945). General theory of natural equivalences. *Transactions of the American Mathematical Society*, 58:231–294.

Freyd, P. J. (1964). *Abelian categories*, volume 1964. Harper & Row New York.

Goguen, J. A. (1967). L-fuzzy sets. *Journal of mathematical analysis and applications*, 18(1):145–174.

Hagino, T. (1987). *Categorical programmer language*. PhD thesis, Dissertation]. Edinburgh: University of Edinburgh.

Kondo, M. (2006). On the structure of generalized rough sets. *Information Sciences*, 176(5):589–600.

Lawvere, F. W. (1963). Functorial semantics of algebraic theories. *Proceedings of the National Academy of Sciences*, 50(5):869–872.

Lawvere, F. W. (1966). The category of categories as a foundation for mathematics. In *Proceedings of the Conference on Categorical Algebra: La Jolla 1965*, pages 1–20. Springer.

Mac Lane, S. (2013). *Categories for the working mathematician*, volume 5. Springer Science & Business Media.

Mahato, S. and Tiwari, S. (2020). On bijective correspondence between fuzzy reflexive approximation spaces and fuzzy transformation systems. *New Mathematics and Natural Computation*, 16(02):291–304.

Mi, J.-S., Leung, Y., Zhao, H.-Y., and Feng, T. (2008). Generalized fuzzy rough sets determined by a triangular norm. *Information Sciences*, 178(16):3203–3213.

Močkoř, J. (2018). Any f-transform is defined by a powerset theory. In *International Workshop on Fuzzy Logic and Applications*, pages 55–66. Springer.

Močkoř, J. (2019). F-transforms and semimodule homomorphisms. *Soft Computing*, 23(17):7603–7619.

Močkoř, J. (2020). Powerset theory of fuzzy soft sets. *International Journal of Fuzzy Logic and Intelligent Systems*, 20(4):298–315.

Močkoř, J. (2023). Elements of relational power set theories for semiring-valued fuzzy structures. In Massanet, S., Montes, S., Ruiz-Aguilera, D., and González-Hidalgo, M., editors, *Fuzzy Logic and Technology, and Aggregation Operators*, pages 61–73, Cham. Springer Nature Switzerland.

Morsi, N. N. and Yakout, M. M. (1998). Axiomatics for fuzzy rough sets. *Fuzzy sets and Systems*, 100(1-3):327–342.

Pawlak, Z. (1982). Rough sets. *International journal of computer & information sciences*, 11:341–356.

Pei, D. (2004). Fuzzy logic algebras on residuated lattices. *Southeast Asian Bulletin of Mathematics*, 28(3).

Perfilieva, I., Singh, A. P., and Tiwari, S. (2017). On the relationship among f-transform, fuzzy rough set and fuzzy topology. *Soft Computing*, 21(13):3513–3523.

Poll, E. and Zwanenburg, J. (2001). From algebras and coalgebras to dialgebras. *Electronic Notes in Theoretical Computer Science*, 44(1):289–307.

Qin, K., Yang, J., and Pei, Z. (2008). Generalized rough sets based on reflexive and transitive relations. *Information Sciences*, 178(21):4138–4141.

Radzikowska, A. M. and Kerre, E. E. (2002). A comparative study of fuzzy rough sets. *Fuzzy sets and systems*, 126(2):137–155.

Radzikowska, A. M. and Kerre, E. E. (2005). Fuzzy rough sets based on residuated lattices. In *Transactions on Rough Sets II: Rough Sets and Fuzzy Sets*, pages 278–296. Springer.

Rodabaugh, S. (1999a). Powerset operator foundations for poslat fuzzy set theories and topologies. *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory*, pages 91–116.

Rodabaugh, S. (1999b). Powerset operator foundations for poslat fuzzy set theories and topologies. *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory*, pages 91–116.

Rodabaugh, S. . et al. (2007). Relationship of algebraic theories to powerset theories and fuzzy topological theories for lattice-valued mathematics. *International Journal of Mathematics and Mathematical Sciences*, 2007.

She, Y.-H. and Wang, G.-J. (2009). An axiomatic approach of fuzzy rough sets based on residuated lattices. *Computers & Mathematics with Applications*, 58(1):189–201.

Srivastava, A. K. and Tiwari, S. (2003). On relationships among fuzzy approximation operators, fuzzy topology, and fuzzy automata. *Fuzzy sets and systems*, 138(1):197–204.

Tiwari, S., Sharan, S., and Yadav, V. K. (2014). Fuzzy closure spaces vs. fuzzy rough sets. *Fuzzy information and engineering*, 6(1):93–100.

Tiwari, S. and Srivastava, A. K. (2013). Fuzzy rough sets, fuzzy preorders and fuzzy topologies. *Fuzzy sets and systems*, 210:63–68.

Tiwari, S., Yadav, V. K., Davvaz, B., et al. (2018). A categorical approach to minimal realization for a fuzzy language. *Fuzzy Sets and Systems*, 351:122–137.

Trnková, V. and Goralčík, P. (1969). On products in generalized algebraic categories. *Commentationes Mathematicae Universitatis Carolinae*, 10(1):49–89.

Wang, C. Y. and Hu, B. Q. (2015). Granular variable precision fuzzy rough sets with general fuzzy relations. *Fuzzy Sets and Systems*, 275:39–57.

Wang, Z. (2023). Fundamental properties of fuzzy rough sets based on triangular norms and fuzzy implications: the properties characterized by fuzzy neighborhood and fuzzy topology. *Complex & Intelligent Systems*, pages 1–12.

Wu, W.-Z., Leung, Y., and Shao, M.-W. (2013). Generalized fuzzy rough approximation operators determined by fuzzy implicators. *International Journal of Approximate Reasoning*, 54(9):1388–1409.

Yao, Y. (1998). Constructive and algebraic methods of the theory of rough sets. *Information sciences*, 109(1-4):21–47.

Yao, Y., Mi, J., and Li, Z. (2014). A novel variable precision (θ, σ)-fuzzy rough set model based on fuzzy granules. *Fuzzy Sets and Systems*, 236:58–72.

Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, 8(3):338–353.

Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning—i. *Information sciences*, 8(3):199–249.