Approximations of New MV-Valued Types of Fuzzy Sets

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Abstract: Many of the new types of fuzzy sets, such as intuitionistic, neutrosophic, multi-level or fuzzy soft sets and their combinations, can be transformed into one common type of fuzzy sets, called $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets, with values in a set R that is a common underlying set of complete commutative idempotent semirings $\mathscr{R}$ and $\mathscr{R}^*$. For $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets, the theory of lower and upper approximations by $(\mathscr{R},\mathscr{R}^*)$-relations is defined and the basic properties of these approximations are presented. Using examples of the transformation of some new types of MV-valued fuzzy sets and corresponding fuzzy relations into R-fuzzy sets and R-fuzzy relations, examples of approximation of these new types of fuzzy sets through their fuzzy relations are presented, without having to define these operators separately for each new type of fuzzy set.

1 INTRODUCTION

The paper deals with the application of approximation methods to new types of MV-valued fuzzy sets. The development of fuzzy mathematics and their theory and applications leads to an expansion of new types of fuzzy set with values in various ordered algebraic structures. Let us mention, for example, intuitionistic fuzzy sets (Aggarwal et al., 2019), (Atanassov, 1986), (Atanassov, 1984), (Kozae and et al., 2020), fuzzy soft sets (Aktas and Cagman, 2007), (Maji and et al, 2001), (Maji et al., 2003), (Maji and et al., 2002), (Molodtsov, 1999), (Mushrif et al., 2006), or neutrosophic fuzzy sets (Hu and Zhang, 2019), (James and Mathew, 2021), (Zhang et al., 2018) and their mutual combinations, such as intuitionistic fuzzy soft sets (Agarwal et al., 1013), (Garg and Arora, 2018) and many others. Although all of these fuzzy structures are different from each other, they still use many of the analogous methods and tools, typical for classical fuzzy sets.

The importance of this topic lies mainly in the fact that nowadays the use of these new fuzzy structures for solving specific applications is expanding very quickly, on the one hand, but on the other hand, the theoretical part of these methods is often not fully solved using the tools of these new fuzzy sets. This often leads to the creation of own theories (sometimes not fully correct), mainly motivated by classic fuzzy sets, within individual methods using new fuzzy sets. And this regardless of the fact that a large part of these new fuzzy sets can be isomorphically transformed into one type of parametric fuzzy sets, where parameters are represented as special examples of operations in some semirings. The transformed structures then correspond to standard MV-valued fuzzy sets.

This transformation then makes it possible to use the entire range of theoretical methods of classic MV-valued fuzzy sets and, using the inverse aforementioned isomorphic transformation, to convert them to the appropriate methods in new types of fuzzy sets. And without any need to prove the properties of these transformed operations for individual types of new fuzzy sets.

This possible approach to the unification of new types of fuzzy sets was published in (Močkoř, 2021), where some of these new MV-valued fuzzy sets were transformed into the so-called $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets, that is, the mappings $X \to R$, where R is the (common) underlying set of complete commutative idempotent semirings $\mathscr{R}$ and $\mathscr{R}^*$ with involutive isomorphisms $\neg : \mathscr{R} \to \mathscr{R}^*$. The advantage of this construction is, among other things, that the pair $(\mathscr{R},\mathscr{R}^*)$ explicitly defines the pair of dual constructions, commonly used in the theory of fuzzy sets, such as, for example, upper and lower approximations, upper and lower F-transforms or closure and interior operators.

In this paper, we focus on building the theory of

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approximations of $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets and on illustrative examples of how different types of approximations for new types of MV-valued fuzzy sets can be defined using this theory. For this purpose, we also use the elementary theory of monads in categories, as introduced in (Manes, 1976).

2 PRELIMINARIES

In this preliminary section, we introduce the basic definitions and properties of dual pairs of semirings $(\mathscr{R},\mathscr{R}^*)$ and $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets that were presented (in a modified form) in (Močkoř, 2021). The main motivation for the introduction of dual pairs of semirings $(\mathscr{R},\mathscr{R}^*)$ and $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets is that this type of fuzzy sets can be used to transform many of the new MV-valued fuzzy sets in a set X into mappings $X \to R$. And even in cases where the given new type fuzzy set is not a mapping $X \to L$, although traditionally it is called L-fuzzy set. A typical example of this obstacle in the application of classic L-fuzzy sets tools to new types of fuzzy sets is represented by L-fuzzy soft sets. A L-fuzzy soft sets are defined in a space (K,X) where K is the (fixed) set of criteria and X is a set. L-fuzzy soft set is then defined as a pair (E, f), where $E \subseteq K$ and $f : E \to L^X$. This form of fuzzy set requires completely new definitions of operations with these new fuzzy sets.

In this introductory section, we present the definition of $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets, and we also show how some new types of fuzzy sets can be isomorphically transformed into $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets, with the basic operations defined formally in the same way as for L-fuzzy sets. Construction of $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets is based on the notion of a complete commutative idempotent semiring.

Definition 1. *((Berstel and Perrin, 1985)) A complete commutative idempotent semiring (or a semiring, shortly)* $\mathscr{R} = (R,+,\times,0,1)$ *is an algebraic structure with the following properties:*

1. $(R,+,0)$ *is a complete idempotent commutative monoid,*
2. $(R,\times,1)$ *is a commutative monoid,*
3. $x \times \sum^{\mathscr{R}}_{y\in I} y = \sum^{\mathscr{R}}_{y\in I}(x \times y)$ *holds for all* $x \in R, I \subseteq R$, *where* $\sum^{\mathscr{R}} = \sum$ *is the complete operation* $+$ *in* $\mathscr{R}$,
4. $0 \times x = 0$ *holds for all* $x \in R$.

The notion of a semiring homomorphism is defined as a standard homomorphism between algebraic structures. We introduce the dual pair of semirings $(\mathscr{R},\mathscr{R}^*)$ that was originally introduced in modified form in (Močkoř, 2021). The following definition is a simplification of the original definition.

Definition 2. *Let* $\mathscr{R} = (R,+,\times,0,1)$ *and* $\mathscr{R}^* = (R,+^*,\times^*,0^*,1^*)$ *be complete idempotent commutative semirings with the same underlying set R. The pair* $(\mathscr{R},\mathscr{R}^*)$ *is called the dual pair of semirings if there exists a semiring isomorphism* $\neg : \mathscr{R} \to \mathscr{R}^*$ *and the following axioms hold:*

1. $\neg : \mathscr{R} \to \mathscr{R}^*$ *is the involutive isomorphism;*
2. $\forall a \in R, S' \subseteq R \quad a \times^* (\sum_{b\in S'} b) = \sum_{b\in S'}(a \times^* b)$;
3. $\forall a \in R, S' \subseteq R \quad a + (\sum^*_{b\in S'} b) = \sum^*_{b\in S'}(a + b)$, *where* $\sum^*$ *is the complete operation* $+^*$ *in* $\mathscr{R}^*$;
4. $\forall a,b \in R, \quad a+b = a \Leftrightarrow a+^* b = b$.

Using the isomorphism $\neg$, it is easy to see that the following dual statements also hold for arbitrary dual pairs of semirings $(\mathscr{R},\mathscr{R}^*)$:

2'. $\forall a, \in R, S \subseteq R, \quad a \times \sum^*_{b\in S} b = \sum^*_{b\in S}(a \times b)$,

3'. $\forall a, \in R, S \subseteq R, \quad a +^* \sum_{b\in S} b = \sum_{b\in S}(a +^* b)$.

The basic properties of the dual pair of semirings $(\mathscr{R},\mathscr{R}^*)$, including the orderings defined in the underlying set R, are described in the following lemma.

Lemma 1. *Let* $(\mathscr{R},\mathscr{R}^*)$ *be the dual pair of semirings.*

1. *Let the relations* $\leq$ *and* $\leq^*$ *be defined by*

   $$x,y \in R, \quad x \leq y \Leftrightarrow x+y = y, \quad x \leq^* y \Leftrightarrow x+^* y = y.$$

   The following statements hold:

   (a) $\leq$ *and* $\leq^*$ *are the order relations on R,*

   (b) $x \leq^* y \Leftrightarrow x \geq y \Leftrightarrow \neg x \leq \neg y$,

   (c) $x \leq y \Rightarrow x+z \leq y+z, \quad x+^* z \leq y+^* z$,

   (d) $x \leq y \Rightarrow x \times z \leq y \times z, \quad x \times^* z \leq y \times^* z$.
2. $(R,+,\times,\leq)$ *and* $(R,+^*,\times^*,\leq^*)$ *are lattice-ordered semirings, where, for arbitrary* $S \subseteq R$,

   $$sup\, S = \sum_{x\in S} x, \quad inf\, S = \sum^*_{x\in S} x, \text{ in } (R,\leq),$$

   $$sup\, S = \sum^*_{x\in S} x, \quad inf\, S = \sum_{x\in S} x, \text{ in } (R,\leq^*),$$

   where $\sum^*_{x\in S} x$ *is the sum of elements with respect to* $+^*$.

Proof is only a simple application of Definition 2 and will be omitted. ■

The dual pairs of semirings $(\mathscr{R},\mathscr{R}^*)$ with a common underlying set R represent a special value set structure for the so-called $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets, which can be effectively used to transform some of the new fuzzy structures into the shape of classic fuzzy sets in the set X, i.e. in the mappings $X \to R$.

The following definition was introduced in (Močkoř, 2021).

Definition 3. *(Močkoř, 2021) Let* $(\mathscr{R},\mathscr{R}^*)$ *be a dual pair of semirings and X be a set.*

1. *A mapping $s : X \to R$ is called a $(\mathscr{R},\mathscr{R}^*)$-fuzzy set in a set X.*
2. *Operations with $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets are defined by*
   (a) *The intersection $s \sqcap t$ is defined by $(s \sqcap t)(x) = s(x) +^* t(x)$, $x \in X$,*
   (b) *The union $s \sqcup t$ is defined by $(s \sqcup t)(x) = s(x) + t(x)$, $x \in X$,*
   (c) *Complement $\neg s$ is defined by $(\neg s)(x) = \neg(s(x))$,*
   (d) *The external multiplication $\star$ of $(\mathscr{R},\mathscr{R}^*)$-fuzzy set s by an element $a \in R$ is defined by $(a \star s)(x) = a \times s(x)$, $x \in X$,*
   (e) *The order relation $\leq$ between s,t is defined by $s \leq t \Leftrightarrow (\forall x \in X) s(x) \leq t(x)$ where $\leq$ is the order relation defined in Lemma 1.*

The set of all $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets in a set X will be denoted by $(\mathscr{R},\mathscr{R}^*)^X$. In the following proposition, some basic properties of operations with $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets are summarized. It follows that the algebraic structure $((\mathscr{R},\mathscr{R}^*)^X, \sqcap, \sqcup, \neg, \star)$ has analogical properties as the algebraic structure of classical L-fuzzy sets.

Proposition 1. *(Močkoř, 2021) Let $(\mathscr{R},\mathscr{R}^*)$ be a dual pair of semirings. Let X be a set, and let s,t,w be $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets. Then the following statements are valid.*

1. $s \sqcap s = s, s \sqcup s = s$,
2. $s \sqcap t \leq s$, $s \leq s \sqcup t$,
3. $s \sqcap t \leq s \sqcup t$,
4. $s \sqcap (t \sqcup w) = (s \sqcap t) \sqcup (s \sqcap w)$,
5. $s \sqcup (t \sqcap w) = (s \sqcup t) \sqcap (s \sqcup w)$,
6. $a \star (t \sqcup w) = (a \star t) \sqcup (a \star w)$,
7. $a \star (t \sqcap w) = (a \star t) \sqcap (a \star w)$,
8. $\neg(s \sqcup t) = \neg s \sqcap \neg t$, $\neg(s \sqcap t) = \neg s \sqcup \neg t$,
9. $s \leq t \Rightarrow s \sqcup w \leq t \sqcup w, \quad s \sqcap w \leq t \sqcap w$.

As we mentioned in the Introduction, in the paper we will use some basic constructions from the theory of monads, which will be used in the next parts of the paper. Elements of category theory can be found in many publications, such as (Herrlich and Strecker, 2007). By **Set** we denote the category of sets with mappings as morphisms. For more information on monads, see, for example, (Manes, 1976; Herrlich and Strecker, 2007).

Definition 4. *(Manes, 1976) The structure $\mathbf{T} = (T, \Diamond, \eta)$ is a monad in the category* **Set**, *if*

1. *T : **Set** $\to$ **Set** is a mapping of objects,*
2. *η is a system of mappings $\{\eta_X : X \to T(X) | X \in$ **Set**$\}$,*
3. *For each $X,Y,Z \in$ **Set** and any pair of mappings $f : X \to T(Y)$, $g : Y \to T(Z)$, there exists a composition (called Kleisli composition) $g \Diamond f : X \to T(Z)$, which is associative,*
4. *For every mapping $f : X \to T(Y)$, $\eta_Y \Diamond f = f$ and $f \Diamond \eta_X = f$ hold,*
5. *For arbitrary mappings $f : X \to Y$, $g : Y \to T(Z)$, we have $g \Diamond (\eta_Y . f) = g.f : X \to T(Z)$, where . is the composition of morphisms in the category* **Set**.

Remark 1. *Morphisms $X \to T(Y)$ will be denoted by $X \rightsquigarrow Y$ and will be called $\mathbf{T}$-relations (or Kleisli morphisms) from X to Y. The composition of $\mathbf{T}$-relations $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ is defined by $g \Diamond f : X \rightsquigarrow Z$.*

The following proposition presents an example of a monad that is defined by a complete commutative idempotent semiring. This monad will be the key theoretical tool for the approximation of $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets.

Proposition 2. *Let $\mathscr{R} = (R, +, \times, 0, 1)$ be a complete commutative idempotent semiring. The structure $\mathbf{T}_{\mathscr{R}} = (T, \Diamond, \eta)$ is defined by*

1. *The mapping T : **Set** $\to$ **Set** of objects is defined by $T(X) = R^X$,*
2. *For $\mathbf{T}_{\mathscr{R}}$-relations $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ their composition $g \Diamond f : X \rightsquigarrow Z$ is defined by*
$$(g \Diamond f)(x)(z) = \sum_{y \in Y}^{\mathscr{R}} f(x)(y) \times g(y)(z).$$
3. *$\eta = \{\eta_X : X \in$ **Set**$\}$, where η_X is the $\mathbf{T}_{\mathscr{R}}$-relation $X \rightsquigarrow X$ defined by*
$$\eta_X(x)(y) = \begin{cases} 1_R, & x = y, \\ 0_R, & x \neq y, \end{cases}.$$

Then $\mathbf{T}_{\mathscr{R}}$ is the monad in the category **Set**.

Remark 2. 1. *If $(\mathscr{R},\mathscr{R}^*)$ is a dual pair of semirings, according to Proposition 2, there also exists the monad $\mathbf{T}_{\mathscr{R}^*} = (T, \Diamond^*, \eta^*)$, where for $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$,*
$$g \Diamond^* f(x)(z) = \sum_{y \in Y}^{\mathscr{R}^*} f(x)(y) \times^* g(y)(z),$$
$$\eta_X^*(x)(y) = \begin{cases} 1^* = 0, & x = y, \\ 0^* = 1, & x \neq y. \end{cases}$$

2. *$\mathbf{T}_{\mathscr{R}}$-relations will be simply called $\mathscr{R}$-relations and $\mathbf{T}_{\mathscr{R}^*}$-relations will be called $\mathscr{R}^*$-relations. Both relations represent the same morphisms $X \to R^Y$, the difference is only related to their compositions $\Diamond$ and $\Diamond^*$. Both relations will sometimes be called $(\mathscr{R},\mathscr{R}^*)$-relations.*

3. *Instead of category* **Set** *we can consider the Kleisli category* $\mathbf{Set}_{\mathscr{R}}$ *of the monad* $\mathbf{T}_{\mathscr{R}}$*, where objects are the same as in* **Set** *and morphisms are* $\mathbf{T}_{\mathscr{R}}$*-relations* $X \rightsquigarrow Y$*, with* $\Diamond$ *as the composition of morphisms.*

3 EXAMPLES OF TRANSFORMATION OF NEW TYPES OF MV-VALUED FUZZY SETS INTO $(\mathscr{R},\mathscr{R}^*)$-FUZZY SETS

In this section we present examples of transformations of MV-valued fuzzy soft sets, neutrosophic MV-valued fuzzy sets and Ω-level MV-valued fuzzy sets into $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets which extend some results presented in (Močkoř, 2021). We also show that algebras of MV-valued fuzzy soft sets, neutrosophic fuzzy sets or Ω-level MV-valued fuzzy sets are isomorphic to the corresponding algebras of $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets. These isomorphisms enable us to define basic operations with these new MV-valued fuzzy sets in an analogous way to classical L-fuzzy sets. In this section, $L = (L,\vee,\wedge,\oplus,\otimes,\neg,0_L,1_L)$ is a complete MV-algebra.

3.1 MV-valued Fuzzy Soft Sets

Let K be a fixed set of criteria. L-fuzzy soft set in a set X is a pair (E,s), where $E \subseteq K$ and $s : K \to L^X$ such that $s(k) = 0_L$ for $k \in K \setminus E$. Therefore, (E,s) can be interpreted as a mapping $(E,s) : K \to L^X$ such that

$$(E,s)(k)(x) = \begin{cases} s(k), & k \in E, \\ 0_L, & k \notin E. \end{cases}$$

By $FSS(X)$ we denote the set of all L-fuzzy soft sets in a set X (Maji and et al, 2001). For the operations with fuzzy soft sets, see (Aygünoğlu and Aygün, 2009).

We define the dual pair of semirings $(\mathscr{R},\mathscr{R}^*)$ that transform L-fuzzy soft sets into $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets. We use the following notation. For arbitrarily $s \in L^K$ and $E \subseteq K$, the object s_E is defined by

$$s_E : K \to L, \quad \forall e \in K, \quad s_E(e) = \begin{cases} s(e), & e \in E, \\ 0_L, & e \in K \setminus E. \end{cases}$$

For an element $\alpha \in L$, by $\underline{\alpha}$ we denote the constant function $K \to L$ with the only value α. It should be mentioned that because s_E are functions $K \to L$, $s_E = t_F$ iff $s_E(e) = t_F(e)$ for arbitrary $e \in K$. We set

$$R = \{s_E : s \in L^K, E \subseteq K\},$$

and in the set R we define two structures:

1. Let $\mathscr{R} = (R,+,\times,0,1)$ be defined by
   (a) $s_E + t_F := (s_E \vee t_F)_{E\cup F} \in R$,
   (b) $s_E \times t_F = (s_E \otimes t_F)_{E\cup F}$. It should be observed that in that case $(s_E \otimes t_F)_{E\cup F} = (s_E \otimes t_F)_{E\cap F}$ holds,
   (c) $0 = \underline{0_L}_K$, $1 = \underline{1_L}_K$,
2. Let $\mathscr{R}^* = (R,+^*,\times^*,0^*,1^*)$ be defined by
   (a) $s_E +^* t_F = (s_E \wedge t_F)_{E\cup F}$. It should be observed that in that case $(s_E \wedge t_F)_{E\cup F} = (s_E \wedge t_F)_{E\cap F}$ holds,
   (b) $s_E \times^* t_F = (s_E \oplus t_F)_{E\cup F} \in R$,
   (c) $0^* = 1, 1^* = 0$.
3. The mapping $\neg_R : \mathscr{R} \to \mathscr{R}^*$ is defined by

   $$s_E \in R, \quad \neg_R(s_E) := (\neg(s_E))_K \in R,$$

   where $\neg$ is the negation in L.

It is easy to see that the operations $+,+^*$ can be extended to the complete operations $\sum,\sum^*$ by

$$(\sum_{s_E\in S} s_E) = (\bigvee_{s_E\in S} s_E)_{\bigcup_{s_E\in S} E},$$

$$(\sum_{s_E\in S}^{*} s_E) = (\bigwedge_{s_E\in S} s_E)_{\bigcup_{s_E\in S} E} = (\bigwedge_{s_E\in S} s_E)_{\bigcap_{s_E\in S} E},$$

where $S \subseteq R$.

Lemma 2. *$(\mathscr{R},\mathscr{R}^*)$ is the dual pair of semirings with involutive isomorphism $\neg_{\mathscr{R}}$.*

Proof is only a technical verification of properties of $(\mathscr{R},\mathscr{R}^*)$ from Definition 2 and due to the limited scope of the paper it will be omitted. ∎

Proposition 3 (Transformation of L-fuzzy soft sets onto $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets)**.** *Let L be a complete MV-algebra. In the set $FSS(X)$ of all L-fuzzy soft sets, the union $\cup$, intersection $\cap$ and the complement $\neg$ can be defined so that there exist homomorphisms*

$$\Lambda : (FSS(X),\cup,\cap,\neg) \to ((\mathscr{R},\mathscr{R}^*)^X,\sqcup,\sqcap,\neg),$$
$$\Lambda^{-1} : ((\mathscr{R},\mathscr{R}^*)^X,\sqcup,\sqcap,\neg) \to (FSS(X),\cup,\cap,\neg),$$

such that $\Lambda.\Lambda^{-1}.\Lambda = \Lambda$.

Sketch of the proof. We define the mapping $\Lambda : FSS(X) \to (\mathscr{R},\mathscr{R}^*)^X$ by

$$(E,s) \in FSS(X), \quad \Lambda(E,s) : X \to R,$$
$$x \in X, \quad \Lambda(E,s)(x) = s^x_E \in R, \text{ where}$$
$$s^x \in L^K, \quad s^x(k) := s(k)(x), \forall k \in K. \qquad (1)$$

To define the mapping $\Lambda^{-1} : (\mathscr{R},\mathscr{R}^*)^X \to FSS(X)$, let $f \in (\mathscr{R},\mathscr{R}^*)^X$. According to the definition of R, for $x \in X$ we have $f(x) = f^x_{E_x} \in R$, where $f^x \in L^K, E_x \subseteq K$.

The mapping $\Lambda^{-1} : (\mathscr{R},\mathscr{R}^*)^X \to FSS(X)$ is defined by

$$\Lambda^{-1}(f) = (E,s) \in FSS(X),$$
$$E = \bigcap_{x\in X} E_x, \quad s : K \to L^X,$$
$$k \in K, x \in X, s(k)(x) = \begin{cases} f^x(k), & k \in E \subseteq E_x, \\ 0_L, & k \notin E. \end{cases} \quad (2)$$

It is easy to see that $\Lambda^{-1}.\Lambda = id_{FSS(X)}$ and it follows that $\Lambda : FSS(X) \hookrightarrow R^X$ in the injective map. The mapping Λ represents the transformation of MV-valued fuzzy soft sets onto $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets and, conversely, Λ^{-1} is the inverse transformation of $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets onto MV-valued fuzzy soft sets. Using these transformation and inverse transformation we can define basic operations with MV-valued fuzzy soft sets, such as union, intersection or complement in such a way that the resulting algebras of MV-valued fuzzy soft sets and $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets are isomorphic.

For example, we present the constructions of the operations $\cup$ and $\cap$ in $FSS(X)$. Let $(E,s),(F,t) \in FSS(X)$ and let $f = \Lambda(E,s), g = \Lambda(F,t) \in R^X$. Therefore, according to (1), for $x \in X$,

$$f(x) = \Lambda(E,s)(x) = s^x_E, \quad g(x) = \Lambda(F,t)(x) = t^x_F.$$

Using the operation $\sqcup$ in $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets from Definition 3, we can define the operation $\cup$ by

$$(E,s) \cup (F,t) := \Lambda^{-1}(\Lambda(E,s) \sqcup \Lambda(F,t)) = \Lambda^{-1}(f \sqcup g), \quad (3)$$

where according to Definition 3, for $x \in X$ we have

$$(f \sqcup g)(x) = f(x) + g(x) = s^x_E + t^x_F = (s^x_E \vee t^x_F)_{E\cup F} : K \to L.$$

Using the definition (2) of the inverse transformation Λ^{-1}, we obtain

$$\Lambda^{-1}(f \sqcup g) = (E \cup F, q), \quad q : K \to L^X \text{ is such that}$$
$$q(k)(x) = (s^x_E \vee t^x_F)_{E\cup F}(k) = \begin{cases} s(k) \vee t(k). & k \in E \cap F, \\ s(k), & k \in E \setminus F, \\ t(k), & k \in F \setminus E, \\ 0_L, & k \notin E \cup F. \end{cases}$$

The operation $\cap$ can be calculated in a similar way, that is,

$$(E,s) \cap (F,t) = \Lambda^{-1}(\Lambda(E,s) \sqcap \Lambda(F,t)) = \Lambda^{-1}(f \sqcap g) = (G,h),$$
$$x \in X, \quad (f \sqcap g)(x) = f(x) +^* g(x) = s^x_E +^* t^x_F = (s^x_E \wedge t^x_F)_{E\cup F} : K \to L.$$

Therefore, we obtain

$$(E,s) \cap (F,t) = (E \cup F, h), h : K \to L^X \text{ is such that}$$
$$h(k)(x) = (s^x_E \wedge t^x_F)_{E\cup F}(k) = \begin{cases} s(s) \wedge t(k), & k \in E \cap F \\ 0_L, & s \notin E \cap F. \end{cases}$$

Moreover, Λ is the homomorphism of algebras. In fact, it should be observed that for an arbitrary $f \in R^X$ such that it satisfies the following condition (*)

$$(\forall x \in X)(f(x) = s^x_{E_x} \implies E_x = E), \quad (4)$$

the identity $\Lambda.\Lambda^{-1}(f) = f$ also holds. It is easy to see that for arbitrary $(E,s) \in FSS(X)$ this condition (4) is satisfied for $f = \Lambda(E,s) \in R^X$ and it follows that $\Lambda.\Lambda^{-1}.\Lambda(E,s) = \Lambda(E,s)$. Then, for example, for $(E,s),(F,t) \in FSS(X)$, using identity (3), we have

$$\Lambda((E,s) \cup (F,t)) = \Lambda(\Lambda^{-1}(\Lambda(E,s) \sqcup \Lambda(F,t))) = \Lambda\Lambda^{-1}(\Lambda(E,s) \sqcup \Lambda(F,t)) = \Lambda(E,s) \sqcup \Lambda(F,t).$$

Analogously, it can be proven that Λ is a homomorphism for other operations $\cap, \neg$. ■

It should be mentioned that the basic operations on the set $FSS(X)$ defined in Proposition 3 correspond to the operations defined in (Aygünoğlu and Aygün, 2009).

3.2 *MV*-valued Neutrosophic Fuzzy Sets

Recall (James and Mathew, 2021) that a neutrosophic L-fuzzy set is mapping $X \to L^3$. By $NFS(X)$ we denote the set of all neutrosophic L-fuzzy sets in a set X. Operations with neutrosophic L-fuzzy sets f,g, where $f(x) = (\alpha_{x,1}, \alpha_{x,2}, \alpha_{x,3})$, $g(x) = (\beta_{x,1}, \beta_{x,2}, \beta_{x,3})$ for $x \in X$, are defined by

$$f \cup g(x) = (\alpha_{x,1} \vee \beta_{x,1}, \alpha_{x,2} \vee \beta_{x,2}, \alpha_{x,3} \wedge \beta_{x,3}),$$
$$(\neg f)(x) = (\alpha_{x,3}, \neg\alpha_{x,2}, \alpha_{x,1}),$$
$$f \cap g(x) = (\alpha_{x,1} \wedge \beta_{x,1}, \alpha_{x,2} \wedge \beta_{x,2}, \alpha_{x,3} \vee \beta_{x,3}).$$

We define the dual pair of semirings $(\mathscr{R},\mathscr{R}^*)$ that transforms neutrosophic L-fuzzy sets into $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets. Let $R = L^3$.

1. The semiring $\mathscr{R} = (R, +, \times, \mathbf{0}, \mathbf{1})$ is defined by
   (a) $(\alpha,\beta,\gamma) + (\alpha_1,\beta_1,\gamma_1) := (\alpha \vee \alpha_1, \beta \vee \beta_1, \gamma \wedge \gamma_1)$,
   (b) $(\alpha,\beta,\gamma) \times_1 (\alpha_1,\beta_1,\gamma_1) := (\alpha \otimes \alpha_1, \beta \otimes \beta_1, \gamma \oplus \gamma_1)$,
   (c) $\mathbf{0} = (0_L, 0_L, 1_L), \quad \mathbf{1} = (1_L, 1_L, 0_L)$,
2. The semiring $\mathscr{R}^* = (R, +^*, \times^*, \mathbf{0}^*, \mathbf{1}^*)$ is defined by
   (a) $(\alpha,\beta,\gamma) +^* (\alpha_1,\beta_1,\gamma_1) := (\alpha \wedge \alpha_1, \beta \wedge \beta_1, \gamma \vee \gamma_1)$,

(b) $(\alpha,\beta,\gamma)\times^*(\alpha_1,\beta_1,\gamma_1) := (\alpha\oplus\alpha_1,\beta\oplus\beta_1,\gamma\otimes\gamma_1)$,

(c) $\mathbf{0}^* = (1_L,1_L,0_L),\quad \mathbf{1}^* = (0_L,0_L,1_L)$.

Let $\neg_R : \mathscr{R}\to\mathscr{R}^*$ be defined by

$$(\alpha,\beta,\gamma)\in R,\quad \neg_R(\alpha,\beta,\gamma) = (\gamma,\neg\beta,\alpha).$$

Proposition 4 (Transformation of neutrosophic L-fuzzy sets into $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets)**.** *$(\mathscr{R},\mathscr{R}^*)$ is the dual pair of semirings. For arbitrary set X there exists an identity isomorphism $\Lambda = 1_{NSF}$*

$$\Lambda : (NFS(X),\cup,\cap,\neg)\to((\mathscr{R},\mathscr{R}^*)^X,\sqcup,\sqcap,\neg)$$

between the algebra of $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets and the algebra of neutrosophic L-fuzzy sets in a set X.

Proof. The proof is straightforward and will be omitted. ■

3.3 Ω-Level L-Fuzzy Sets

Let L be the complete MV-algebra. If we want to specify how an element $x\in X$ corresponds to the fuzzy set $s\in L^X$, sometimes this value depends on the "observation points" $\alpha\in\Omega$, the points $x\in X$ are observed from. Therefore, instead of a L-fuzzy set $s : X\to L$ we should consider the function $s : X\times\Omega\to L$. For the correct determination of the value $s(x,\alpha)$, it should be assumed that if the positions of two observation points are similar, the observed values should also be close. A similar approach was first discussed by A. Šostak in (Šostak A. et al., 2019), where he introduced the concept of many-level L-fuzzy relations. We show that these fuzzy structures can be transformed into $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets.

Let ρ be a L-fuzzy equivalence relation in Ω. Recall that a L-fuzzy set $f\in L^\Omega$ is called ρ-extensional, if

$$\forall\alpha,\beta\in\Omega,\quad f(\alpha)\otimes\rho(\alpha,\beta)\le f(\beta).$$

For arbitrary $g\in L^\Omega$, the ρ-extensional hull $\overline{g}$ of g is defined by $\overline{g}(\alpha) = \bigvee_{\beta\in\Omega} g(\beta)\otimes\rho(\beta,\alpha)$. The Ω-level L-fuzzy set is defined in the following definition.

Definition 5. *Let X be a set and let (Ω,ρ) be a set with the L-fuzzy equivalence relation ρ. The Ω-level L-fuzzy set s in a set X is a mapping $s : X\times\Omega\to L$, such that for arbitrary $x\in X$, the mapping $s(x,-)\in L^\Omega$ is ρ-extensional.*

The set of all Ω-level L-fuzzy sets in a set X is denoted by $\Omega(X,\rho)$.

We show that if L is the MV-algebra, Ω-level L-fuzzy sets can also be transformed into $(\mathscr{R},\mathscr{R}^*)$-fuzzy set. In fact, let

$$R = \{f\in L^\Omega : f \text{ is } \rho\text{-extensional}\}\subseteq L^\Omega.$$

1. Let $\mathscr{R} = (R,+,\times,0,1)$ be defined by
   (a) $f+g = f\vee g$, where $\vee$ is the supremum in L^Ω,
   (b) $f\times g = \overline{f\otimes g}$, where $f\otimes g$ is defined pointwise in L^Ω,
   (c) $0(\alpha) = 0_L,\quad 1(\alpha) = 1_L$, for arbitrary $\alpha\in\Omega$,
2. Let $\mathscr{R}^* = (R,+^*,\times^*,0^*,1^*)$ be defined by
   (a) $f+^*g = f\wedge g$,
   (b) $f\times^* g = \neg(\overline{\neg(f\oplus g)})$, where $f\oplus g$ is defined point-wise in L^Ω,
   (c) $0^* = 1,\quad 1^* = 0$,

Let $\neg : \mathscr{R}\to\mathscr{R}^*$ be defined by

$$f\in R,\quad (\neg f)(\alpha) = \neg(f(\alpha)).$$

Proposition 5 (Transformation of Ω-level L-fuzzy sets into $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets)**.** *$(\mathscr{R},\mathscr{R}^*)$ is the dual pair of semirings. In the set $\Omega(X,\rho)$ the operations $\cup,\cap,\neg$ can be defined such that there exists the isomorphism of algebras*

$$\Lambda : (\Omega(X,\rho),\cup,\cap,\neg)\to((\mathscr{R},\mathscr{R}^*)^X,\sqcup,\sqcap,\neg).$$

Sketch of the proof. We use the fact that in the MV-algebra the negation of ρ-extensional mappings is also ρ-extensional. Using the properties of operations in MV-algebra it is easy to prove that the definition of $(\mathscr{R},\mathscr{R}^*)$ is correct.

We define two mappings

$$\Lambda : \Omega(X,\rho)\to(\mathscr{R},\mathscr{R}^*)^X,$$
$$\Lambda^{-1} : (\mathscr{R},\mathscr{R}^*)^X\to\Omega(X,\rho),$$

where for $x\in X,\alpha\in\Omega$,

$$s\in\Omega(X,\rho),\quad \Lambda(s)(x) := s(x,-) : \Omega\to L,$$
$$f\in R^X,\quad \Lambda^{-1}(f)(x,\alpha) := f(x)(\alpha).$$

Λ and Λ^{-1} are mutually inverse mappings, as follows from the following:

$$\Lambda.\Lambda^{-1}(f)(x) = \Lambda(\Lambda^{-1}(f))(x) =$$
$$\Lambda^{-1}(f)(x,-) = f(x),$$
$$\Lambda^{-1}.\Lambda(s)(x,\alpha) = \Lambda^{-1}(\Lambda(s))(x,\alpha) =$$
$$\Lambda(s)(x)(\alpha) = s(x,\alpha).$$

The mapping Λ represents the transformation of Ω-level L-fuzzy sets into $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets and Λ^{-1} is the inverse transformation. Using these transformations, we can define basic operation $\cap$,$\cup$ and $\star$.

For example, if $x\in X,\alpha\in\Omega,s,t\in\Omega(X,\delta)$, the operation $\cup$ on $\Omega(X,\delta)$ can be defined by

$$(s\cup t)(x,\alpha) := \Lambda^{-1}(\Lambda(s)\sqcup\Lambda(t))(x,\alpha) =$$
$$(\Lambda(s)\sqcup\Lambda(t))(x,-)(\alpha) =$$
$$(\Lambda(s)(x)+\Lambda(t)(x))(\alpha) = s(x,\alpha)\vee t(x,\alpha).$$

Furthermore, if $r \in R, s \in R^X$, $r \star s$ also represents a Ω-level L-fuzzy set $t \in \Omega(X,\rho)$ defined by

$$t := \Lambda^{-1}(r \star s)(x,-)(\alpha) = (r \star s)(x)(\alpha) = (r \times s(x))(\alpha) = \overline{(r \otimes s(x))}(\alpha) = \bigvee_{\beta \in \Omega} r(\beta) \otimes s(x,\beta) \otimes \rho(\beta,\alpha).$$

If we want to calculate the external multiplication $\lambda.s$, where $\lambda \in L, s \in \Omega(X,\rho)$, we can take the constant function $\underline{\lambda} \in R$ with the only value λ. In that case,

$$\lambda.s(x,\alpha) = \Lambda^{-1}(\underline{\lambda} \star \Lambda(s))(x,\alpha) = \Lambda^{-1}(\overline{\underline{\lambda} \otimes s(x,-)})(\alpha) = \bigvee_{\beta \in \Omega} \underline{\lambda}(\beta) \otimes s(x,\beta) \otimes \rho(\beta,\alpha) = \lambda \otimes \bigvee_{\beta \in \Omega} s(x,\beta) \otimes \rho(\beta,\alpha) = \lambda \otimes s(x,\alpha).$$

It is easy to see that Λ^{-1} and Λ are isomorphisms of the corresponding structures. ■

4 EXAMPLES OF TRANSFORMATION OF NEW TYPES OF MV-VALUED FUZZY RELATIONS INTO $(\mathscr{R},\mathscr{R}^*)$-FUZZY RELATIONS

Fuzzy type relations are basic structures for approximation of fuzzy sets, including new types of fuzzy sets. In this section we show two examples of transformation of fuzzy relation in new MV-valued types of fuzzy sets into $(\mathscr{R},\mathscr{R}^*)$-relations. In this section $L = (L, \vee, \wedge, \oplus, \otimes, \neg, 0_L, 1_L)$ is a complete MV-algebra.

The notion of the L-fuzzy soft relation was introduced, for example, in (Sut, 2012). We repeat this definition.

Definition 6. *(Sut, 2012) Let X,Y,Z be sets.*

1. *A L-fuzzy soft relation from X to Y (denoted $X \multimap Y$) is the L-fuzzy soft set (E,r) in the Cartesian product $X \times Y$.*
2. *If (E,r) and (F,s), respectively, are L-fuzzy soft relations in $X \times Y$ and $Y \times Z$, respectively, their composition $(F,s) \circ (E,r) = (E \cap F, s \diamond r) : X \multimap Z$, where $s \diamond r : E \cap F \to L^{X \times Z}$ is defined by*
$$(s \diamond r)(k)(x,z) = \bigvee_{y \in Y} r(k)(x,y) \otimes s(k)(y,z).$$
for $k \in E \cap F, x \in X, y \in Z$.
3. *For a set X, $1_X : X \multimap X$ is defined by $1_X = (K,\Delta)$, where $\Delta : K \to L^{X \times X}$ is defined by $\Delta(k)(x,y) = \begin{cases} 1_L, & x = y, \\ 0_L, & x \neq y. \end{cases}$*

The following proposition holds.

Proposition 6 (Transformation of L-fuzzy soft relations into $(\mathscr{R},\mathscr{R}^*)$-fuzzy selations)**.** *Let $(\mathscr{R},\mathscr{R}^*)$ be the dual pair of semirings defined in Section 3.1. There exists the embedding homomorphism*

$$H : \mathbf{FSS} \to \mathbf{Set}_{\mathscr{R}}$$

of the category **FSS** *of sets with L-fuzzy soft relations as morphisms into the Kleisli category* $\mathbf{Set}_{\mathscr{R}}$ *defined in Remark 2.*

Sketch of the proof. Let $(E,r) : X \multimap Y$ be a morphism in the category **FSS**. The embedding functor $H : \mathbf{FSS} \to \mathbf{Set}_{\mathscr{R}}$ is defined by

$$H(X) = X, \quad H(E,r) : X \rightsquigarrow Y,$$
$$(\forall x \in X, y \in Y) H(E,r)(x)(y) = r_E^{xy} \in R,$$
$$r^{xy} : K \to L, \quad r^{xy}(k) = r(k)(x,y), k \in K.$$

For $x,y \in X$, we have

$$H(1_X)(x)(y) = H(K,\Delta)(x)(y) = \Delta_K^{xy} = \eta_X(x)(y) = 1_{H(X)}(x)(y).$$

For illustration, we show that H respects the composition of the morphisms. In fact, let $(E,r) : X \multimap Y$ and $(F,s) : Y \multimap Z$ be morphisms in **FSS**. For $x \in X, z \in Z$ we obtain the following:

$$H((F,s) \circ (E,r))(x)(z) = H(F \cap E, s \diamond r)(x)(z) = (s \diamond r)_{E \cap F}^{xz} : K \to L,$$
$$k \in E \cap F, \quad (s \diamond r)_{E \cap F}^{xz}(k) = \bigvee_{y \in Y} r_E^{xy}(k) \otimes s_F^{yz}(k) =$$
$$(\bigvee_{y \in Y} (r_E^{xy} \otimes s_F^{yz})_{E \cap F})_{E \cap F} = (\sum_{y \in Y}^{\mathscr{R}} r_E^{xy} \times s_F^{yz})(k) =$$
$$\sum_{y \in Y}^{\mathscr{R}} H(E,r)(x)(y) \times H(F,s)(y)(z) = (H(F,s) \Diamond H(E,r))(x)(z).$$

Therefore, H is the functor and it is easy to see that H is the embedding.

■

Neutrosophic fuzzy relations were introduced, for example, in (Salama et al., 2014).

Definition 7. *1. Neutrosophic L-fuzzy relation from X to Y (also denoted by $X \multimap Y$) are neutrosophic L-fuzzy sets in $X \times Y$.*

2. *For neutrosophic L-fuzzy relations $r : X \multimap Y$ and $s : Y \multimap Z$, their composition $s \circ r : X \multimap Z$ is defined by*

$$r(x,y) = (\alpha^1_{x,y}, \alpha^2_{x,y}, \alpha^3_{x,y}),$$
$$s(y,z) = (\beta^1_{y,z}, \beta^2_{y,z}, \beta^3_{y,z}),$$
$$(s \circ r)(x,z) =$$
$$\left(\bigvee_{y \in Y} \alpha^1_{x,y} \otimes \beta^1_{y,z}, \bigvee_{y \in Y} \alpha^2_{x,y} \otimes \beta^2_{y,z}, \bigwedge_{y \in Y} \alpha^3_{x,y} \oplus \beta^3_{y,z} \right).$$

3. *For a set X, the unit morphisms $1_X : X \multimap X$ are defined by*

$$1_X(x,y) = \begin{cases} (1_L, 1_L, 0_L), & x = y, \\ (0_L, 0_L, 1_L), & x \neq y. \end{cases}$$

Proposition 7 (Transformation of neutrosophic L-fuzzy relations into $(\mathscr{R},\mathscr{R}^*)$-fuzzy relations)**.** *Let $(\mathscr{R},\mathscr{R}^*)$ be the dual pair of semirings defined in Section 3.2. There exists the identity isomorphic functor $H = 1_{\mathbf{NSF}}$,*

$$H : \mathbf{NFS} \to \mathbf{Set}_{\mathscr{R}}$$

between the category **NFS** *of sets with neutrosophic L-fuzzy relations as morphisms and the Kleisli category* $\mathbf{Set}_{\mathscr{R}}$ *defined in Remark 2.*

The **proof** of this proposition is straightforward and will be omitted.

■

An analogous situation concerns Ω-level L-fuzzy sets. Ω-level L-fuzzy relations are defined in the following definition.

Definition 8. *1. Ω-level L-fuzzy relations from X to Y (denoted by $X \multimap Y$) are Ω-level L-fuzzy sets in a set $X \times Y$.*

2. *For Ω-level L-fuzzy relations $r : X \multimap Y$ and $s : Y \multimap Z$, their composition $s \circ r : X \multimap Z$ is defined by*

$$(s \circ r)(x,y,\alpha) =$$
$$\bigvee_{b \in \Omega} \bigvee_{y \in Y} r(x,y,\beta) \otimes s(y,z,\beta) \otimes \rho(\beta,\alpha).$$

3. *For a set X, the unit morphism $1_X : X \mapsto X$ is defined by $1_X(x,y,\alpha) = \eta_X(x)(y)$.*

The following proposition holds, where L is again the complete MV-algebra.

Proposition 8 (Transformation of Ω-level L-fuzzy relations into $(\mathscr{R},\mathscr{R}^*)$-fuzzy relations)**.** *Let $(\mathscr{R},\mathscr{R}^*)$ be the dual pair of semirings defined in Section 3.3. There exists the isomorphic functor*

$$H : \Omega(\rho) \to \mathbf{Set}_{\mathscr{R}}$$

between the category $\Omega(\delta)$ of sets with Ω-level L-fuzzy relations as morphisms and the Kleisli category $\mathbf{Set}_{\mathscr{R}}$ *defined in Remark 2*

The technical proof will be omitted.

5 EXAMPLES OF APPROXIMATIONS OF NEW TYPES OF MV-VALUED FUZZY SETS

Basic operations with classic L-fuzzy sets certainly include approximation of fuzzy sets by fuzzy relations. This operation makes it possible to transform fuzzy sets into a simpler form, and thus facilitate the work with fuzzy sets modified in this way. The advantage of this operation is also that in some cases a reverse process can also be defined, which restores the original fuzzy set from the simplified fuzzy set. Typical examples of approximations using fuzzy relations are, for example, rough fuzzy sets or F-transform operations. It is therefore natural to deal with the issue of approximations for new types of fuzzy set. For these purposes, we can again use the transformation of new fuzzy structures into $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets and the inverse transformation back to the original types of fuzzy sets. This can be done in the following steps:

Step 1 Using the transformation morphisms Λ from Section 3, we transform new types of MV-valued fuzzy sets f into $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets, that is, we need to calculate $\Lambda(f) \in (\mathscr{R},\mathscr{R}^*)^X$,

Step 2 Using the transformation functors H from Section 4, we transform new types of MV-valued fuzzy relations Q into morphisms $H(Q) : X \rightsquigarrow Y$ (that is, $H(Q) : X \to R^Y$) in Kleisli category $\mathbf{Set}_{\mathscr{R}}$ (i.e., $(\mathscr{R},\mathscr{R}^*)$-fuzzy relations).

Step 3 Calculate the upper approximation $H(Q)^{\uparrow}(\Lambda(f))$ of $\Lambda(f)$ by the $(\mathscr{R},\mathscr{R}^*)$-relation $H(Q)$, where $H(Q)^{\uparrow}$ (and similarly $H(Q)^{\downarrow}$) is defined in a way similar to the upper and lower approximations defined for classical L-fuzzy sets (see Definition 9.

Step 4 Using the inverse transformations Λ^{-1} from Section 3, for arbitrary new MV-valued fuzzy set f in a set X we obtain new MV-valued fuzzy sets $Q^{\uparrow}(f)$ and $Q^{\downarrow}(f)$ in the set Y that are approximations of f, that is,

$$Q^{\uparrow}(f) := \Lambda^{-1}(H(Q)^{\uparrow}(\Lambda(f))),$$
$$Q^{\downarrow}(f) := \Lambda^{-1}(H(Q)^{\downarrow}(\Lambda(f))).$$

Therefore, to apply the above-mentioned procedure, we need to define the notion of the upper and lower approximations of $(\mathscr{R},\mathscr{R}^*)$ fuzzy sets by $(\mathscr{R},\mathscr{R}^*)$-fuzzy relations. This is done in the following definition, and it represents, in fact, an analogy of these notions defined for classical L-fuzzy sets.

Because this construction uses the composition of $(\mathscr{R},\mathscr{R}^*)$-relations, we need to distinguish between two types of these approximations, depending on the type of Kleisli composition $\Diamond$ or $\Diamond^*$ we use.

Definition 9. *Let $(\mathscr{R},\mathscr{R}^*)$ be the dual pair of semirings with the involutive isomorphism $\neg$ and let $S : X \rightsquigarrow Y$ be a $(\mathscr{R},\mathscr{R}^*)$-relation.*

1. *The upper approximation defined by the $(\mathscr{R},\mathscr{R}^*)$-relation S is the mapping $S^{\uparrow} : R^X \to R^Y$, such that*
$$S^{\uparrow} = S\Diamond 1_{R^X}.$$
2. *The lower approximation defined by the $(\mathscr{R},\mathscr{R}^*)$-relation S is the mapping $S^{\downarrow} : R^X \to R^Y$, such that*
$$S^{\downarrow} = (\neg S)\Diamond^* 1_{R^X}.$$

For illustration, the following basic properties of these approximation operators can be simply proven.

Lemma 3. *Let $(\mathscr{R},\mathscr{R}^*)$ be the dual pair of semirigs with the involutive isomorphism $\neg$ and let $S : X \rightsquigarrow Y$ be a $(\mathscr{R},\mathscr{R}^*)$-relation. Let $a \in \mathscr{R}$ and $s,t,s_i \in \mathscr{R}^X$, $i \in I$.*

1. $S^{\uparrow}(\bigsqcup_{i\in I} s_i) = \bigsqcup_{i\in I} S^{\uparrow}(s_i), S^{\downarrow}(\bigsqcap_{i\in I} s_i) = \bigsqcap_{i\in I} S^{\downarrow}(s_i)$,
2. $S^{\uparrow}(a \star s) = a \star S^{\uparrow}(s)$, $S^{\downarrow}(a \star^* s) = a \star^* S^{\downarrow}(s)$,
3. $s \le t \Rightarrow S^{\downarrow}(s) \le S^{\downarrow}(t), S^{\uparrow}(s) \le S^{\uparrow}(t)$,
4. $S^{\downarrow}(s) = \neg(S^{\uparrow}(\neg s))$, $S^{\uparrow}(s) = \neg(S^{\downarrow}(\neg s))$.
5. *Let $R : X \rightsquigarrow Y$ and $S : Y \rightsquigarrow Z$ be $(\mathscr{R},\mathscr{R}^*)$-relations. We have*
$$S^{\uparrow}.R^{\uparrow} = (S\Diamond R)^{\uparrow}, \quad S^{\downarrow}.R^{\downarrow} = (S\Diamond R)^{\downarrow}.$$

In the next three subsections we present examples of Steps 1-4 for neutrosophic, fuzzy soft sets, and Ω-level MV-valued fuzzy sets, respectively. These examples illustrate the possibilities of applying the classical theory of L-fuzzy sets to new types of fuzzy sets, without having to derive this theory or prove its properties for individual types of new fuzzy sets.

5.1 Approximations of Neutrosophic L-Fuzzy Sets

In Sections 3 and 4 we show that for L= complete MV-algebra, neutrosophic L-fuzzy sets and neutrosophic L-fuzzy relation can be equivalently expressed as $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets and $(\mathscr{R},\mathscr{R}^*)$-fuzzy relations, that is, according to Proposition 4, the algebra of neutrosophic L-fuzzy sets in a set X is isomorphic to the algebra of $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets in a set X, where $(\mathscr{R},\mathscr{R}^*)$ was introduced in Section 3. According to Proposition 7, the neutrosophic L-fuzzy relation $Q : X \multimap Y$ can be equivalently defined as the $(\mathscr{R},\mathscr{R}^*)$-relation $Q : X \to R^Y$.

Let us consider a neutrosophic MV-fuzzy set f and neutrosophic MV-fuzzy relation $Q : X \multimap Y$ from X to Y. According to the definition of neutrosophic fuzzy sets, for $x,y \in X$ we set $f(x) = (\alpha_x,\beta_x,\gamma_x)$ and $Q(x)(y) = (\alpha_{xy},\beta_{xy},\gamma_{xy})$. We show how Steps 1-4 are realized.

Step 1, Step 2

These are the trivial steps because according to Proposition 4 and Proposition 7, Λ is the identity mapping and H is the identity functor. Therefore, $\Lambda(f) = f$, $H(Q) = Q$.

Step 3

According to the definition of operations in $(\mathscr{R},\mathscr{R}^*)$ in Section 3, the upper and lower approximations $Q^{\uparrow}(s), Q^{\downarrow}(s)$ of s in a set Y are defined by

$$\forall y \in Y, \quad Q^{\uparrow}(s)(y) = \sum_{x\in X}^{\mathscr{R}} s(x) \times Q(x)(y) =$$
$$\left(\bigvee_{x\in X} \alpha_x \otimes \alpha_{xy}. \bigvee_{x\in X} \beta_x \otimes \beta_{xy}, \bigwedge_{x\in X} \gamma_x \oplus \gamma_{xy} \right),$$

$$\forall y \in Y, \quad Q^{\downarrow}(s)(y) = \sum_{x\in X}^{\mathscr{R}^*} s(x) \times^* \neg Q(x)(y) =$$
$$\left(\bigwedge_{x\in X} \alpha_x \oplus \gamma_{xy}, \bigwedge_{x\in X} \beta_x \oplus \neg\beta_{xy}, \bigvee_{x\in X} \gamma_x \otimes \alpha_{xy} \right).$$

Step 4

This is also trivial step, because Λ^{-1} is the identity map. Hence, the results of this step are idetical to the results from Step 3.

5.2 Approximations of L-Fuzzy Soft Sets

We show how the Steps 1-4 can be realized for L-fuzzy soft sets, where L is the complete MV-algebra. In that case the situation is more complicated, because Λ is not the identity map and H is not the identity functor.

Let K be the set of criteria and let X be a set. We use the notation form Section 3.1. and Section 4. Let us consider a L-fuzzy soft set $f = (E,s)$ and L-fuzzy soft relation $Q = (F,q) : X \multimap Y$.

Step 1

According to Proposition 3, the transformation of f into $(\mathscr{R},\mathscr{R}^*)$-fuzzy set is defined by $\Lambda(f) = \Lambda(E,s) : X \to R$, where for $x \in X$ we have

$$\Lambda(E,s)(x) = s_E^x \in R,$$
$$k \in K, \quad s_E^x(k) = \begin{cases} s(k)(x), & k \in E, \\ 0_L, & k \in K \setminus E. \end{cases}$$

Step 2
Recall that a L-fuzzy soft relation $Q = (F,q)$ is a L-fuzzy soft set in the set $X \times Y$. That is, q is a mapping $K \to L^{X\times Y}$, such that $q(k)(x,y) = 0_L$, if $k \in K \setminus E$. According to Proposition 6 it can be transformed into the $(\mathscr{R},\mathscr{R}^*)$-relation $H(Q) = H(F,q) : X \to R^Y$, such that for $x \in X, y \in Y$,

$$H(F,q)(x)(y) = q_F^{xy} \in R,$$
$$q^{xy}(k) := \begin{cases} q(k)(x,y) \in L, & k \in F, \\ 0_L, & k \in K \setminus F. \end{cases}$$

Step 3
We need to calculate the upper and lower approximations $H(F,q)^{\uparrow}(\Lambda(E,s))$ and $H(F,q)^{\downarrow}(\Lambda(E,s))$ of $\Lambda(E,s)$ by the $(\mathscr{R},\mathscr{R}^*)$-relation $H(F,q)$ according to Definition 9. We have

$$H(F,q)^{\uparrow}(\Lambda(E,s))(x)(k) =$$
$$\sum_{z\in X}^{\mathscr{R}} (\Lambda(E,s)(z) \times H(F,q)(z)(x))(k) =$$
$$\sum_{z\in X}^{\mathscr{R}} (s_E^z \times q_F^{zx})_{E\cap F}(k) =$$
$$\begin{cases} \bigvee_{z\in X}(s_E^z(k) \otimes q_F^{zx}(k)), & k \in E\cap F \\ 0_L, & k \notin E\cap F, \end{cases} =$$
$$\begin{cases} \bigvee_{z\in X}(s(k)(z) \otimes q(k)(z)(x)), & k \in E\cap F \\ 0_L, & k \notin E\cap F, \end{cases}$$

Analogically, we obtain the lower approximation:

$$H(F,q)^{\downarrow}(\Lambda(E,s))(x)(k) =$$
$$((\neg_R H(F,q))\Diamond^* 1_{R^X})(\Lambda(E,s))(x)(k) =$$
$$\sum_{y\in X}^{\mathscr{R}^*} (\Lambda(E,s))(z) \times^* \neg_R H(F,q)(z)(x))(k) =$$
$$\sum_{z\in X}^{\mathscr{R}^*} (s_E^z \times^* \neg_R q_F^{zx})_{E\cap F}(k) =$$
$$\sum_{z\in X}^{\mathscr{R}^*} (s_E^z \oplus (\neg(q_F^{zx}))_{E\cup F}(k) =$$
$$\begin{cases} \bigwedge_{z\in X}(s_E^z(k) \oplus (\neg q_F^{zx}(k)), & k \in E\cup F, \\ 0_L, & k \notin E\cup F \end{cases} =$$
$$\begin{cases} \bigwedge_{z\in X}(s(k)(z) \oplus \neg q(k)(z,x)), & k \in E\cup F, \\ 0_L, & k \notin E\cup F. \end{cases}$$

Step 4
We use the inverse transformation Λ^{-1} from Proposition 3 to obtain the L-fuzzy soft sets $(F,q)^{\uparrow}(E,s) := \Lambda^{-1}(H(F,q)^{\uparrow}(\Lambda(E,s)))$ and $(F,q)^{\downarrow}(E,s) := \Lambda^{-1}(H(F,q)^{\downarrow}(\Lambda(E,s)))$ We obtain the following formulas for upper and lower approximations of L-fuzzy soft sets by L-fuzzy soft relations.

$$(F,q)^{\uparrow}(E,s)(k)(x) =$$
$$\begin{cases} \bigvee_{z\in X} s(k)(z) \otimes q(k)(z)(x), & k \in E\cap F, \\ 0_L, & k \notin E\cap F \end{cases}$$
$$(F,q)^{\downarrow}(E,s)(k)(s) =$$
$$\begin{cases} \bigwedge_{z\in X} s(k)(z) \oplus \neg q(k)(z,x), & k \in E\cup F, \\ 0_L, & k \notin E\cup F \end{cases}.$$

5.3 Approximations of Ω-Level L-Fuzzy Soft Sets

We show how the steps 1-4 can be realizded for Ω-valued L-fuzzy soft sets. We use the notation of Sections 3.3 and 4. Let ρ be the L-fuzzy equivalence relation in the set Ω and let $f \in \Omega(X,\rho)$, $f : X \times \Omega \to L$ be an Ω-level L-fuzzy sets and let $Q : X \multimap Y$ be a Ω-level L fuzzy relation, that is, $Q : X \times Y \times \Omega \to L$.
Step 1
According to Proposition 5, the transformation $\Lambda(f)$ of f into the $(\mathscr{R},\mathscr{R}^*)$-fuzzy set $X \to R$ is defined by

$$x \in X, \quad \Lambda(f)(x) = f(x,-) \in R.$$

Step 2
The transformation of Ω-level L-fuzzy relation Q is defined by Proposition 8 by

$$H(Q) : X \to R^Y,$$
$$x \in X, y \in Y, \quad H(Q)(x)(y) := Q(x,y,-) \in R.$$

Step 3
According to definition of operations in $(\mathscr{R},\mathscr{R}^*)$ in Section 3.3, the upper and lower approximations of $\Lambda(f)$ by $H(Q)$ are defined for $y \in Y, \alpha \in \Omega$ by

$$H(Q)^{\uparrow}(\Lambda(f))(y)(\alpha) =$$
$$\left(\sum_{x\in X}^{\mathscr{R}} (\Lambda(f)(x) \times H(Q)(x,y)\right)(\alpha) =$$
$$\bigvee_{\beta\in\Omega} \bigvee_{x\in X} f(x,\beta) \otimes Q(x,y,\beta) \otimes \rho(\beta,\alpha), \quad (5)$$
$$H(Q)^{\downarrow}(\Lambda(f))(y)(\alpha) =$$
$$\left(\sum_{x\in X}^{\mathscr{R}^*} (\Lambda(f)(x) \times^* \neg H(Q)(x,y)\right)(\alpha) =$$
$$\bigwedge_{x\in X} \neg(\overline{\neg(s(x,-) \oplus \neg Q(x,y,-)})(\alpha) =$$
$$\bigwedge_{x\in X} \neg \left(\bigvee_{\beta\in\Omega} \neg s(x,\beta) \otimes Q(x,y,\beta) \otimes \rho(\alpha,\beta) \right) =$$
$$\bigwedge_{x\in X} \bigwedge_{\beta\in\Omega} s(x,\beta) \oplus \neg Q(x,y,\beta) \oplus \neg\rho(\alpha,\eta). \quad (6)$$

Step 4
According to Proposition 5, the inverse transformation Λ^{-1} is define such that

$$Q^{\uparrow}(f)(y,\alpha) = \Lambda^{-1}(H(Q)^{\uparrow}(\Lambda(f)))(y,\alpha) = H(Q)^{\uparrow}(\Lambda(f))(y)(\alpha) = (5),$$
$$Q^{\downarrow}(f)(y,\alpha) = \Lambda^{-1}(H(Q)^{\downarrow}(\Lambda(f)))(y,\alpha) = H(Q)^{\downarrow}(\Lambda(f))(y)(\alpha) = (6).$$

6 CONCLUSIONS

In this short paper, we show that some types of new L-fuzzy sets with values in complete MV -algebras L (such as neutrosophic L -fuzzy sets, L -fuzzy soft sets, intuitionistic L-fuzzy sets or multi-level L-fuzzy sets and some others) can be transformed into the so-called $(\mathscr{R},\mathscr{R}^*)$-fuzzy sets, where $(\mathscr{R},\mathscr{R}^*)$ is a pair of commutative complete idempotent semirings with involutive isomorphisms between them.

Using this value structure $(\mathscr{R},\mathscr{R}^*)$, the theories of these new L-fuzzy sets can be defined in a unified way, without having to prove the properties of this theory for individual types of these new L-fuzzy sets. For illustration, we have shown how the theory of upper and lower approximations can be defined using the corresponding types of new L-fuzzy relations for neutrosophic MV-valued fuzzy sets, MV-valued fuzzy soft sets, and for Ω-level MV-valued fuzzy sets. In a completely analogous way, this theory can also be defined, for example, for intuitionistic MV-valued fuzzy sets or for combinations of these new types of fuzzy sets. Moreover, if we consider some new types of pairs $(\mathscr{R},\mathscr{R}^*)$, we can obtain completely new types of L-fuzzy sets.

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