Fractional Order-Sliding-Mode Controller for Regulation of a Nonlinear Chemical Process with Variable Delay

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Abstract: The present work shows the application of a new controller based on combining the fractional order calculus concepts with the sliding mode theory to a non-linear system with variable delay. The power of fractional-order calculus is used to identify the real process and represent it as a reduced-order model. From this model, the controller is developed using the sliding-mode control procedure. An SMC based on FOPDT and one based on fractional calculus are compared using some performance indicators to assess performance quantitatively.

1 INTRODUCTION

The chemical and biochemical engineering field has a lot of control problems, one of which is the primary focus of current work: regulating variable time delay processes. In real applications, plant behavior is often affected by unexpected dynamics, outside disturbances, etc. (Obando et al., 2023). Disturbances, model errors, unmodeled dynamics, elevated time delay, and poorly defined plant characteristics reduce the efficacy of conventional regulation schemes even for linear time-invariant systems in chemical processes, especially at the industrial scale.

When designing a process controller, the control actions become slow if it has a time delay, decreasing the total process performance. Furthermore, in some cases, it has been seen that delays produce instability in the system, as in (Prado et al., 2022), leading to the search for more advanced control proposals. Thus, controlling dynamical systems with delay time has become a major topic in control theory.

The variable structure control has a method called the sliding mode control (SMC). SMC is a simple and reliable method for creating controllers for linear and non-linear processes, according to (Camacho and Smith, 2000; Utkin et al., 2020). SMC is one of the most widely used techniques for managing dynamical systems because it is relatively insensitive to uncertainties(Camacho and Smith, 2000; Espín et al., 2022; Utkin et al., 2020). The traditional sliding mode control always maintains a predesigned sliding variable $S(t)$, which is generated by a so-called reaching condition, such as $S(t)\dot{S(t)} < 0$, to be zero under high-frequency switching (Utkin et al., 2020).

Substantial research has been published on the application of SMC in industrial engineering process control; we can name some of them here(Espín et al., 2022; Xiao and Li, 2016; Dimassi et al., 2019; Rasul and Pathak, 2016; Sardella et al., 2020; Siddiqui et al., 2020; Herrera et al., 2020; Salinas et al., 2018; Kadu et al., 2018). Although few studies on SMC design employ fractional order calculus techniques for chemical processes, we can highlight some of them (Di Teodoro et al., 2023; Di Teodoro et al., 2022; Ullah and Mohammad, 2022; Allahem et al., 2022; Mehri and Tabatabaei, 2021; Haghighi and Ziaratban, 2020; Ardjal et al., 2021).

The current work illustrates the application of a novel controller based on fusing notions from sliding mode theory and fractional order calculus to a non-linear system with variable delay. First, the actual process is represented as a reduced-order model using the strength of fractional-order calculus. Then, the sliding-mode control approach is used to create the controller from this model. Finally, some performance measures were used to evaluate and compare an SMC based on FOPDT and an SMC based on fractional calculus.

The paper is divided as follows: in Section two, some fundamentals are described; Section three shows the methodology of design; in Section four, the

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results by simulations are presented; and finally, the conclusion.

2 BACKGROUND

2.1 Sliding Mode Control

The SMC is a variable structure controller created utilizing nonlinear control techniques. It is a robust control that reacts appropriately to nonlinear systems under unknown conditions. Furthermore, it is immune to changes in modeling parameters. The SMC considers a sliding surface that allows the controlled variable to transition from an initial state to the desired final state. For this reason, its control law includes a continuous component to move on the sliding surface and a discontinuous part for the reachability phase. The motion of the system on a sliding surface is known as the sliding mode (Utkin et al., 2020).

The SMC control law $U(t)$, as expressed in (1):

$$U(t) = U_{eq}(t) + U_D(t) \tag{1}$$

$U_{eq}(t)$: It is obtained from the equivalent control method (Utkin et al., 2020). Keep the controlled variable on the sliding surface $\sigma(t) = 0$. The equivalent control is deduced, considering that the derivative of the surface is zero.

The sliding condition is given by:

$$\frac{d\sigma(t)}{dt} = 0 \tag{2}$$

Combined with the previous equation and the model of the process system, $U_{eq}(t)$ is obtained.

$U_D(t)$: It is the discontinuous control part; it allows the system to reach the sliding surface. It incorporates a nonlinear element that includes the switching element of the control law; therefore, $U_D(t)$ contains the switching element and is given by:

$$U_D(t) = K_D sign(\sigma(t)) \tag{3}$$

K_D is a tuning parameter responsible for the reaching mode.

SMC transitions on the sliding surface cause chattering. Chattering excites system dynamics not modeled, causes vibration of the actuator with wear, and degrades performance. One way to reduce such impacts is to smooth the non-linear switching function using soft functions such as the sigmoid function (Camacho and Smith, 2000). Therefore, the discontinuous controller utilizes the sigmoid function:

$$U_D(t) = K_D \frac{\sigma(t)}{|\sigma(t)| + \delta} \tag{4}$$

The convergence condition, often known as attractiveness, ensures that the system dynamics will always converge on the sliding surface(Utkin et al., 2020). It is necessary to formulate a Lyapunov function $V(t) > 0$ with finite energy. The candidate Lyapunov function is defined as follows.

$$V(t) = 1/2\sigma^2(t) \tag{5}$$

It is sufficient to make sure that the derivative of the function $V(t)$ is negative for it to be possible for the function $V(t)$ to be reduced. Therefore, the condition of convergence can be written as follows:

$$\frac{dV(t)}{dt} = \frac{d\sigma(t)}{dt}\sigma(t) < 0 \tag{6}$$

It states that if the projection of the system trajectories on the sliding surface is stable, then the system is stable (Li et al., 2010).

2.2 Briefs About Fractional Calculus

2.2.1 Definitions of Caputo and Riemann-Liouville Derivative and Their Connection

Definition 1. *The Riemann − Liouville fractional integral of order $\alpha > 0$ is given by (see (Kilbas et al., 2006; Miller and Ross, 1993; Podlubny, 1994; Kilbas et al., 1993))*

$$\left(I^{\alpha}_{a^+}h\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{h(t)}{(x-t)^{1-\alpha}}\,dt, \quad x > a. \tag{7}$$

We denote by $I^{\alpha}_{a^+}(L_1)$ the class of functions h, represented by the fractional integral (7) of a summable function, that is, $h = I^{\alpha}_{a^+}\varphi$, where $\varphi \in L_1(a,b)$. A description of this class of functions is given in (Kilbas et al., 2006; Kilbas et al., 1993; Abbas et al., 2023; Patel et al., 2023) ($L_1[a,b]$ space can be defined as a space of measurable functions for which the absolute value is Lebesgue-integrable).

Definition 2. *Let $\left(D^{\alpha}_{a^+}h\right)(x)$ denote the fractional Riemann–Liouville derivative of order $\alpha > 0$, where $h \in L_1(a,b)$ (see (Kilbas et al., 2006; Miller and Ross, 1993; Podlubny, 1994; Kilbas et al., 1993; Abbas et al., 2023))*

$$\left({}_{RL}D^{\alpha}_{a^+}h\right)(x) = \left(\frac{d}{dx}\right)^s \frac{1}{\Gamma(s-\alpha)}\int_a^x \frac{h(t)}{(x-t)^{\alpha-s+1}}\,dt, \tag{8}$$

$$s = [\alpha] + 1, x > a, \tag{9}$$

where $[\alpha]$ denotes the integer part of α and Γ is the gamma function.

When $0 < \alpha < 1$, then (8) takes the form

$$\left({}_{RL}D^{\alpha}_{a^+}h\right)(x) = \frac{d}{dx}\left(I^{1-\alpha}_{a^+}h\right)(x). \tag{10}$$

Example 1.

$$_{RL}D^{\alpha}_{a^+}f(x-a)^{\gamma} = \begin{cases} 0, & \gamma=\alpha-1, \\ \frac{\gamma+1}{\gamma-\alpha+1}(x-a)^{\gamma-\alpha}, & otherwise \end{cases}.$$

With $\alpha \in (0,1)$, $a>0$, $k \in \mathbb{N}$ and $\gamma > -1$, and an appropriate f. (See (Ceballos et al., 2020; Ceballos et al., 2022; Kilbas et al., 1993))

Definition 3. *Let $\alpha \geq 0$ and $m=[\alpha]$. Then, we can define the operator $_cD^{\alpha}_{a^+}$ by $_cD^{\alpha}_{a^+}f := I^{m-\alpha}_{a^+}\left(\frac{d}{dx}\right)^m f$, when $\left(\frac{d}{dx}\right)^m f \in L_1[a,b]$.*

Example 2. $_cD^{\alpha}_{a^+}(x-a)^{\beta}=0$ *if* $\beta \in \{0,1,2,\ldots,m-1\}$.

Lemma 1. *Let $\alpha \geq 0$ and $m=[\alpha]+1$. Suppose that f is such that $_cD^{\alpha}_{a^+}$ and $_{RL}D^{\alpha}_{a^+}$ exists. Then*

$$_cD^{\alpha}_{a^+}f =_{RL} D^{\alpha}_{a^+}f - \sum_{k=0}^{m-1}\frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}\left(\frac{d}{dx}\right)^k f(a).$$

See (Kilbas et al., 2006; Podlubny, 1994; Kilbas et al., 1993))

consequently, we have the following lemma:

Lemma 2. *Let $\alpha \geq 0$ and $m=[\alpha]+1$. Suppose that f is such that $_cD^{\alpha}_{a^+}$ and $_{RL}D^{\alpha}_{a^+}$ exists.*

Then $_cD^{\alpha}_{a^+}f =_{RL} D^{\alpha}_{a^+}f = D^{\alpha}_{a^+}f$. If and only if $\left(\frac{d}{dx}\right)^k f(a)=0$ for all $k=0,...,m-1$.

The semigroup property for the composition of fractional derivatives does not hold in general (see (Podlubny, 1994, Sect. 2.3.6)). In fact, the property:

$$D^{\alpha}_{a^+}\left(D^{\gamma}_{a^+}h\right) = D^{\alpha+\gamma}_{a^+}h \tag{11}$$

holds whenever

$$h^{(j)}(a^+)=0, \qquad j=0,1,\ldots,s-1, \tag{12}$$

and $h \in AC^{s-1}([a,b])$, $h^{(s)} \in L_1(a,b)$ and $s=[\gamma]+1$ ($AC^s([a,b])$ denotes the class of functions h, which are continuously differentiable in the segment $[a,b]$, up to order $s-1$ and $h^{(s-1)}$ is absolutely continuous in $[a,b]$).

Remark 1. *In this paper, we use $\mathscr{D}^{\alpha}_{a^+}$ to identify the fractional operator because we assume that the initial conditions are equal to zero.*

2.2.2 Laplace Transform of Fractional Derivatives

Definition 4. *Let $F(s) := (\mathscr{L}f)(s)$, i.e., say, the Laplace transform of the function f and $\alpha > 0$ (See([(Kilbas et al., 1993)])).*

$$\left(\mathcal{L}\,\mathscr{D}^{\alpha}_{0^+}f\right)(s) = s^{\alpha}F(s) - \sum_{j=0}^{l-1} f^{(j)}(0^+)s^{\alpha-j-1}$$

where $l=[\alpha]+1$ and $f^{(j)}(0^+) = \frac{d^j}{dt^j}f(t)|_{t\to 0^+}$, $j=0,...,l-1$. Clearly, if $\alpha \in (0,1)$ then

$$\left(\mathcal{L}\,\mathscr{D}^{\alpha}_{0^+}f\right)(s) = s^{\alpha}F(s) - f(0^+)s^{\alpha-1} \tag{13}$$

2.2.3 The Use of the Mittag-Leffler for $G(s)$

In order to develop our model, we will make use of the two-parameter Mittag-Leffler function. This function is defined classically as

Definition 5. *Let $z \in \mathbb{C}$ and $\alpha > 0$. The two-parameter Mittag-Leffler function is given by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad \forall z \in \mathbb{C}. \tag{14}$$

Remark 2. *Note that $E_{\alpha,\beta}(z) > 0$ for all $z \in \mathbb{C}^+$ and when $\beta=1$, then $E_{\alpha,\beta}(z) = E_{\alpha}(z)$ This function is uniformly convergent over $\mathbb{C}$. For $\alpha,\beta \geq 0$ If $n=1$ and $r=1$. $E_{1,1}(z)=E_1(z)=exp(z)$. (see (Kilbas et al., 2006; Podlubny, 1994; Kilbas et al., 1993))*

3 CONTROL DESIGN APPROACH IN THE CAPUTO SENSE

This section presents the mathematical development of the sliding mode control based on a FOPID surface. Since SMC is a model-based controller, let us use an empirical model of the actual process. Using the reaction curve procedure, the nonlinear process can be approximated for a first-order plus dead time (FOPDT) model (Smith and Corripio, 2005).

$$\frac{X(s)}{U(s)} = \frac{Ke^{-t_0 s}}{\tau s+1}. \tag{15}$$

3.1 Time Delay Approximation

A first problem encounter in the realization of this fractional controller is the time delay part (Camacho and Smith, 2000). In the complex settings, the natural exponential function e^z can be represented as the well–known Taylor series centered at 0 which is absolutely convergent for $z \in \mathbb{C}$. A simple linear approximation is $1+z$ where the absolute error $|e^z-1-z|$ equals the Taylor remainder R_1. On the other hand, we have that $e^{-z}=\frac{1}{e^z}$ thus we can approximate e^{-z} by $\frac{1}{1+z}$. Indeed, to see this let the approximation error, $\left|e^{-z}-\frac{1}{1+z}\right|$, be bounded by $\varepsilon_{max}>0$. Then, $\left|e^{-z}-\frac{1}{1+z}\right| \leq \frac{|z|}{|1+z|}\max\left\{1,e^{-Re\{z\}}\right\}$ $=\varepsilon_{max}$.

For example, the time delay $e^{-t_0 s}$ can be approximated to $\frac{1}{t_0 s+1}$ and, in turn, the FOPDT transfer function is as follows:

$$\frac{Ke^{-t_0 s}}{\tau s+1} \approx \frac{K}{(\tau s+1)(t_0 s+1)}, \tag{16}$$

that has a second order differential equation form (Camacho and Smith, 2000).

Remark 3. *A discussion that seems interesting to us about the role of the fractional parameter in the exponential function is a future work. A further conjecture is that we believe that the parameter β is not necessary. Which makes the calculation of the Laplace inverse much simpler.*

3.2 Fractional Transfer Functions

Let $\alpha,\beta \in \mathbb{R}$, $r,n \geq 0$. Then, we define the fractional transfer function plus dead time as

$$\frac{X(s)}{U(s)} = \left(\frac{K}{\tau(s^{\alpha})+1}\right)\cdot\left[\frac{1}{E_{n,r}(t_o(s^{\beta}))}\right] \tag{17}$$

where α and β can be interpreted as orders of fractional derivatives and integrals. Note that transfer function (15) is obtained when α and β approach to 1^-.

Transfer function (17) can be approximated by (see sec. 3.1 above)

$$\frac{X(s)}{U(s)} = \left(\frac{K}{\tau(s^{\alpha})+1}\right)\cdot\left[\frac{1}{E_{n,r}(t_o(s^{\beta}))}\right] \approx \left(\frac{K}{\tau(s^{\alpha})+1}\right)\cdot\frac{\Gamma(n)\Gamma(n+r)}{\Gamma(n+r)+\left(t_o(s^{\beta})\right)\Gamma(n)}. \tag{18}$$

and can be changed to a fractional differential equation form. Using this approximation in equation (17) we obtain

$$\left[\Gamma(n)\, s^{\alpha+\beta}+\left(\tfrac{1}{t_0}\,\Gamma(n+r)\right)s^{\alpha}+\left(\tfrac{1}{\tau}\,\Gamma(n)\right)s^{\beta}+\tfrac{1}{\tau t_0}\Gamma(n+r)\right]X(s) = \frac{1}{\tau t_0}\left[K\,\Gamma(n)\,\Gamma(n+r)\right]U(s) \tag{19}$$

Using definition 4 whenever the function X satisfies the conditions of lemma 2 , then

$$\left(\mathscr{D}^{\alpha+\beta}_{0^+}X\right)(t)+\left[\frac{1}{t_0}\,\frac{\Gamma(n+r)}{\Gamma(n)}\right]\left(\mathscr{D}^{\alpha}_{0^+}X\right)(t) + \left[\frac{1}{\tau}\right]\left(\mathscr{D}^{\beta}_{0^+}X\right)(t)+\left[\frac{1}{\tau t_0}\,\frac{\Gamma(n+r)}{\Gamma(n)}\right]X(t)=\frac{1}{\tau t_0}\left[K\,\Gamma(n+r)\right]U(t) \tag{20}$$

3.3 Fractional Sliding Surface

Let us define the fractional sliding surface according to (18) and (20). In the following, we assume the conditions of lemma 2.

Definition 6. *Let $\alpha,\beta \in \mathbb{R}$, $m = [\alpha+\beta]$, $e(t) \in AC^{m-1}([a,b])$ and $e^{(m)} \in L_1(a,b)$. Then, the fractional slinding surface, $S_{\alpha,\beta}$, is defined as*

$$\left(S_{\alpha,\beta}\, e\right)(t) := \left(\mathcal{D}^{\alpha+\beta}_{0^+}+\lambda_1\mathcal{D}^{\alpha}_{0^+}+\lambda_2\mathcal{D}^{\beta}_{0^+}+\lambda_0\right)\left(I^1\, e\right)(t). \tag{21}$$

where $\lambda_0,\lambda_1,\lambda_2$ are selected such that $S_{\alpha,\beta}=0$, giving a closed-loop stable response.

Remark 4. *In the classical sense (Camacho and Smith, 2000), our operator converge to* (22)

$$\left(S\, e\right)(t) = \left(\frac{d}{dt}+\lambda\right)^n\int e(t)\,dt. \tag{22}$$

when $n=2$ as α,β approach to 1^-. The binomial theorem can be used to generalize n. This is a fascinating topic for further research. Just emphasize that fractional computation for outside-interval parameters $(0,1)$ is much more difficult to calculate.

Remark 5. *Note that the definitions 2 and 3 allow us to commute the classical integral operator I^1 in the fractional sliding surface operator in definition 6, i.e.,*

$$\left(\mathcal{D}^{\alpha+\beta}_{0^+}+\lambda_1\mathcal{D}^{\alpha}_{0^+}+\lambda_2\mathcal{D}^{\beta}_{0^+}+\lambda_0\right)I^1 = I^1\left(\mathcal{D}^{\alpha+\beta}_{0^+}+\lambda_1\mathcal{D}^{\alpha}_{0^+}+\lambda_2\mathcal{D}^{\beta}_{0^+}+\lambda_0\right). \tag{23}$$

Classical derivation of operator 6 will be needed in the following:

$$\tfrac{d}{dt}\left(S_{\alpha,\beta}\, e\right)(t) = \left(\mathcal{D}^{\alpha+\beta}_{0^+}+\lambda_1\mathcal{D}^{\alpha}_{0^+}+\lambda_2\mathcal{D}^{\beta}_{0^+}+\lambda_0\right)[e(t)]. \tag{24}$$

where we have used the result in remark 5 followed by the application of the Fundamental Theorem of Calculus.

The next theorem shows how we can construct the fractional equivalent control law $U^{\alpha,\beta}_{eq}(t)$.

Theorem 1. *Let $\alpha,\beta \in \mathbb{R}$, $m=[\alpha+\beta]$, $e(t) \in AC^{m-1}([a,b])$ and $R^{(m)}(t), X^{(m)}(t) \in L_1(a,b)$. Then, the fractional control law $u^{\alpha,\beta}_{eq}(t)$ based in the fractional sliding surface defined in 6 is*

$$u^{\alpha,\beta}_{eq}(t) = \frac{\tau t_0}{K\Gamma(n+r)}\left(\begin{array}{c}\left(\mathcal{D}^{\alpha+\beta}_{0^+}R\right)(t)+\lambda_1\left(\mathcal{D}^{\alpha}_{0^+}R\right)(t)+\\ \lambda_2\left(\mathcal{D}^{\beta}_{0^+}R\right)(t)+\lambda_0 R(t)+\\ \left(\dfrac{1}{t_0}\,\dfrac{\Gamma(n+r)}{\Gamma(n)}-\lambda_1\right)\left(\mathcal{D}^{\alpha}_{0^+}X\right)(t)+\\ \left(\tfrac{1}{\tau}-\lambda_2\right)\left(\mathcal{D}^{\beta}_{0^+}X\right)(t)+\\ \left(\dfrac{1}{\tau t_0}\,\dfrac{\Gamma(n+r)}{\Gamma(n)}-\lambda_0\right)X(t)\end{array}\right) \tag{25}$$

Let $e(t) = R(t) - X(t)$. First, we apply operator (24) to function $e(t)$ to obtain

$$\frac{d}{dt}\left(S_{\alpha,\beta}\, e\right)(t) = \frac{d}{dt}\left(S_{\alpha,\beta}\, R\right)(t) - \frac{d}{dt}\left(S_{\alpha,\beta}\, X\right)(t). \tag{26}$$

by the linearity of this fractional operator. Next, $\frac{d}{dt}\left(S_{\alpha,\beta}\, e\right)(t) = 0$ guides us, combining equation (20) to

$$\left(\mathcal{D}_{0^+}^{\alpha+\beta} R\right)(t) + \lambda_1 \left(\mathcal{D}_{0^+}^{\alpha} R\right)(t) + \lambda_2 \left(\mathcal{D}_{0^+}^{\beta} R\right)(t) + \lambda_0 R(t) +$$
$$\left(\mathscr{D}_{0^+}^{\alpha+\beta} X\right)(t) + \left[\frac{1}{t_0}\,\frac{\Gamma(n+r)}{\Gamma(n)}\right]\left(\mathscr{D}_{0^+}^{\alpha} X\right)(t) +$$
$$\left[\frac{1}{\tau}\right]\left(\mathscr{D}_{0^+}^{\beta} X\right)(t) + \left[\frac{1}{\tau t_0}\,\frac{\Gamma(n+r)}{\Gamma(n)}\right] X(t) = \left(\mathcal{D}_{0^+}^{\alpha+\beta} X\right)(t) +$$
$$\lambda_1 \left(\mathcal{D}_{0^+}^{\alpha} X\right)(t) + \lambda_2 \left(\mathcal{D}_{0^+}^{\beta} X\right)(t) + \lambda_0 X(t) +$$
$$\frac{1}{\tau t_0}\left[K\,\Gamma(n+r)\right] U(t) \tag{27}$$

The result follows.

Remark 6. *If $R : \mathbb{R} \to \mathbb{R}$ such that $R = c$, where $c \in \mathbb{R}$. Then $\dot{R} = 0$, consequently, $D_{0^+}^{\alpha} R(t) = 0$ and $\mathcal{D}_{0^+}^{\beta} R\,(t) = 0$.*

The derivatives of the reference value can be ignored (Camacho and Smith, 2000) without affecting the control performance, which allows a simpler controller.
Thus, the resulting equivalent controller law is given as follows:

$$u_{eq}^{\alpha,\beta}(t) = \frac{\tau t_0}{K\,\Gamma(n+r)} \begin{pmatrix} \lambda_0 R(t) + \\ \left(\frac{1}{t_0}\,\frac{\Gamma(n+r)}{\Gamma(n)} - \lambda_1\right)\left(\mathcal{D}_{0^+}^{\alpha} X\right)(t) + \\ \left(\frac{1}{\tau} - \lambda_2\right)\left(\mathcal{D}_{0^+}^{\beta} X\right)(t) + \\ \left(\frac{1}{\tau t_0}\,\frac{\Gamma(n+r)}{\Gamma(n)} - \lambda_0\right) X(t) \end{pmatrix} \tag{28}$$

The previous equation can be simplified by letting,

$$\frac{1}{t_0}\,\frac{\Gamma(n+r)}{\Gamma(n)} = \lambda_1 \tag{29}$$

and

$$\frac{1}{\tau} = \lambda_2 \tag{30}$$

The sliding (continuous) control law is:

$$u_{eq}^{\alpha,\beta}(t) = \frac{\tau t_0}{K\,\Gamma(n+r)}\left(\lambda_0 e(t) + \left(\frac{1}{\tau t_0}\,\frac{\Gamma(n+r)}{\Gamma(n)}\right) X(t)\right) \tag{31}$$

The reaching (discontinuous) control law $U_D^{\alpha,\beta}(t)$ is defined as follows:

$$U_D^{\alpha,\beta}(t) = K_{D(\alpha,\beta)}\,\frac{S_{\alpha,\beta}(t)}{\left|S_{\alpha,\beta}(t)\right| + \delta_{\alpha,\beta}} \tag{32}$$

Where $K_{D(\alpha,\beta)}$ is the switching gain.

Once the continuous and discontinuous parts of the SMC, based on the fractional reduced order, have been obtained; the total fractional order SMC law can be represented as follows:

$$U^{\alpha,\beta}(t) = u_{eq}^{\alpha,\beta}(t) + K_{D(\alpha,\beta)}\,\frac{S_{\alpha,\beta}(t)}{\left|S_{\alpha,\beta}(t)\right| + \delta_{\alpha,\beta}}. \tag{33}$$

Until now, to our knowledge, there are no tuning equations for $K_{D(\alpha,\beta)}$ neither $\delta_{\alpha,\beta}$; hence in this paper they are tuned by trial and error.

3.4 Fractional Results Using $r = \alpha$ and $n = \beta = 1$

$$\begin{aligned} M\, U_{eq}^{\alpha,1}(t) &= \left[\mathscr{D}_{0^+}^{\alpha+1} + \lambda_1 \mathscr{D}_{0^+}^{\alpha} + \lambda_2 \mathscr{D}_{0^+}^{1} + \lambda_0\right] R(t) \\ &\quad + \left[\frac{\Gamma(\alpha+1)}{t_0} - \lambda_1\right]\left(\mathscr{D}_{0^+}^{\alpha} X\right)(t) \\ &\quad + \left[\frac{1}{\tau} - \lambda_2\right]\left(\mathscr{D}_{0^+}^{1} X\right)(t) + \left[\frac{\Gamma(\alpha+1)}{\tau t_0} - \lambda_0\right] X(t) \end{aligned} \tag{34}$$

Where $M = \frac{K}{\tau t_0}\,\Gamma(\alpha+1)$ and $\Gamma(1) = 0! = 1$

3.5 Stability Condition

The dynamics of the system will always converge on the sliding surface according to the convergence condition; to reach the sliding surface, it is necessary to create a Lyapunov candidate function $V(t) > 0$ with finite energy. We use the Mittag–Leffler stability theorem to do this and get the corresponding systems' asymptotic stability.

Theorem 2 (See Theorem 5.1 in (Li et al., 2010))**.** *Let $x = 0$ be an equilibrium point for the system*

$$D_{0^+}^{\alpha} x(t) = f(x(t)), \quad t \geq t_0,$$

where $0 < \alpha \leq 1$ and $I \subset \mathbb{R}$ be a domain containing the origin $f \in C^1(I)$. Let $V(t, x(t)) : [0, \infty) \times I \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that

i) $A||x||^a \leq V(t, x(t)) \leq B||x||^b$,
ii) $D_{0^+}^{\alpha} V(t, x(t)) \leq -F||x||^{ab}$

where $t \geq 0$, $x \in I, \alpha \in (0,1), A, B, F, a,$ and b. are arbitrary positive constants. Then $x = 0$ is Mittag–Leffler stable. If the assumptions hold globally on $\mathbb{R}$, then $x = 0$ is globally Mittag–Leffler stable.

$$V(t, x(t)) = x(t)^{(2)\alpha-1} E_{\alpha,\alpha}(-\lambda x(t)^{(2)\alpha-1}). \tag{35}$$

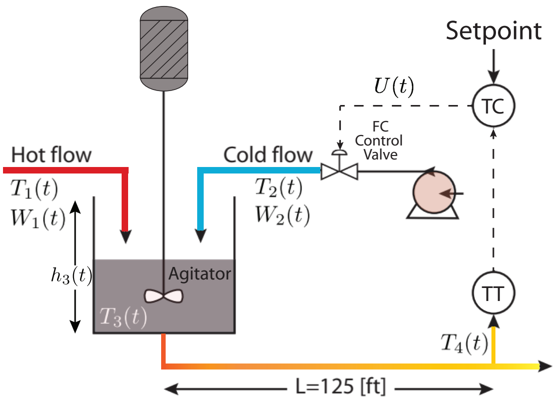

Figure 1: Mixing Tank process.

Where $\alpha \in (0,1)$, $E_{\alpha,\beta}$ is the Mittag-Leffler function defined in (14).
This function is locally Lipschitz any interval of $\mathbb{R}$ and satisfies

(i $0 < V(t,x(t)) \leq B||x||^b$ for $x \in I \subset \mathbb{R}$ and any $B \in \mathbb{R}$. See (Coloma et al., 2021; Ceballos et al., 2020; Ceballos et al., 2022)

(ii $D^{\alpha}_{0^+}V(t,x(t)) = -\lambda V(t,x(t)) \leq -F||x||^{ab}$ for any $F,a,b \in \mathbb{R}$, see (Kilbas et al., 2006).

in consequence $x = 0$ is Mittag–Leffler stable, using the Remark 4.4 in (Li et al., 2010) (Mittag–Leffler stability and Generalized Mittag–Leffler stability imply asymptotic stability) we guarantee asymptotic stability of our system.

4 RESULTS AND DISCUSSION

4.1 Mixing Tank Process

The mixing tank shown in Fig. 1 consists of the simultaneous entry of the hot $W_1(t)$ and cold flow $W_2(t)$ into the process with temperatures $T_1(t)$ and $T_2(t)$, respectively. The output $T_4(t)$ is the temperature of the mixture measured at a point 125 ft downstream of the mixing tank. The Fail-Closed (FC) actuator regulates the cold stream to maintain the desired temperature T_3 within the mixing tank. The control objective consists of maintaining the required mixing temperature $T_3(t)$ despite disturbances in the hot flow $W_1(t)$. The following determines the time delay between the tank and the sensor's position.

$$T_4(t) = T_3(t - t_0) \tag{36}$$

with, transportation lag or delay time:

$$t_0 = \frac{LA\rho}{W_1(t) + W_2(t)} \tag{37}$$

where A is the cross section of the pipe, the length of the pipe L and the density ρ of the contents of the mixing tank. A complete description of the non-linear model and parameters in a stationary state and the system variables can be found in (Camacho and Smith, 2000).

4.2 Identification and Validation of FOPDT and FO-FOPDT Models

The reaction curve method is used to identify an approximate model of the non-linear mixing tank process. Thus, 10% of the input of the process $(m(t))$ is made. Figure 2 shows the responses of the tank mix process, the approximate FOPDT model (Smith and Corripio, 2005) and the approximate fractional FO-FOPDT model (Gude and García Bringas, 2022). Equations (38) and (39) show the approximate models:

$$G(s)_{FOPDT} = \frac{-0.8207e^{-4.206s}}{2.227s + 1} \tag{38}$$

$$G(s)_{FO-FOPDT} = \frac{-0.8207e^{4.142s}}{2.262s^{1.01} + 1} \tag{39}$$

As can be seen in the Eqs. (38) and (39). The controllability relation $\left(\frac{t_0}{\tau} > 1\right)$, which makes it difficult to control (Obando et al., 2023). Also, it can be considered as a process with a long delay.

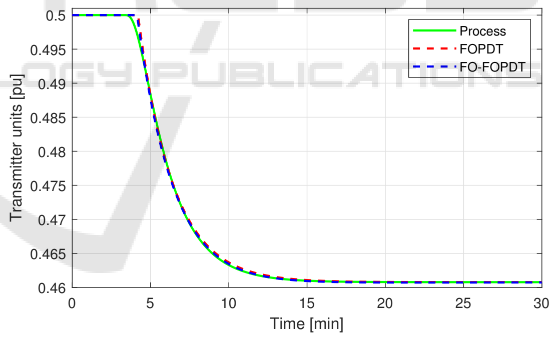

Figure 2: Open Loop Output Responses for step process input.

The mean square error (MSE) is used to validate the models obtained. Where: $MSE_{FOPDT} = 1.4192x10^{-7}$ and $MSE_{F0-FOPDT} = 9.3819x10^{-8}$. Although both indices are low, the MSE of the fractional model was the lowest.

To analyze the performance of the FO-SMC controller, it is compared with a conventional SMC controller presented in (Camacho and Smith, 2000). FO-SMC controller parameters λ_0, λ_1 and λ_2 are tuned according to Section 3. SMC controller parameters and the FO-FOPDT K_D and δ parameters are tuned according to (Camacho and Smith, 2000). These parameters are shown in Table 1.

Table 1: Controller design parameters.

Parameter	SMC	FO-SMC
λ_0	0.1179	0.1171
λ_1	0.6867	0.2425
λ_2	–	0.4419
K_D	0.3833	0.3925
δ	0.7059	0.6894

4.3 Disturbance Rejection Test

In this test, from the operating point $T_3(t) = 150[^oF]$, the variations in steps from 250 $\left[\frac{lb}{min}\right]$ to 150 $\left[\frac{lb}{min}\right]$ in the hot flow $W_1(t)$.

Due to the distance of the pipe $L = 120[ft]$, the outlet temperature is measured downstream. Furthermore, due to the effect of the disturbance, the variation of $W_1(t)$ causes the time delay of the processes to increase stepwise from 3.58 to 4.9$[min]$, as shown in Fig. 3.

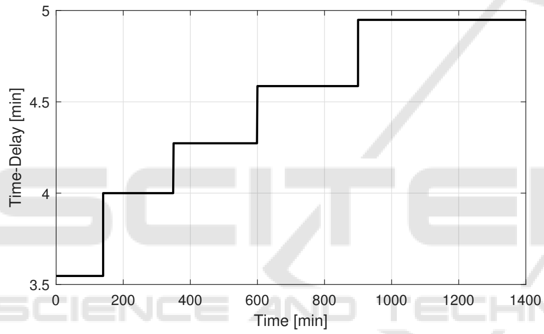

Figure 3: Dead Time variation.

Figure 4 shows the temperature responses for the disturbance rejection test. It can be seen that for the last two disturbances, the response of the FO-SMC controller presents a slight improvement.

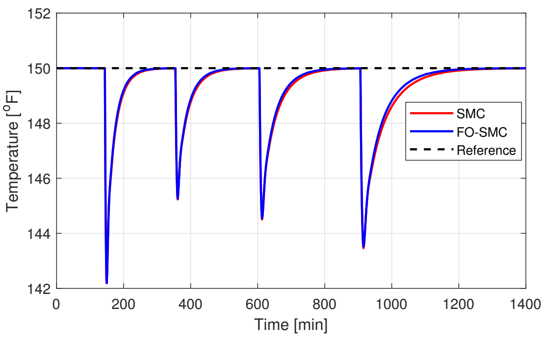

Figure 4: Temperature responses.

The control actions for both controllers have very similar behavior, as shown in Fig. 5.

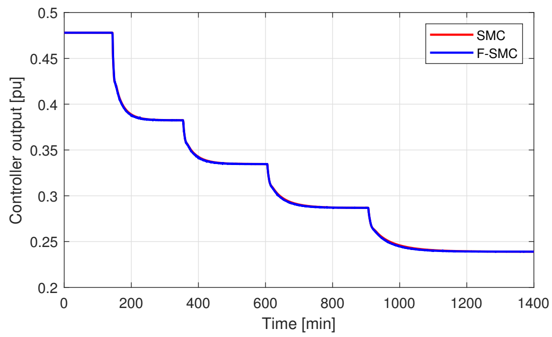

Figure 5: Action Controllers.

4.4 Performance Indices Comparison

In Fig. 6 the comparison of the performance indices of the SMC controller and the FO-SMC is shown. It can be seen that the performance values obtained for both controllers are very similar. However, the FO-SMC controller presents slightly lower ISE and IAE values.

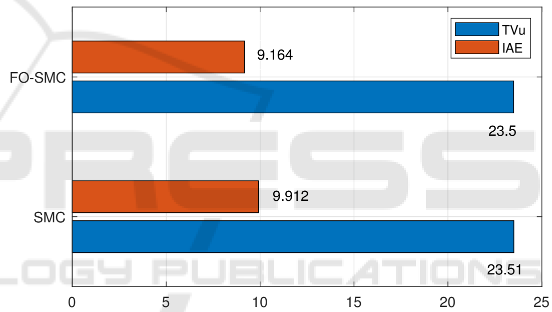

Figure 6: Comparative of performance indexes of the controllers.

5 CONCLUSION

The paper applied a novel SMC based on a fractional-order model to a variable time-delayed nonlinear process. The results showed that the proposed approach is promising, requiring more exploration of the tuning equations and the effects of λ and β to reach the sliding surface and reduce chattering.

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