Measuring and Ranking Bipolarity via Orthopairs

Zoltán Ernő Csajbók[a]
Department of Health informatics, Faculty of Health Sciences, University of Debrecen
Sostoi ut 2-4, HU-4406 Nyıregyhaza, Hungary

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Abstract: Orthopairs, i.e., disjoint sets, are reasonable means to represent bipolar information. Bipolarity has different models; we use the well-known Dubois-Prade typology. Of course, bipolarity can also carry uncertainty. In this paper, we investigate mainly the bipolarity of type II. In Pawlak's rough set theory, this bipolarity type, with its uncertainty, can be modeled naturally. The "positive" and "negative" sets form an orthopair whose two sets can be approximated by rough sets separately. Rough sets represented by nested sets can be considered an interval set structure. With the help of counting measure, interval numbers can be assigned to the nested sets. Then, relying on interval arithmetic, taking into account the uncertain nature of bipolarity, the degree of bipolarity can be measured, and the positive and negative sets ranked.

1 INTRODUCTION

As Cacioppo, Gardner, and Berntson say in their seminal paper (Cacioppo et al., 1997),

> To be sure, there are in fact bipolarities and dichotomies in the world. (p. 6)

Indeed, bipolarity can be found in many natural and social science fields, even as a feature of human thinking.

Two sides of bipolar information are usually provided with *positive* and *negative* labels. "Positive" and "negative" claim nothing else that the two sides are well separated; nevertheless, they cannot completely be unrelated (Dubois and Prade, 2006). Bipolarity may also carry uncertainty.

Orthopair and its different generalizations are reasonable means to represent bipolar information. Of course, bipolarity may carry uncertainty as well. Bipolarity arises naturally in Pawlak's rough set theory (Pawlak, 1982; Pawlak, 1991; Pawlak and Skowron, 2007).

According to the Dubois and Prade typology (Dubois and Prade, 2006; Dubois and Prade, 2008), orthopair modeling of "Type II: Symmetric bivariate bipolarity" can be interpreted naturally within the rough set theory (see, also (Ciucci, 2011)).

In Pawlak's rough set theory, rough sets represented by nested sets can be considered an interval set structure to represent nonnumeric uncertainty on the model of real interval numbers (Wong et al., 2013; Yao, 2009; Yao and Wong, 1997; Yao and Li, 1996). Then, with the help of counting measure, interval numbers can be assigned to rough sets represented by nested sets. With these interval numbers, the "size" of bipolarity can be measured with different methods, considering its uncertain nature.

In medical practice, it is often the case that one or more diseases have almost identical clinical symptoms. Examples include the common cold and flu. Likewise, the symptoms of hypothyroidism (caused by an "underactive" thyroid gland) and hyperthyroidism (caused by an "overactive" thyroid gland) are also closely related.

The general practitioner (GP) makes a presumptive diagnosis based on the clinical symptoms and then refers the patient to a specialist. In this paper, we focus on a possible numerical analysis of the presumptive diagnoses. It is important to consider the two presumptive diagnoses, how much they differ, and how "stronger" one is than the other.

A set of patients based on two different diagnoses can be divided into two mutually exclusive sets, i.e., they form an orthopair. Then, a numerical comparison of the two presumptive diagnoses can be made using the proposed calculations. It relies on the combination of rough set theory and interval arithmetic.

In Section 2, basic notations and notions of rough sets are summarized. Section 3 and Section 4

[a] https://orcid.org/0000-0002-6357-0233

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overview different representations of rough sets and the basic facts about interval arithmetic. Section 5 presents the typology of bipolarity; the modeling and measuring of the bipolarity of Type II; the ranking of positive and negative reference sets; and the modeling of the bipolarity of Type III. Section 6 gives an illustrative example.

2 BASIC NOTATIONS

Let U and V be two nonempty sets.

A function f is denoted by $f : U \to V, u \mapsto f(u)$ with domain U and codomain V; $u \mapsto f(u)$ is the assignment or mapping rule of f. V^U denotes the set of all functions from U into V.

For any $S \subseteq U$, $f(S) = \{f(u) \mid u \in S\} \subseteq V$ is the direct image of S. $f(U)$ is the range of f.

Let $S \subseteq U$. S^c is the complement of S with respect to U. If $f \in V^U$, the complement of $f(S)$ with respect to V is denoted by $f^c(S)$ instead of $(f(S))^c$.

$\mathcal{P}(U)$ is the set of subsets of U, that is the power set of U.

Let $\mathbb{R}$ represent the real numbers.

The sets S_1 and S_2 ($S_1, S_2 \in \mathcal{P}(U)$) are commonly called disjoint if $S_1 \cap S_2 = \emptyset$. The family $\mathcal{S}$ of sets is called disjoint if any two distinct sets in $\mathcal{S}$ are disjoint. In this case, $\cup\mathcal{S}$ is referred to as a disjoint union and specially denoted by $\uplus\mathcal{S}$.

If $a, b \in \mathbb{R}$ and $a \le b$, $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$ and $]a,b[= \{x \in \mathbb{R} \mid a < x < b\}$ denote closed and open intervals. It is easy to interpret the open-closed $]a,b]$ and closed-open $[a,b[$ intervals.

$(\cdot,\cdot)$ denotes an ordered pair.

$|\cdot|$ is the cardinality of a set.

3 ROUGH SETS

Let U be a nonempty set.

$\mathrm{PAS}(U) = (U, \mathcal{B}, \mathcal{D}_{\mathcal{B}}, \ell, u)$ is a *Pawlak's approximation space* if

- $\mathcal{B} = \Pi(U)$ is a partition of U; its equivalence classes are called *base sets*.
- $\mathcal{D}_{\mathcal{B}}$ is defined with the following inductive definition:
  - $\emptyset \in \mathcal{D}_{\mathcal{B}}$, $\mathcal{B} \subseteq \mathcal{D}_{\mathcal{B}}$;
  - if $D_1, D_2 \in \mathcal{D}_{\mathcal{B}}$, then $D_1 \cup D_2 \in \mathcal{D}_{\mathcal{B}}$.

  The members of $\mathcal{D}_{\mathcal{B}}$ are called *definable sets*.
- *Lower* and *upper approximation operators* ℓ and u are defined as
  - $\ell : \mathcal{P}(U) \to \mathcal{D}_{\mathcal{B}}, S \mapsto \cup\{B \in \mathcal{B} \mid B \subseteq S\}$;
  - $u : \mathcal{P}(U) \to \mathcal{D}_{\mathcal{B}}, S \mapsto \cup\{B \in \mathcal{B} \mid B \cap S \neq \emptyset\}$.

The *boundary* operator derived from lower and upper operators is also defined on $\mathcal{P}(U)$:

$$bnd : \mathcal{P}(U) \to \mathcal{D}_{\mathcal{B}}, S \mapsto u(S) \setminus \ell(S).$$

It is easy to check that $bnd(S)$ is definable indeed.

It is straightforward that $u(S) = \ell(S) \uplus bnd(S)$.

The sets in Pawlak's approximation spaces are characterized with the following notions. A set $S \in \mathcal{P}(U)$ is

- *crisp* (*exact*) if $\ell(S) = u(S)$, i.e., $bnd(S) = \emptyset$;
- *rough* (*inexact*) if it is not exact, i.e., $bnd(S) \neq \emptyset$.

An important feature of Pawlak's approximation spaces is that the exactness and definability coincide. Therefore these two terms can be used synonymously.

For each $S \in \mathcal{P}(U)$, the lower and upper approximation pair (ℓ, u) divides the universe U into three mutual disjoint regions:

- $POS(S) = \ell(S)$ — *positive region* of S;
  if $u \in POS(S)$, it is said that u is an *positive example* of S.
- $NEG(S) = U \setminus u(S) = u^c(S)$ — *negative region* of S;
  if $u \in NEG(S)$, it is said that u is a *negative example* of S.
- $BN(S) = bnd(S)$ — *borderline region* of S;
  if $u \in BN(S)$, it is said that u is an *abstained example* of S.

Knowledge of algebraic aspects of rough sets was summarized by Banerjee and Chakraborty in their comprehensive study (Banerjee and Chakraborty, 2004). From now on, especially different definitions of rough sets and partly their formalism are based on (Banerjee and Chakraborty, 2004).

Let $\mathrm{PAS}(U) = (U, \mathcal{B}, \mathcal{D}_{\mathcal{B}}, \ell, u)$ be a finite Pawlak approximation space.

Let $S_1, S_2 \in \mathcal{P}(U)$. It is said that S_1 and S_2 are *roughly equal*, in notation $S_1 \approx S_2$, if $\mathsf{l}(S_1) = \mathsf{l}(S_2)$ and $\mathsf{u}(S_1) = \mathsf{u}(S_2)$. It is straightforward that $\approx$ is an equivalence relation on $\mathcal{P}(U)$. Then the *rough sets* are the equivalence classes of $\mathcal{P}(U)/\approx$.

For the detailed structure of $\mathcal{P}(U)/\approx$, see (Banerjee and Chakraborty, 2004; Bonikowski, 1992). Here we present below its most relevant properties for this paper.

The equivalent class containing $S \in \mathcal{P}(U)$ is denoted by $[\![S]\!]$. By definition, for all $S' \in [\![S]\!]$, $S' \approx S$, i.e., $\ell(S') = \ell(S)$ and $u(S') = u(S)$, consequently $\ell(S) \subseteq S' \subseteq u(S)$.

A set $S \in \mathcal{P}(U)$ may be exact or rough. Let us see what happens in these two cases.

1. *S is exact.*
   Clearly, S is exact if and only if $|[\![S]\!]| = 1$ and $[\![S]\!] = \{S\}$. Moreover, $\ell(S) = S = u(S) \in [\![S]\!]$.
2. *S is rough.*
   If S is rough, i.e., $\ell(S) \neq u(S)$, then $S \in [\![S]\!]$ by definition, but
   - $\ell(S) \notin [\![S]\!]$, because $\ell(\ell(S)) = \ell(S)$, but $u(\ell(S)) = \ell(S) \neq u(S)$;
   - $u(S) \notin [\![S]\!]$, because $\ell(u(S)) = u(S) \neq \ell(S)$, though $u(u(S)) = u(S)$.

   In sum, if $S \in \mathcal{P}(U)$ is rough, for all $S' \in [\![S]\!]$, $\ell(S) \subsetneq S' \subsetneq u(S)$.

Remark 1. It is worth paying attention to the terms that have evolved historically. A set is *rough* if its boundary is not the empty set. A *rough set* is an equivalence class from $\mathcal{P}(U)/\approx$. □

There are additional equivalent representations of rough sets; namely, for each $S \in \mathcal{P}(U)$,

(1) $[\![S]\!] \in \mathcal{P}(U)/\approx$,

(2) $(\ell(S), u(S))$,

(3) $(\ell(S), u^c(S))$,

(4) $(\ell(S), bnd(S))$.

are rough sets. These four definitions are equivalent to each other in the sense that for any $S \in \mathcal{P}(U)$ the equivalent class $[\![S]\!]$ in $\mathcal{P}(U)/\approx$, and the entities $(\ell(S), u(S))$, $(\ell(S), u^c(S))$, and $(\ell(S), bnd(S))$ are identifiable ((Banerjee and Chakraborty, 2004), pp. 158-159).

By definition, $\ell(S) \cap u^c(S) = \emptyset$ and $\ell(S) \cap bnd(S) = \emptyset$ hold in the cases of representations (2) and (3). That is, in these approaches, orthopairs represent rough sets. However, for our purposes, choosing (2) will be appropriate. In this case, rough sets are represented by nested pairs of sets. However, not every pair (S_1, S_2) $(S_1, S_2, \in \mathcal{P}(U), S_1 \subseteq S_2)$ forms a rough set.

Proposition 1. *Let $S_1, S_2, \in \mathcal{P}(U), S_1 \subseteq S_2$. The pair (S_1, S_2) is a rough set of the form $(\ell(S), \mathrm{u}(S))$ for a set S $(S_1 \subseteq S \subseteq S_2)$ if and only if S_1 and S_2 are definable and $S_2 \setminus S_1$ does not contain any singleton base set.*

Proof. See, (Marek and Truszczyński, 1999), Proposition 3.2. □

4 BASIC NOTIONS OF INTERVAL ARITHMETIC

In this section, basic definitions, notations, facts, and partly their formalism concerning interval numbers are based mainly on (Moore et al., 2009), and partly on (Alefeld and Mayer, 2000; Hickey et al., 2001; Sengupta and Pal, 2000).

An *interval number* or *interval* is simply a closed real interval of the form

$$a = [a^l, a^u] = \{x \in \mathbb{R} \mid a^l \leq x \leq a^u\},$$

where a^l and a^u denote the left and right endpoints of the interval a, respectively.

If $a^l = a^u$, i.e., the interval a is *degenerate*, and it is identified with the real number $a = a^l = a^u$.

Some frequently used special terms for an interval number a are the following.

- $m(a) = \frac{1}{2}(a^l + a^u)$ is the *midpoint* or *center* of a;
- $w(a) = a^u - a^l$ is the *width* or *diameter* of a.

Two intervals $a = [a^l, a^u]$ and $b = [b^l, b^u]$ are said to be equal, in notation $a = b$, if $a^l = b^l$ and $a^u = b^u$.

Let $\odot \in \{+, -, \cdot, /\}$ be a binary operation of the four elementary binary operations on $\mathbb{R}$, i.e., addition, subtraction, multiplication, and division, respectively. Then the following general formula

$$a \odot b = \{x \odot y \mid x \in a, y \in b\}$$

defines four binary operations on the set of interval numbers. Their endpoint formulas are the following (Moore et al., 2009):

$$\begin{aligned}
a + b &= [a^l + b^l, a^u + b^u] \\
a - b &= a + (-b) = [a^l - b^u, a^u - b^l], \\
&\quad \text{where } -b = [-b^u, -b^l] \\
a \cdot b &= [\min\{a^l b^l, a^l b^u, a^u b^l, a^u b^u\}, \\
&\quad \max\{a^l b^l, a^l b^u, a^u b^l, a^u b^u\}] \\
a/b &= a \cdot (1/b), \\
&\quad \text{where } 1/b = [1/b^u, 1/b^l] \ (0 \notin b).
\end{aligned}$$

For nonnegative intervals a and b $(0 \leq a^l \leq a^u$, $0 \leq b^l \leq b^u)$, formulae for multiplication and division are simplified to:

- $a \cdot b = [a^l b^l, a^u b^u]$;
- $a/b = [a^l/b^u, a^u/b^l]$,
  provided in addition that $0 < b^l$.

Let $\lambda \in \mathbb{R}$ be a real number. Then, multiplication with a scalar λ can be defined as a special case of the multiplication: $\lambda \cdot a = \lambda[a^l, a^u] = [\lambda, \lambda] \cdot [a^l, a^u]$.

5 MAIN RESULTS: MODELING AND MEASURING OF BIPOLARITY

Orthopair and its different generalizations are reasonable means to represent bipolar information, and they are widely used to model uncertainty (Campagner and Ciucci, 2017; Ciucci, 2011; Marek and Truszczyński, 1999; Yager and Alajlan, 2017). On the other hand, in rough set theory, bipolarity arises naturally: positive/negative, positive/boundary, and negative/boundary regions.

For a comprehensive discussion of orthopairs, their generalizations, and their connection with rough sets, see (Ciucci, 2011; Gehrke and Walker, 1992; Marek and Truszczyński, 1999; Pagliani, 1998), and the references therein.

5.1 Typology of Bipolarity

Two sides of bipolar information are called *positive* and *negative* aspects. "Positive" and "negative" just mean that the two sides are separated in one way or another. Nevertheless, they cannot be completely unrelated, additional relationships between them may be supposed as well (Dubois and Prade, 2006).

Bipolarity has several forms depending on the nature of the link between its two sides. Dubois and Prade gave the typology of the following forms (Dubois and Prade, 2006; Dubois and Prade, 2009).

Representation of bipolarity relies on some *characteristics* (data, information, opinion, response) of entities (objects, persons, notions).

Bipolarity of Type I Symmetric (homogeneous) univariate bipolarity:

- the evaluation is either totally positive or totally negative;
- positive and negative aspects are mutually exclusive and evaluated simultaneously;
- the evaluation relies on same data.

This is the most constrained form.

Bipolarity of Type II Symmetric (homogeneous) bivariate bipolarity:

- the evaluation is not necessarily totally positive nor totally negative;
- positive and negative aspects have a duality relation, and they are evaluated separately;
- the evaluation relies on the basis of the same data.

This is a looser form.

Bipolarity of Type III Asymmetric (heterogeneous) bivariate bipolarity:

- the evaluation is neither totally positive nor totally negative;
- positive and negative aspects are evaluated separately, a duality relation between them is not required;
- the evaluation does not rely on the same data.

This is the loosest form.

5.2 Modeling and Measuring of Bipolarity of Type II

Bipolarity of type II can be modeled with Pawlak's approximation spaces within rough set theory in an appropriate manner.

Let $\mathrm{PAS}(U) = (U, \mathcal{B}, \mathcal{D}_{\mathcal{B}}, \ell, u)$ be a finite Pawlak's approximation space.

Let $\langle A_+, A_- \rangle$, be an orthopair, viz. $A_+, A_- \in \mathcal{P}(U)$ and $A_+ \cap A_- = \emptyset$. A_+ and A_- are called the *positive reference set* and *negative reference set*, respectively.

Based on A_+ and A_- as two separate entities, let us form two distinct rough sets in their own right. Let us choose the nested pair rough set representation:

$$RS_{A_+} = (\ell(A_+), u(A_+)) \text{ and } RS_{A_-} = (\ell(A_-), u(A_-)).$$

Measure is a mathematical device which reflects some sorts of "size" of sets. The simple so-called *counting measure* accounts for the size of sets as the number of their elements. It plays a key role in rough set theory.

Applying counting measure, interval numbers can be assigned to the former rough sets:

$$RS_{A_+} \mapsto [|\ell(A_+)|, |u(A_+)|] = a_+ = [A_+^{\ell}, A_+^{u}]$$

and

$$RS_{A_-} \mapsto [|\ell(A_-)|, |u(A_-)|] = a_- = [A_-^{\ell}, A_-^{u}]$$

In the above formulae, to avoid heavy notations, the simplified symbols A_+^{l}, A_+^{u}, and A_+^{bnd} have been introduced instead of $|\ell(A_+)|$, $|u(A_+)|$, and $|bnd(A_+)|$. Similar notations have been introduced for A_- as well. In addition, the intervals $[A_+^{\ell}, A_+^{u}]$ and $[A_-^{\ell}, A_-^{u}]$ have been denoted by a_+ and a_-.

Many diverse methods have been proposed to compare interval numbers, for their historical overview, see, (Xu and Chen, 2008). For the comparison between two interval numbers, Facchinetti et al. (Facchinetti et al., 1998), Xu and Da (Xu and L. Da, 2002), and Wang et al. (Wang et al., 2005) have been, respectively, proposed three so-called possibility-degree formulae. It turned out that these three formulae are equivalent ((Xu and Chen, 2008), Theorem 2). This paper will use the formula proposed by Xu and Da in (Xu and L. Da, 2002) because it is the most appropriate for our purposes.

Definition 1 ((Xu and L. Da, 2002), Definition 2.3). Let $a = [a^l, a^u]$ and $b = [b^l, b^u]$ be two real interval numbers. The *possibility degree* of *a over b*, in notation $p(a \geq b)$, is defined by

$$
\begin{aligned}
& p(a \geq b) \\
& = \max\left\{1-\max\left\{\frac{b^u-a^l}{(a^u-a^l)+(b^u-b^l)},0\right\},0\right\} \\
& = \max\left\{1-\max\left\{\frac{b^u-a^l}{w(a)+w(b)},0\right\},0\right\} \qquad (1)
\end{aligned}
$$

Similar formula is defined for the *possibility degree of b over a* by

$$
\begin{aligned}
& p(b \geq a) \\
& = \max\left\{1-\max\left\{\frac{a^u-b^l}{(a^u-a^l)+(b^u-b^l)},0\right\},0\right\} \\
& = \max\left\{1-\max\left\{\frac{a^u-b^l}{w(a)+w(b)},0\right\},0\right\} \qquad (2)
\end{aligned}
$$

□

The following theorem summarises the most important properties of the possibility degree just defined.

Theorem 1 ((Xu and L. Da, 2002), Theorem 2.1). *Let $a = [a^l, a^u]$ and $b = [b^l, b^u]$ be two real interval numbers. The following properties for the possibility degree of a over b (b over a) hold:*

1. $0 \leq p(a \geq b) \leq 1$,
   $0 \leq p(b \geq a) \leq 1$.
2. $p(a \geq b) + p(b \geq a) = 1$
3. $p(a \geq b) = 1$ *if and only if* $b^u \leq a^l$,
   $p(b \geq a) = 1$ *if and only if* $a^u \leq b^l$.
4. $p(a \geq b) = 0$ *if and only if* $a^u \leq b^l$,
   $p(b \geq a) = 0$ *if and only if* $b^u \leq a^l$.
5. $p(a \geq a) = \frac{1}{2}$.
6. $p(a \geq b) \geq \frac{1}{2}$ *if and only if* $a^u + a^l \geq b^u + b^l$.
   Especially, $p(a \geq b) = \frac{1}{2}$ *if and only if* $a^u + a^l = b^u + b^l$.

Properties (3) and (4) mean that the possibility degree of a over b are equal to 0 or 1 if and only if they do not have a common area re*gardless of the distance between a and b*. Similar property holds for the possibility degree of b over a.

Let us consider the possibility degree of reference sets over each other.

Possibility degree of the positive reference over negative reference set and the negative reference set over positive reference set can be calculated. With the above notations:

$$p(a_+ \geq a_-) \text{ and } p(a_- \geq a_+).$$

Proposition 2. *Let* $\mathrm{PAS}(U)$ *be a Pawlak's approximation space and* (A_+, A_-) *be an orthopair. Then*

1. $p(a_+ \geq a_-) = 1$ *if and only if* $|u(A_-)| \leq |\ell(A_+)|$.
2. $p(a_+ \geq a_-) = 0$ *if and only if* $|u(A_+)| \leq |\ell(A_-)|$.

Proof.

$$
\begin{aligned}
p(a_+ \geq a_-) &= p([A_+^\ell, A_+^u] \geq [A_-^\ell, A_-^u]) \\
&= p([|\ell(A_+)|, |u(A_+)|] \\
&\qquad \geq [|\ell(A_-)|, |u(A_-)|])
\end{aligned}
$$

Hence, statements *1.* and *2.* follow from Theorem 1/3, and Theorem 1/4, respectively. □

The statement

$$p(a_+ \geq a_-) = 1 \text{ if and only if } |u(A_-)| \leq |\ell(A_+)|$$

means that the possibility degree of the positive reference set A_+ over negative reference set A_- is equal to 1 if and only if the cardinality of the upper approximation of the negative reference set $u(A_-)$ is less than or equal to the cardinality of the lower approximation of the positive reference set $\ell(A_+)$.

The statement

$$p(a_+ \geq a_-) = 0 \text{ if and only if } |u(A_+)| \leq |\ell(A_-)|$$

means that the possibility degree of the positive reference set A_+ over negative reference set A_- is equal to 0 if and only if the cardinality of the upper approximation of the positive reference set $u(A_+)$ is less than or equal to the cardinality of the lower approximation of the negative reference set $\ell(A_-)$.

Proposition 3. *Let* $\mathrm{PAS}(U)$ *be a Pawlak's approximation space and* (A_+, A_-) *be an orthopair. Then*

$$p(a_+ \geq a_-) = \frac{1}{2}$$

if and only if

$$|u(A_+)| - |u(A_-)| = |\ell(A_-)| - |\ell(A_+)|.$$

Proof.

$$
\begin{aligned}
p(a_+ \geq a_-) &= p([A_+^\ell, A_+^u] \geq [A_-^\ell, A_-^u]) \\
&= p([|\ell(A_+)|, |u(A_+)|] \\
&\qquad \geq [|\ell(A_-)|, |u(A_-)|]) = \frac{1}{2}
\end{aligned}
$$

by Theorem 1/6

$$
\begin{aligned}
&\Leftrightarrow \quad |u(A_+)| + |\ell(A_+)| = |u(A_-)| + |\ell(A_-)| \\
&\Leftrightarrow \quad |u(A_+)| - |u(A_-)| = |\ell(A_-)| - |\ell(A_+)|
\end{aligned}
$$

□

Let $|u(A_+)| - |u(A_-)| = |\ell(A_-)| - |\ell(A_+)| = K$.

Then $p(a_+ \geq a_-) = \frac{1}{2}$ can be interpreted in the following way:

The possibility degree of the positive reference set over the negative reference set is equal to $\frac{1}{2}$ if

- *provided* $K = 0$:
  - The elements of lower approximations of positive and negative reference sets are equal;
  - and the elements of upper approximations of positive and negative reference sets are also equal at the same time.
- *provided* $K > 0$:
  - The upper approximation of the positive reference set as much many more elements than the elements of the upper approximation of the negative reference set
  - as the elements of the lower approximation of the negative reference set has more elements than the elements of the lower approximation of the positive reference set.
- *provided* $K < 0$: The interpretation in this case can be done in a similar way as in case $K > 0$.

The above interpretations are reasonable, and coincide with our intuitive approach.

5.3 Ranking of Positive and Negative Reference Sets

To rank the positive and negative reference sets, both of them must be compared with themselves and one with the other. Specifically, the following quantities must be formed (Xu and L. Da, 2002):

$$p(a_+ \geq a_+), p(a_+ \geq a_-),$$
$$p(a_- \geq a_+), p(a_- \geq a_-).$$

where

- $p(a_+ \geq a_+), p(a_+ \geq a_-), p(a_- \geq a_+), p(a_- \geq a_-) \geq 0$;
- $p(a_+ \geq a_+) = p(a_- \geq a_-) = \frac{1}{2}$ (Theorem 1, 5.);
- $p = (a_+ \geq a_-) + p(a_- \geq a_+) = 1$ (Theorem 1, 2.).

These quantities can be arranged in the matrix of the form

$$P = \begin{bmatrix} p(a_+ \geq a_+) & p(a_+ \geq a_-) \\ p(a_- \geq a_+) & p(a_- \geq a_-) \end{bmatrix}.$$

Let us sum these quantities lines by lines as follows:

$$p_+ = p(a_+ \geq a_+) + p(a_+ \geq a_-),$$
$$p_- = p(a_- \geq a_+) + p(a_- \geq a_-).$$

Then the positive and negative reference sets can be ranked in increasing or descending order according to the quantities p_+ and p_- naturally.

In (Wang et al., 2005) Wang et al. proposed another ranking method for interval numbers.

Definition 2 ((Wang et al., 2005), Definition 2). Let $a = [a^l, a^u]$ and $b = [b^l, b^u]$ be two real interval numbers.

- If $p(a \geq b) > p(b \geq a)$, it is said that *a superior to b* to the degree of $p(a \geq b)$, in notation $a \succ_{p(a \geq b)} b$.
- If $p(a \geq b) = p(b \geq a) = \frac{1}{2}$, it is said that a is *indifferent to b*, in notation $a \sim b$.
- If $p(b \geq a) > p(a \geq b)$, it is said that a is *inferior to b* to the degree $p(b \geq a)$, in notation $a \prec_{p(b \geq a)} b$. □

The matrix P can be applied to compare the positive and negative reference sets concerning Definition 2.

Definition 3. The *positive reference set* is

- *superior to the negative reference set* to the degree $p(a_+ \geq a_-)$ if
$$a_+ \succ_{p(a_+ \geq a_-)} a_-;$$
- *indifferent to the negative reference set* if
$$a_+ \sim a_-;$$
- *inferior to the negative reference set* to the degree $p(a_- \geq a_+)$ if
$$a_- \succ_{p(a_- \geq a_+)} a_+.$$

□

Remark 2. Roughly speaking,

- $a_+ \succ_{p(a_+ \geq a_-)} a_-$ means that the possibility degree of the positive reference set over the negative reference set is greater than the possibility degree of the negative reference set over the positive reference set to the degree $p(a_+ \geq a_-)$;
- $a_+ \sim a_-$ means that the possibility degree of the positive reference set over the negative reference set is equal to the possibility degree of the negative reference set over the positive reference set;
- $a_- \succ_{p(a_- \geq a_+)} a_+$ means that the possibility degree of the negative reference set over the positive reference set is greater than the possibility degree of the positive reference set over the negative reference set to the degree $p(a_- \geq a_+)$. □

5.4 An Illustrative Example

In practice, the equivalence relation of objects comes from the characteristics of objects. After Pawlak, objects and their attributes (characteristics) arranged in a table is called the *Information Table* or *Information System* (Pawlak, 1981; Pawlak, 1982; Ciucci, 2011).

Definition 4 ((Pawlak, 1981), pp. 205-206). An *Information System* is a structure $IS = (U, A, V, F)$, where

- U is a finite, nonempty set of *objects*;
- A is a finite, nonempty set of *attributes*;
- $V = \cup_{a \in A} V_a$, where V_a is the set of all *possible values* that can be observed for an attribute $a \in A$ concerning all objects from U;
- F is an *information function* $F : U \times A \to V$, where F assigns a value $F(u,a) \in V$ to any pair (u,a) $(u \in U, a \in A)$.

If the function F is total, the system is called *complete*, otherwise, it is *incomplete*. □

Due to the finiteness of the information system, the information function can be given by a finite table. The columns of the table are labeled with attributes, and the rows with objects. Of course, the order of columns and rows in the table is insignificant. In addition, some attributes can share a set of values.

Next, we define a binary relation on U.

Definition 5. Let $B \subseteq A$ be a subset of attributes. Two objects $x, y \in U$ are called *indiscernible* with respect to B, in notation xI_By, if $F(x,a) = F(y,a) (a \in B)$. □

It is easy to check that the indiscernibility relation I_B is an equivalence one. The equivalence class containing x is $[\![x]\!]_{I_B} = \{y \in U \mid xI_By\}$. The partition generated by I_B is denoted by Π_B, and the set of definable sets is $\mathcal{D}_B$. Thus Pawlak's approximation space is

$$\mathrm{PAS}(U) = (U, \Pi_B, \mathcal{D}_B, \ell_B, u_B).$$

Turning to the example, let us consider the symptoms of thyroid dysfunctions. We deal with only hypothyroidism and hyperthyroidism thyroid disorders (Ladenson and Kim, 2011).

An "underactive" thyroid gland (which releases too much hormone) causes the symptoms of hypothyroidism, and an "overactive" thyroid gland (which does not produce enough hormone) causes the symptoms of hyperthyroidism. Their clinical symptoms are closely related.

Table 1, Information Table, summarizes some patients' observed clinical symptoms concerning thyroid dysfunction. Expanding this table, the last two columns, based on these clinical symptoms, contain presumptive diagnoses made by a general practitioner. A patient may develop a hypothyroidism or hyperthyroidism thyroid disorder, perhaps neither of them. The possible symptoms have been compiled based on GP experiences but simplified here for illustrative purposes.

The set of objects, here patients, is:

$$U = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}\}.$$

The possibly presumptive diagnoses are: `Hypothyroidism: yes, no`, `Hyperthyroidism: yes, no`.

The observed clinical symptoms are included in the attribute set

$$A = \{\texttt{Weight change}, \texttt{Oedema}, \texttt{Tachycardia}, \texttt{Increased sweating}, \texttt{Weakness}, \texttt{Morbid psychomotor activity}\}. \quad (3)$$

The possible values of the attributes are

- $V_{\texttt{Weight change}} = \{\texttt{loss}, \texttt{gain}, \texttt{unchanged}\}$.
- $V_{\texttt{Oedema}} = \{\texttt{yes}, \texttt{no}\}$.
- $V_{\texttt{Tachycardia}} = \{\texttt{yes}, \texttt{no}\}$.
- $V_{\texttt{Increased sweating}} = \{\texttt{yes}, \texttt{no}\}$.
- $V_{\texttt{Weakness}} = \{\texttt{yes}, \texttt{no}\}$.
- $V_{\texttt{Morbid psychomotor activity}} = \{\texttt{excitement}, \texttt{slowness}, \texttt{unchanged}\}$.

$$\begin{aligned} V = & V_{\texttt{Weight change}} \cup V_{\texttt{Oedema}} \\ & \cup V_{\texttt{Tachycardia}} \cup V_{\texttt{Increased sweating}} \\ & \cup V_{\texttt{Weakness}} \cup V_{\texttt{Morbid psychomotor activity}} \\ & = \{\texttt{loss}, \texttt{gain}, \texttt{unchanged}, \texttt{yes}, \texttt{no}, \\ & \quad \texttt{excitement}, \texttt{slowness}\} \end{aligned}$$

Let S_{hypo} ("positive reference set") and S_{hyper} ("negative reference set") be the sets of patients who demonstrably suffer from hypothyroidism and hyperthyroidism:

$$S_{hypo} = \{P_1, P_2, P_4, P_7, P_8\}, S_{hyper} = \{P_3, P_5, P_6, P_9\}.$$

The sets S_{hypo} and S_{hyper} form an orthopair because $S_{hypo} \cap S_{hyper} = \emptyset$.

Let choose the attribute set

$$B = \{\texttt{Weight change}\} \subseteq A.$$

Then the equivalence relation with respect to B is I_B with $P_i I_B P_j$ if $F(P_i, \texttt{Weight change}) = F(P_j, \texttt{Weight change})$ $(i, j = 1, 2, \ldots, 10)$. Hence, the partition of U generated by I_B is

$$\Pi_B = \{\{P_1, P_6, P_7\}, \{P_5, P_9\}, \{P_2, P_3, P_4, P_8, P_{10}\}\},$$

reflecting the weight change being "gain," "loss," and "unchanged", respectively.

Let consider $RS_{S_{hypo}} = (\ell_B(S_{hypo}), u_B(S_{hypo}))$.

$$\begin{aligned} \ell_B(S_{hypo}) &= \emptyset, \\ u_B(S_{hypo}) &= \{P_1, P_6, P_7\} \cup \{P_2, P_3, P_4, P_8, P_{10}\} \\ &= \{P_1, P_2, P_3, P_4, P_6, P_7, P_8, P_{10}\}, \end{aligned}$$

Table 1: Clinical symptoms of thyroid dysfunction and presumptive diagnoses.

No.	Weight change	Oedema	Tachy-cardia	Increased sweating	Weakness	Morbid psychomotor activity	Hypothy-roidism	Hyperthy-roidism
P_1	gain	no	no	no	no	unchanged	yes	no
P_2	unchanged	yes	yes	yes	yes	unchanged	yes	no
P_3	unchanged	no	yes	no	no	excitement	no	yes
P_4	unchanged	no	yes	no	yes	slowness	yes	no
P_5	loss	no	yes	no	yes	unchanged	no	yes
P_6	gain	no	no	no	yes	unchanged	no	yes
P_7	gain	no	yes	no	no	excitement	yes	no
P_8	unchanged	no	no	no	no	unchanged	yes	no
P_9	loss	yes	yes	yes	yes	unchanged	no	yes
P_{10}	unchanged	no	no	no	no	excitement	no	no

Then,

$$a_{hypo} = [a^l, a^u] = [|\ell_B(S_{hypo})|, |u_B(S_{hypo})|] = [0, 8].$$

Let consider $RS_{S_{hyper}} = (\ell_B(S_{hyper}), u_B(S_{hyper}))$.

$$\ell_B(S_{hyper}) = \{P_5, P_9\},$$
$$u_B(S_{hyper}) = U.$$

Then,

$$b_{hyper} = [b^l, b^u] = [|\ell_B(S_{hyper})|, |u_B(S_{hyper})|] = [2, 10].$$

With the Eqn. (1), we can calculate:

$$\begin{aligned} & p(a_{hypo} \geq b_{hyper}) \\ = \ & \max\left\{1 - \max\left\{\frac{b^u - a^l}{(a^u - a^l) + (b^u - b^l)}, 0\right\}, 0\right\} \\ = \ & \max\left\{1 - \max\left\{\frac{10 - 0}{(8 - 0) + (10 - 2)}, 0\right\}, 0\right\} \\ = \ & \max\left\{1 - \max\left\{\frac{10}{16}, 0\right\}, 0\right\} = \frac{6}{10} = \frac{3}{5}. \end{aligned}$$

According to Theorem 1, Properties 2,

$$p(b_{hyper} \geq a_{hypo}) = 1 - p(a_{hypo} \geq b_{hyper}) = \frac{2}{5}.$$

These results can be interpreted as follows. With respect to our knowledge represented in Table 1 and partitioning U by `Weight change`, the overall contribution of the clinical symptoms weight change to the presence of

- hypothyroidism has the possibility degree $\frac{3}{5}$,
- hyperthyroidism has the possibility degree $\frac{2}{5}$.

It must be noted that, at this stage of the study, this interpretation focuses purely on the mathematical relationships without entering into medical issues.

5.5 Modeling Bipolarity of Type III

Type III bipolarity can be modeled by two distinct *general* set approximation spaces *over the same universe*. Pawlak's approximation space is a classical one and has many generalizations. Here we only describe the generalisation that we need.

$\mathrm{GAS}(U) = (U, \mathcal{B}, \mathcal{D}_{\mathcal{B}}, \ell, u)$ is a finite general approximation space if

- U is finite nonempty set;
- $\mathcal{B}$ is not a partition of U but covers it: $\cup\mathcal{B} = U$;
- base system $\mathcal{D}_{\mathcal{B}}$ is strictly union type defined with the following inductive definition:
  - $\emptyset \in \mathcal{D}_{\mathcal{B}}$, $\mathcal{B} \subseteq \mathcal{D}_{\mathcal{B}}$;
  - if $D_1, D_2 \in \mathcal{D}_{\mathcal{B}}$, then $D_1 \cup D_2 \in \mathcal{D}_{\mathcal{B}}$.
- Lower and upper approximation operators ℓ and u are defined as
  - $\ell : \mathcal{P}(U) \to \mathcal{D}_{\mathcal{B}}$, $S \mapsto \cup\{B \in \mathcal{B} \mid B \subseteq S\}$;
  - $u : \mathcal{P}(U) \to \mathcal{D}_{\mathcal{B}}$, $S \mapsto \cup\{B \in \mathcal{B} \mid B \cap S \neq \emptyset\}$.

To model Type III Bipolarity, let us define *two distinct base systems* for the independent description of positive and negative reference sets on the universe U. Applying the creation rules of $\mathrm{GAS}(U)$, we obtain two approximation spaces with different structures:

- $\mathrm{GAS}_+(U) = (U, \mathcal{B}_+, \mathcal{D}_{\mathcal{B}_+}, \ell_+, u_+)$,
- $\mathrm{GAS}_-(U) = (U, \mathcal{B}_-, \mathcal{D}_{\mathcal{B}_-}, \ell_-, u_-)$.

The measuring and ranking of positive and negative reference sets can be done by accordingly modifying the general procedure in the two different approximation spaces.

6 CONCLUSION

In this paper, measuring the extent of bipolarity has been proposed with the help of interval arithmetic.

Working in finite Pawlak approximation space, the uncertain nature of bipolarity approximating the positive and negative reference sets with rough sets has also been modeled. The proposed methods can be extended to any family of mutual disjoint sets.

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