Lyapunov Function Computation for Linear Switched Systems: Comparison of SDP and LP Approaches

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Abstract: For a switched system, of which each subsystem is linear, the exponential stability of the origin is equivalent to the existence of a common Lyapunov function (CLF) for all the subsystems. A popular approach to search for the latter is by means of solving a linear matrix inequality (LMI) using semidefinite programming (SDP). Another approach is to use linear programming (LP). The contribution of this work is twofold. First, we compare the SDP approach to the LP approach, with and without a certain preconditioning of system matrices. And, second, we present a software tool to visualise the conditions for a CLF. As the problem of investigating the stability of the origin is a very difficult one and sufficient and necessary conditions using the system matrices are only known for exponential stability of planar systems, a tool to visualise the original data in some meaningful form is potentially of great use for the full understanding of the problem.

1 INTRODUCTION

In science and engineering, one often uses ordinary differential equations (ODEs) for modelling, that is, the temporal behaviour of the state variables is quantified through a differential equation of the following form,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{1}$$

If system (1) has an equilibrium point $\mathbf{x}_0$, that is, $\mathbf{f}(\mathbf{x}_0) = 0$, which, without loss of generality, can be assumed to be at the origin, then one often studies the linearised system given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A := D\mathbf{f}(0), \quad \mathbf{x} \in \mathbb{R}^n. \tag{2}$$

According to the Hartman-Grobman theorem (Grüne and Junge, 2016), its solutions have the same qualitative behaviour close to the origin as system (1) whenever the real parts of the eigenvalues of $D\mathbf{f}(0)$ are nonzero; here $D\mathbf{f}(0) \in \mathbb{R}^{n\times n}$ denotes the Jacobian-matrix of $\mathbf{f}$ at the origin.

In different engineering applications, such as power electronics, automotive control, robotics, and air traffic control (Veer and Poulakakis, 2020), it is useful to consider a switched linear system of the following form,

$$\dot{\mathbf{x}} = A_m\mathbf{x}, \quad m \in \mathcal{P}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{3}$$

The switching between the systems $\dot{\mathbf{x}} = A_m\mathbf{x}$ is modelled through switching signal $\sigma\colon [0,\infty) \to \mathcal{P}$. Moreover, in hybrid systems, switching is due to the interaction between the dynamics of individual continuous-time subsystems and the discrete-time dynamics of the switching (Goebel et al., 2012). Another common source of switched systems is uncertainty quantification in continuous-time systems and the associated differential inclusions (Aubin and Cellina, 1984; Clarke, 1990). In this paper, we will only consider arbitrary switching between a finite number of systems; that is, $\mathcal{P}$ is finite and $\sigma\colon [0,\infty) \to \mathcal{P}$ is arbitrary, except for the (technical) assumption that the number of switching times is finite on any finite time-interval.

The transition matrix for system (2) is $t \mapsto \exp[At]$; that is, a solution starting at $\xi \in \mathbb{R}^n$ at time zero will be at $\exp[At]\xi$ at time t. Correspondingly, the transition matrix for (3) with switching signal σ is

$$t \mapsto \exp[A_{\sigma(t_k)}(t-t_k)] \cdot \exp[A_{\sigma(t_{k-1})}(t_k - t_{k-1})] \cdot \cdots \exp[A_{\sigma(t_1)}(t_2-t_1)] \cdot \exp[A_{\sigma(t_0)}(t_1-t_0)],$$

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where the $t_0, t_1, t_2, \ldots, t_k$ are the switching times and

$$t > t_k > t_{k-1} > \ldots > t_1 > t_0 = 0.$$

Asymptotic stability of system (2) is equivalent to exponential stability; that is, there exist constants $\alpha > 0$ and $c \geq 1$ such that

$$\|\phi(t,\xi)\| \leq c\|\xi\| \exp[-t\alpha] \quad \text{for all } t \geq 0, \tag{4}$$

where $\phi(t,\xi) := \exp[At]\xi$. Note that the origin is asymptotically stable, if and only if the real-parts of all eigenvalues of A are negative. In this case, matrix A is said to be Hurwitz.

The origin is said to be exponentially stable for switched system (3) under arbitrary switching if and only if, for every switching signal σ with a finite number of switching times on every finite interval, there exist constants $\alpha > 0$ and $c \geq 1$ such that inequality (4) holds for

$$\phi_\sigma(t,\xi) := \exp[A_{\sigma(t_k)}(t - t_k)] \cdot \exp[A_{\sigma(t_{k-1})}(t_k - t_{k-1})] \cdots \exp[A_{\sigma(t_1)}(t_2 - t_1)] \cdot \exp[A_{\sigma(t_0)}(t_1 - t_0)]\xi.$$

For planar systems ($n = 2$) and switching only between two different systems ($|\mathcal{P}| = 2$), the exponential stability of the origin can be characterised through matrices A_1 and A_2, where $\mathcal{P} = \{1,2\}$. The origin is exponentially stable if and only if all pairwise convex combinations of A_1, A_2, A_1^{-1}, and A_2^{-1} are Hurwitz (Cohen and Lewkowicz, 1993; Liberzon, 2003). However, unless the matrices A_m have some very special structure, for instance, if they all commute (Agrachev and Liberzon, 2001), determining stability remains an open problem.

A sufficient condition for exponential stability of the origin of (3) is the existence of a quadratic common Lyapunov function (QCLF) for the subsystems. Computing a QCLF numerically by solving the linear matrix inequality (LMI) (Khalil, 2002; Boyd et al., 1994) given by

$$P \succ 0, \quad PA_m + A_m^{\mathrm{T}} P \prec 0, \quad \forall m \in \mathcal{P}, \tag{5}$$

is a frequent approach. Here, $P \succ 0$ means that $P \in \mathbb{R}^{n \times n}$ is symmetric and positive definite . However, even in the case $n = |\mathcal{P}| = 2$ it is possible that an arbitrarily switched system has an exponentially stable equilibrium at the origin, but there does not exist a QCLF for the system. Numerous methods to compute different kinds of common Lyapunov function (CLF) have been proposed in the literature; for example, see (Giesl and Hafstein, 2015). One method is to use linear programming (LP) to parametrise piecewise linear Lyapunov functions (Brayton and Tong, 1979; Brayton and Tong, 1980; Blanchini, 1995; Blanchini and Carabelli, 1994; Blanchini and Miani, 2008; Wang and Michel, 1996; Polanski, 1997; Polanski, 2000; Ohta, 2001; Ohta and Tsuji, 2003; Yfoulis and Shorten, 2004; Andersen et al., 2023), another one is to use semidefinite programming (SDP) to parametrise higher-order polynomial Lyapunov functions (Prajna and Papachristodoulou, 2003; Piccini et al., 2023). Regarding the theoretical foundations on the existence of CLFs of specific kind, see for example (Molchanov and Pyatnitskiy, 1986; Molchanov and Pyatnitskiy, 1989; Goebel et al., 2006; Mason et al., 2006; Mason et al., 2022).

The remainder of the paper is organised as follows. In Section 2, we first introduce the SDP approach and then apply it to an important problem from the literature in Section 2.1. Since we find only a few solutions and often encounter numerical problems, in Section 2.2, we create a large set of switched systems of lower dimension to test the approach. In Section 3, we repeat the investigation of Section 2.2 using the LP approach and compare the results obtained from the two approach. In Section 4, we introduce the LP approach. Our LP approach heavily relies on preconditioning the problem, for which we use a visualisation app that we developed and present it in Section 5. Finally, we conclude the paper in Section 6.

2 NUMERICAL STUDY: SDP

In this section, we investigate the stability of all switched linear systems defined by (3) and all the subsets $\mathcal{P} \neq \emptyset$ of a given superset Q. Specifically, for each switched system, we use SDP to search for a QCLF that solves the LMIs given by

$$P - \varepsilon I_n \succeq 0, \tag{6}$$

$$A_m^T P + PA_m + \varepsilon I_n \preceq 0, \quad \forall m \in \mathcal{P} \subseteq Q. \tag{7}$$

Here, I_n is the n-dimensional identity matrix and $\varepsilon > 0$ is a small constant.

We use two simple tricks to speed up the computation. First, if there is not a solution for $\mathcal{P}$ then there cannot be a solution for any superset $\mathcal{P}^* \supseteq \mathcal{P}$. Second, a necessary condition for a solution to exist for a particular subset $\mathcal{P}$, is that any combination $\widetilde{A} := \sum_{m \in \mathcal{P}} \lambda_m A_m$, $\lambda_m > 0$ for all $m \in \mathcal{P}$, is Hurwitz. The reason for this is that

$$\widetilde{A}^T P + P\widetilde{A} \prec 0 \tag{8}$$

has a solution $P \succ 0$, if and only if $\widetilde{A}$ is Hurwitz, and

$$\widetilde{A}^T P + P\widetilde{A} = \sum_{m \in \mathcal{P}} \lambda_m \left(A_m^T P + PA_m\right). \tag{9}$$

Thus, if $\widetilde{A}$ is not Hurwitz then we know that there cannot be a solution to (6)-(7), because otherwise the left-hand-side of (9) would be negative definite ($\prec$) with

Table 1: Overview of 5D results. PT: Solution reported and passed test, FP: False positive (solution reported but did not pass test), I: Infeasible, O: Numerical issues, lack of progress, and other, and T: Total number of LMIs problems. The effect of not considering supersets of a set $\mathcal{P}$, for which the LMI problem is infeasible, greatly reduces the number of computations, which is reflected in the low numbers reported under T. The tolerance parameter $\varepsilon = 10^{-3}$ appears reasonable and $\varepsilon = 10^{-16}$ does not. However, more solutions are found with $\varepsilon = 10^{-16}$.

Solver	ε	PT	FP	I	O	T
MOSEK		5	0	6	9	20
SDPT3	10^{-3}	13	29	0	4	46
SeDuMi		0	0	1	9	10
MOSEK		8	999	0	0	1007
SDPT3	10^{-16}	18	1	7	30	56
SeDuMi		0	0	0	10	10

the $P \succ 0$ from the solution, which is a contradiction to there not being a solution $P \succ 0$ to (8). Therefore, as a cheap test to eliminate unnecessary computations, for subset $\mathcal{P}$, we first check if $\sum_{m\in\mathcal{P}} A_m$ is Hurwitz, i.e. $\lambda_m = 1$ for all $m \in \mathcal{P}$. If it is not Hurwitz then we do not need to attempt to solve (6)-(7) for this particular $\mathcal{P}$.

We use YALMIP (Löfberg, 2004), a MATLAB (MATLAB, 2022) toolbox, and search for a solution for all possible $\mathcal{P} \subseteq \{1,2,\ldots,20\}$ using the solvers SeDuMi (Sturm, 1999), MOSEK (MOSEK ApS, 2019), and sdpt3 (Toh et al., 1999). All solvers are used with their default parameters and tolerances. We additionally verify that the solution satisfies (5), to identify false positives, that is, whether a solver reports a solution that does not satisfy the constraints.

2.1 Badly Conditioned 5D Systems

First, we study ten five-dimensional systems from (Bakhshande et al., 2020). These switched systems are high-gain observers and the system matrices are very badly conditioned. Computationally, these systems are extremely challenging. However, as such high-gain observers are used in applications with great success, understanding their dynamics is of much interest.

The results can be seen in Table 1. With $\varepsilon = 10^{-3}$, a reasonable value to ensure positive definiteness, SPDT3 finds 13 solutions, MOSEK finds 5, and SeDuMi does not find a solution. None of the solvers reports a solution that does not fulfil (5), that is, a false positive. With $\varepsilon = 10^{-16}$, a value that is close to machine precision, we get unexpected results: SPDT3 finds 18 solutions and MOSEK finds 8, SeDuMi does not find a solution. With the tolerance parameter ε set that unreasonably low, we expected the solvers to deliver $P \approx 0$ as a feasible solution and we certainly did not expect them to find more true solutions than with the reasonable tolerance parameter $\varepsilon = 10^{-3}$. Notably, SeDuMi did not report a single false positive and SDPT3 only 1. More in accordance to our expectation MOSEK reported all LMIs feasible; note that with $\varepsilon = 0$ they are with the trivial solution $P = 0$.

2.2 Planar Systems

We generated twenty linear systems in the plane, constructed as follows. We define vectors

$$\theta = \begin{bmatrix} 0 & \frac{9\pi}{40} & \frac{9\pi}{20} & \frac{27\pi}{40} & \frac{9\pi}{10} \end{bmatrix},$$
$$d = \begin{bmatrix} 1 & 2.1544 & 4.6416 & 10 \end{bmatrix},$$

and the counter-clockwise rotation function

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Vector θ is linearly spaced on $[0, 0.9\pi]$ and d is logarithmically, base 10, spaced on $[1, 10]$. For $j = 1,2,3,4,5$ and $k = 1,2,3,4$, we then define matrices

$$V_j = \begin{bmatrix} R(\theta_j)\begin{bmatrix} 1 \\ 1 \end{bmatrix} & R\left(\theta_j + \frac{\pi}{3}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix}$$
$$\text{and} \quad E_k = \begin{bmatrix} -1 & -d_k \\ d_k & -1 \end{bmatrix}.$$

Finally, we set

$$A_{k+4(j-1)} = V_j E_k V_j^{-1} \tag{10}$$

and consider the switched systems (3) with these A_m and $\emptyset \neq \mathcal{P} \subseteq \mathcal{Q} = \{1,2,\ldots,20\}$.

Note that the eigenvalues of $A_{k+4(j-1)}$ are $\lambda = -1 \pm d_k\sqrt{-1}$ and, thus, the origin is exponentially stable for each of the linear systems with the same rate of attraction. The rotation speed, however, is different for different k. Trajectories for these systems, starting at $(1,1)$, are drawn in Fig. 1.

For some subsets $\emptyset \neq \mathcal{P} \subseteq \mathcal{Q}$, the origin is an exponentially stable equilibrium of the corresponding switched system, for example, for all singe element $\mathcal{P}$; however, for other subsets $\mathcal{P}$ the origin is unstable. The results of our computations are summarised in Table 2. For the reasonable tolerance parameter $\varepsilon = 10^{-3}$, all solvers deliver identical results. We also tried the (unreasonable) tolerance parameter $\varepsilon = 10^{-16}$ that surprisingly delivered more useful solutions for the badly conditioned 5D systems. However, for the planar systems the results are less interesting. MOSEK and SDPT3 report all LMIs as feasible, however, not simply by reporting $P = 0$ as a solution as one might expect for such a low tolerance and, indeed, do find all 1279 solutions. SeDuMi is unaffected by the low tolerance and delivers exactly the

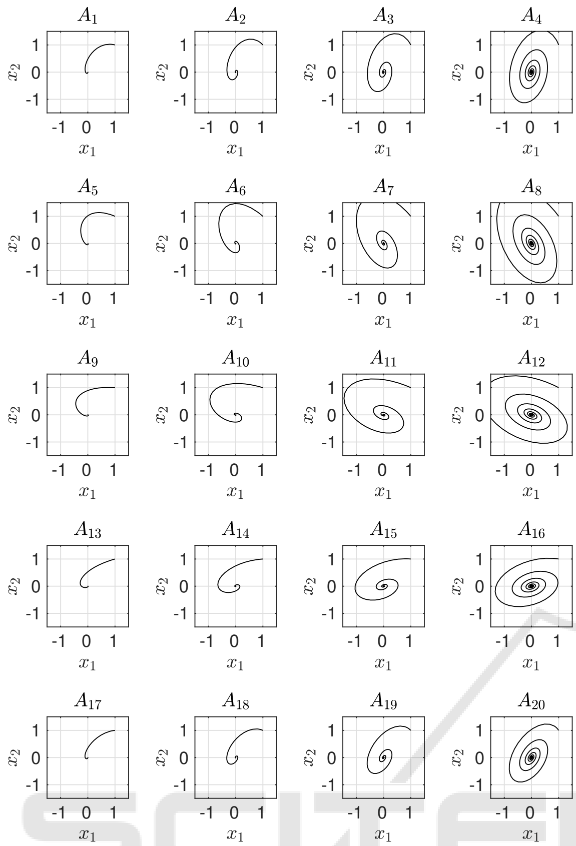


Figure 1: Trajectories of the systems $\dot{\mathbf{x}} = A_m\mathbf{x}$, $m = 1,2,\dots,20$, starting at $(1,1)$. The matrices A_m are defined in (10).

same results as with $\varepsilon = 10^{-3}$. One way to interpret these results, together with the results for the badly conditioned 5D-systems, is that one should use a tolerance that seems unreasonably low, like $\varepsilon = 10^{-16}$, and then verify the reported solutions. Spotting the false positives is computationally very cheap and one might get solutions that are missed with a seemingly more reasonable tolerance parameter.

3 NUMERICAL STUDY: COMPARING LP AND SDP

As discussed in the introduction, the switched linear system (3) with an exponentially stable equilibrium at the origin does not necessarily possess a QCLF. To investigate how limiting QCLFs are in practice, we compared the results from the last section for the planar systems with the LP approach from (Andersen et al., 2023) to search for a common piecewise linear Lyapunov function. The LP problems were solved using Gurobi (Gurobi Optimization, LLC, 2023). At least in theory, there always exists a piecewise linear Lyapunov function for the system (3), if the origin is exponentially stable, cf. (Goebel et al., 2006; Mason et al., 2006; Mason et al., 2022). Thus a priori, we expected that the LP approach will be able to compute CLFs for more subsets $\mathcal{P} \subseteq \mathcal{Q} = \{1,2,\dots,20\}$ than the LMI approach.

Table 3 compares the performance between the two approaches. As we can see for $|\mathcal{P}| > 1$, the LP approach is always able to generate more common piecewise linear Lyapunov function than the SDP approach and the difference is larger, the more elements $\mathcal{P}$ has. Indeed, for $|\mathcal{P}| = 2$, the SDP approach can affirm the stability of 75% of the systems that the LP approach successfully asserts as stable, but this ratio decreases to 50% for $|\mathcal{P}| = 3$ and 33% for $|\mathcal{P}| = 4$; for $|\mathcal{P}| = 8$, the ratio is down to 2% and, for systems with $|\mathcal{P}| > 8$, the SDP approach is not able to affirm the stability of any system.

Comparing the approaches with three-dimensional systems form the appendix we get similar results (see Table 4).

4 PRECONDITIONING FOR THE LP APPROACH

In the LP approach from (Andersen et al., 2023), to search for a piecewise linear CLF for the switched linear system (3), a neighbourhood of the origin must first be triangulated. Then a LP is created, whose variables are the values of the piecewise linear CLF to be parametrised for the system. The number of the constraints of the LP problem is the number of simplices in the triangulation times the dimension. The vertices of the simplices of the triangulation are put on the integer grid $\mathbf{z} \in \mathbb{Z}^n$, $\|\mathbf{z}\|_\infty = K$, where $K \in \mathbb{N}$ is a parameter that determines the resolution, or fineness, of the triangulation. The triangulation with resolution parameter K is denoted $\mathcal{T}_K$. To count the total number of simplices in the triangulation $\mathcal{T}_K$ for a given dimension n, note that $\|\mathbf{z}\|_\infty = K$ has 2^n sides. To see this, note that by fixing one component z_i of $\mathbf{z}$ to K or $-K$ we have fixed the side. Since $\mathbf{z}$ has n components, there are 2^n ways to do this. Each side then has $(2K)^{n-1}$ hypercubes of dimension $n-1$ and each hypercube leads to $(n-1)!$ different simplices. Thus, the number of simplices as a function of dimension n and resolution parameter K is given by

$$\text{Number of simplices} = 2^n(2K)^{n-1}(n-1)!.$$

We can see that the number of simplices grows very fast with the dimension (curse of dimensionality). For a more visual image of the number of simplices needed, see Table 5.

Table 2: Overview of 2D results. PT: Solution reported and passed test, FP: False positive (solution reported but did not pass test), I: Infeasible, and T: Total number of LMIs problems. The tolerance parameter $\varepsilon = 10^{-3}$ appears reasonable and $\varepsilon = 10^{-16}$ does not. Note the effect of the simplification of not considering supersets of a set $\mathcal{P}$, for which the LMI problem is infeasible. With $\varepsilon = 10^{-3}$ only 1,366 LMIs have to be solved instead of $2^{20} - 1 = 1,047,296$, i.e. for every subsets $\mathcal{P} \neq \emptyset$ of $\mathcal{Q}$.

Solver	ε	PT	FP	I	T
MOSEK		1279	0	87	1366
SDPT3	10^{-3}	1279	0	87	1366
SeDuMi		1279	0	87	1366
MOSEK		1279	1,047,296	0	1,048,575
SDPT3	10^{-16}	1279	1,047,296	0	1,048,575
SeDuMi		1279	0	87	1366

Table 3: Ratio, in percentage, of the two-dimensional problems from Section 2.2 successfully solved using SDP to problems successfully solved using LP.

# of matrices	# sys. solved using SDP	# sys. solved using LP	SDP / LP	LP / tot.	total # of sys.
1	20	20	100%	100%	20
2	104	142	73.24%	74.74%	190
3	260	522	49.81%	45.79%	1,140
4	370	1092	33.88%	22.54%	4,845
5	316	1458	21.67%	9.404%	15,504
6	160	1261	12.69%	3.253%	38,760
7	44	696	6.322%	0.8978%	77,520
8	5	233	2.146%	0.1850%	125,970
9	0	42	0%	0.02501%	167,960
10	0	3	0%	0.001624%	184,756
11	0	0	⋆	0%	167,960

If the LP approach from (Andersen et al., 2023) is to be applicable in dimensions larger than $n = 3$ or $n = 4$, then some reduction in the number of simplices is clearly needed. A promising preconditioning approach was presented in (Andersen et al., 2023).

The idea is to make a coordinate transform such that $V(\mathbf{x}) = \|\mathbf{x}\|_2$ is closer to fulfilling the conditions of a Lyapunov function. To compute such a coordinate transform we first solve the Lyapunov equation

$$A_m^T P_m + P_m A_m = -I_n \quad \forall m \in \mathcal{P}.$$

Since the individual subsystems $\dot{\mathbf{x}} = A_m \mathbf{x}$ have an exponentially stable equilibrium at the origin, the matrices A_m are Hurwitz and these equations all have a positive definite solution $P_m \succ 0$. Thus, for each subsystem $\dot{\mathbf{x}} = A_m \mathbf{x}$, the function $V_m(\mathbf{x}) = \mathbf{x}^T P_m \mathbf{x}$ is a quadratic Lyapunov function. We then define the matrix

$$R := \sum_{m \in \mathcal{P}} \lambda_m P_m$$

as a convex combination of the P_m, that is, $\lambda_m \geq 0$ for all $m \in \mathcal{P}$ and $\sum_{m \in \mathcal{P}} \lambda_m = 1$.

Next, we use the coordinate transform $\mathbf{x} \mapsto R^{\frac{1}{2}}\mathbf{x}$; that is, we replace the matrices A_m by $\widetilde{A}_m := R^{\frac{1}{2}} A_m R^{-\frac{1}{2}}$. We call this procedure *preconditioning* of the switched system (3). Note that if $|\mathcal{P}| = 1$ then $V(\mathbf{x}) = \sqrt{\mathbf{x}^T I_n \mathbf{x}} = \|\mathbf{x}\|_2$ is a Lyapunov function for the transformed system. In Figure 2 we show how trajectories of two systems are altered through preconditioning the switched system.

In many cases, one can compute a piecewise linear CLF for switched system (3) with fewer simplices, that is, with a lower K in the triangulation $\mathcal{T}_K$. However, it is not transparent how to choose the λ_m in the convex combination. In Fig. 3, we investigate this for two examples from the three-dimensional linear systems outlined in the appendix. In both cases, we consider two systems, $\dot{\mathbf{x}} = A_r \mathbf{x}$ and $\dot{\mathbf{x}} = A_s \mathbf{x}$, the coordinate transform

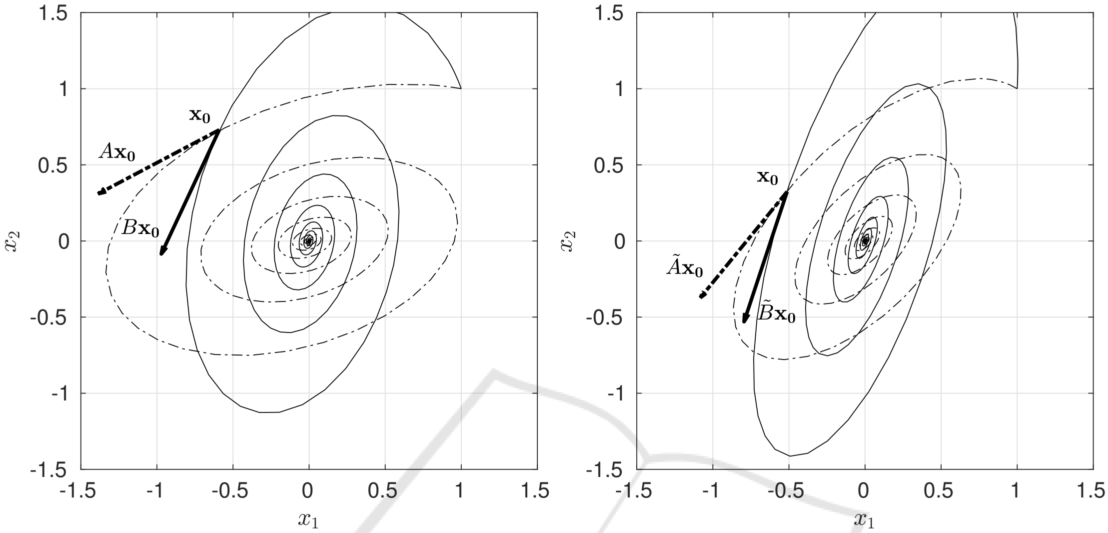
$$R_\lambda = \lambda P_r + (1 - \lambda) P_s,$$

which corresponds to $\lambda_r = \lambda$ and $\lambda_s = 1 - \lambda$, and report the lowest resolution parameter K, for which we could compute a piecewise linear CLF for the system.

For $r = 6$ and $s = 9$, we see the typical case: Preconditioning allows us to compute a Lyapunov function using considerably fewer simplices. However, there are also cases, where preconditioning is not of advantage or where it is even counterproductive, as it is for $r = 3$ and $s = 12$, shown in the lower panel of Fig. 3. However, usually the preconditioning has the

Table 4: Ratio, in percentage, of the three-dimensional problems from the Appendix successfully solved using SDP to problems successfully solved using LP.

# of matrices	# sys. solved using SDP	# sys. solved using LP	SDP / LP	LP / tot.	total # of sys.
1	12	12	100%	100%	12
2	9	10	90%	15.15%	66
3	1	3	33.33%	1.36%	220
4	0	0	⋆	0	495



(a) Angle is 37.11°. (b) Angle is now 20.97°.

Figure 2: Exemplary angle between vector fields $\mathbf{x} \mapsto A\mathbf{x}$ and $\mathbf{x} \mapsto B\mathbf{x}$ (a) and the transformed (or preconditioned) vector fields $\mathbf{x} \mapsto \tilde{A}\mathbf{x}$ and $\mathbf{x} \mapsto \tilde{B}\mathbf{x}$ (b) at a point $\mathbf{x}_0$.

Table 5: Some examples of the number of simplices in the triangulation $\mathcal{T}_K$ in different dimensions n.

n	K	Number of simplices in $\mathcal{T}_K$
2	5	40
2	10	80
2	50	400
2	100	800
3	5	1,600
3	10	6,400
3	50	160,000
3	100	640,000
4	5	96,000
4	10	768,000
4	50	96,000,000
4	100	768,000,000
5	5	7,680,000
5	10	122,880,000
5	50	$7.68 \cdot 10^{10}$
5	100	$1.2288 \cdot 10^{12}$

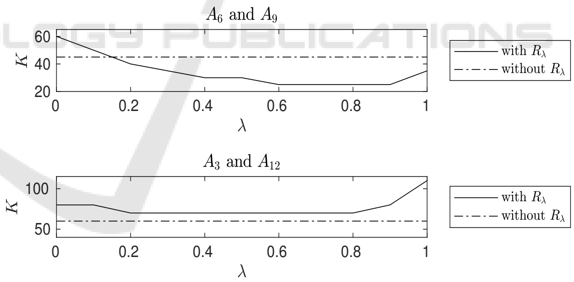


Figure 3: Lowest resolution factor K that delivers a piecewise linear CLF as a function of λ for two example sets for three-dimensional systems. The number of simplices needed is $64K^2$. While $64 \cdot 45^2 = 129,600$ simplices are needed without preconditioning, we get a solution with $64 \cdot 25^2 = 40,000$ simplices for $\lambda \in [0.6, 0.9]$ for $\mathcal{P} = \{6, 9\}$ (upper figure). For $\mathcal{P} = \{3, 12\}$, we have the untypical case that the preconditioning is counterproductive (lower figure). The first example possesses a QCLF while the second does not.

effect of reducing the number of simplices needed, but how to chose the optimal parameter λ remains an open problem. Note that this problem is of much practical value for systems of higher dimensions, where the number of simplices possible in the LP is the limiting factor. Understanding what effect the preconditioning of the systems has, is one of the main reasons for developing the app AngleAnalysis discussed in the next section.

5 THE AngleAnalysis APP

As known from the literature and seen in the preceding sections, the stability of the origin of (3) for a given set of matrices, is far from transparent. To better visualise the problem for two given matrices $A, B \in \mathbb{R}^{n\times n}$, $n = 2, 3$, we developed the application AngleAnalysis, which visualises the angles given by

$$\angle(A\mathbf{x}, B\mathbf{x}) := \arccos\left(\frac{\langle A\mathbf{x}, B\mathbf{x}\rangle}{\|A\mathbf{x}\|\,\|B\mathbf{x}\|}\right)$$

between vector fields $\mathbf{x} \mapsto A\mathbf{x}$ and $\mathbf{x} \mapsto B\mathbf{x}$ (see Fig. 2 (a)).

In the following, we explain in some detail how this angle relates to the stability of the origin, which is equivalent to the existence of a CLF for the systems $\dot{\mathbf{x}} = A\mathbf{x}$ and $\dot{\mathbf{x}} = B\mathbf{x}$. Consider (3) and the definition of a Lyapunov function. Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ is such that $V(\bar{\mathbf{x}}) = 0$ if and only if $\bar{\mathbf{x}} = 0$ is an equilibrium point, and the directional derivative along any system trajectory is strictly negative; that is, $\langle \nabla V(\mathbf{x}), A_m\mathbf{x}\rangle < 0$ for all $m \in \mathcal{P}$ and $\mathbf{x} \neq 0$.

Intuitively, since a CLF must fulfil $\langle \nabla V(\mathbf{x}), A\mathbf{x}\rangle < 0$ and $\langle \nabla V(\mathbf{x}), B\mathbf{x}\rangle < 0$, that is, $\angle(\nabla V(\mathbf{x}), A\mathbf{x}) > 90^\circ$ and $\angle(\nabla V(\mathbf{x}), B\mathbf{x}) > 90^\circ$, for all $\mathbf{x} \neq 0$, this condition is more difficult to satisfy for vector fields $\mathbf{x} \mapsto A\mathbf{x}$ and $\mathbf{x} \mapsto B\mathbf{x}$ if $\angle(A\mathbf{x}, B\mathbf{x})$ tends to be large. In other words, the smaller the angle between $A\mathbf{x}$ and $B\mathbf{x}$ for a particular $\mathbf{x} \in \mathbb{R}^n$, the more "space" there is to put the gradient $\nabla V(\mathbf{x})$, and it should be easier to construct a function $V : \mathbb{R}^n \to [0, \infty)$, whose gradient fulfills $\angle(\nabla V(\mathbf{x}), A\mathbf{x}) > 90^\circ$ and $\angle(\nabla V(\mathbf{x}), B\mathbf{x}) > 90^\circ$, for all $\mathbf{x} \neq 0$.

It is also helpful to look at the two extreme cases, when there is a point $\hat{\mathbf{x}} \in \mathbb{R}^n$, for which the angle between $A\hat{\mathbf{x}}$ and $B\hat{\mathbf{x}}$ is 180°, and when the angle is 0° for all $\mathbf{x} \in \mathbb{R}^n$. In the former case, there exists a constant $c > 0$ such that $A\hat{\mathbf{x}} = -cB\hat{\mathbf{x}}$, and a Lyapunov function V such that

$$\langle \nabla V(\mathbf{x}), A\mathbf{x}\rangle < 0 \quad \text{and} \quad \langle \nabla V(\mathbf{x}), B\mathbf{x}\rangle < 0$$

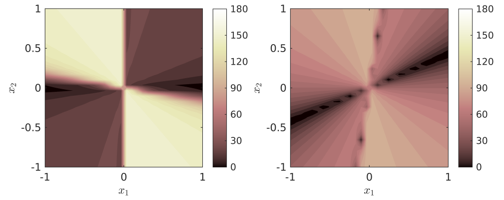
cannot exists, since assuming that there is one leads to the following contradiction:

$$\begin{aligned}\langle \nabla V(\mathbf{x}), A\mathbf{x}\rangle &= \langle \nabla V(\mathbf{x}), -cB\mathbf{x}\rangle \\ &= -c\,\langle \nabla V(\mathbf{x}), B\mathbf{x}\rangle .\end{aligned}$$

Hence, there cannot exist a CLF for the systems $\dot{\mathbf{x}} = A\mathbf{x}$ and $\dot{\mathbf{x}} = B\mathbf{x}$. In the other extreme case, if $\angle(A\mathbf{x}, B\mathbf{x}) = 0^\circ$ for all $\mathbf{x} \in \mathbb{R}^n$ then every Lyapunov function for $\dot{\mathbf{x}} = A\mathbf{x}$ is a Lyapunov function for $\dot{\mathbf{x}} = B\mathbf{x}$ and vice versa.

The AngleAnalysis app takes in any two matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ and visualises the angle $\angle(A\mathbf{x}, B\mathbf{x})$ through a heat-map, where dark colors indicate small angles and bright colors indicate large angles. For $n = 2$, the heat-map is plotted in the square $[-1, 1]^2$. For $n = 3$, the heat-map is plotted in the square $[-1, 1]^2 \times \{z\}$, where the parameter $z = x_3$ can be continuously varied in the interval $[-1, 1]$.

As discussed in Section 4, effect of the preconditioning on the LP problem is not intuitive. For some systems, for example the ones in (Andersen et al., 2023), the app gives clear indication of whether one can expect a reduction in the number of simplices, see Figure 4. However, in other cases the stability properties or preconditioning benefits are not transparent. Figures 5 and 6 show the angle heat map without preconditioning and with two different preconditionings. The resulting angle statistics can be found in tables 6 and 7. In neither case the app gives a clear indication of whether one can expect stability under arbitrary switching. The reason is that even though it is locally not difficult to put gradients $\nabla V(\mathbf{x})$ that fulfil $\angle(\nabla V(\mathbf{x}), A\mathbf{x}) > 90^\circ$ and $\angle(\nabla V(\mathbf{x}), B\mathbf{x}) > 90^\circ$, then $V : \mathbb{R}^n \to [0, \infty)$ has to be a continuous function on $\mathbb{R}^n$, and the problem of the existence of a CLF is a highly non-local one.



(a) (b)

Figure 4: Angle visualisation for matrices from (Andersen et al., 2023) with $a = 0.1$ and $b = 13.25$ (a) without preconditioning, (b) with $\lambda_a = 0.2$ and $\lambda_b = 0.8$. A piecewise linear CLF was obtained for the preconditioned system using a much lower resolution K than without.

Table 6: Maximum, minimum, mean and standard deviation of the angles between the trajectories of A_6 and A_9 at $(x, y, z) \in [-1, 1]^3$ where $x = x_1$, $y = x_2$ and $z = x_3$ are 30 points, evenly spaced on the interval.

λ_6	λ_9	Max	Min	Mean	Std	K
N/A	N/A	176.0	2.97	77.5	43.0	45
0.7	0.3	176.5	1.02	105.3	30.4	25
0	1	176.8	0.83	82.5	37.7	60

Table 7: Maximum, minimum, mean and standard deviation of the angles between the trajectories of A_3 and A_{12} at $(x, y, z) \in [-1, 1]^3$ where $x = x_1$, $y = x_2$ and $z = x_3$ are 30 points, evenly spaced on the interval.

λ_3	λ_{12}	Max	Min	Mean	Std	K
N/A	N/A	143.0	1.04	74.2	25.2	60
0.7	0.3	136.3	1.14	70.0	20.0	70
1	0	151.9	1.38	64.2	23.0	110

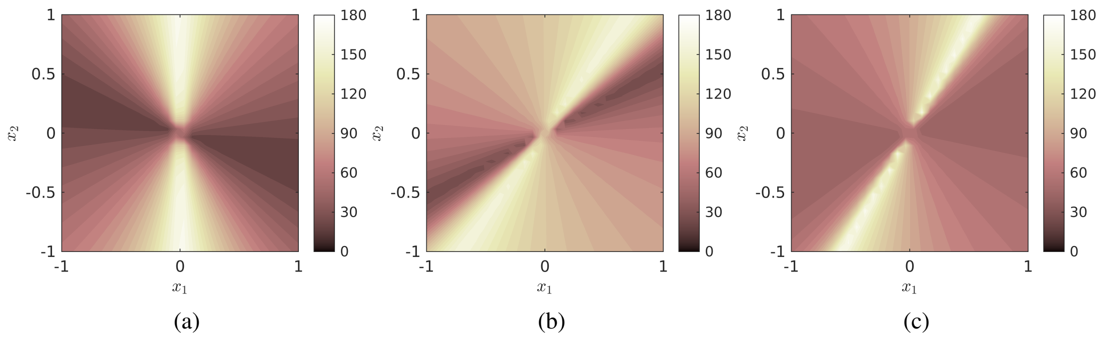


Figure 5: Angle visualisation for A_3 and A_{12} at $z = 0$. (a) Without preconditioning, (b) with $\lambda_6 = 0.7$ and $\lambda_9 = 0.3$, (c) with $\lambda_6 = 0$ and $\lambda_9 = 1$. Corresponding angle data can be found in Table 6.

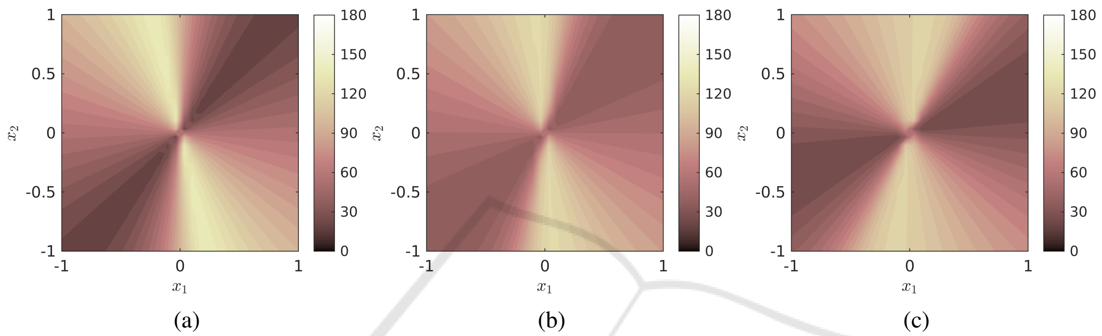


Figure 6: Angle visualisation for A_3 and A_{12} at $z = 0$. (a) Without preconditioning, (b) with $\lambda_3 = 0.7$ and $\lambda_{12} = 0.3$, (c) with $\lambda_3 = 1$ and $\lambda_{12} = 0$. Corresponding angle data can be found in Table 7.

6 CONCLUSIONS

We compared different semidefinite programming (SDP) solvers for computing quadratic common Lyapunov functions (QCLFs) for a variety of switched linear systems. In particular we tried different parameters $\varepsilon > 0$ to force positive definiteness $A \succ 0$ through $A \succeq \varepsilon I_n$ and got the somewhat surprising results that very small ε performed better in some examples. We compared QCLF to piecewise linear CLF computed by solving linear programming (LP) problems. As expected the LP approach is able to assert stability for more switched systems than the SDP approach. The disadvantage of the LP approach is that a triangulation of the state-space is needed, which results in a very high number of simplices in higher dimensions. To leverage this problem a certain preconditioning of the systems has been suggested. However, how to chose the parameters of the preconditioning is far from transparent. We developed the app AngleAnalysis to get visual information about the effect of the preconditioning and gave examples of its use.

In summary, we presented state-of-the art approaches to determine the stability of switched systems. While the SDP approach is faster, it solves the problem far less often than the more time-intensive approach using LP. As mentioned in the introduction, extending the SDP approach to higher-order polynomial Lyapunov functions seems promising, as it is less conservative. Finally, as shown in this paper preconditioning greatly accelerates the LP approach. Thus, in the future, we will continue investigating efficient methods for obtaining suitable preconditioning, in particular, for switched systems of higher order.

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APPENDIX

We generated 12 three-dimensional matrices to examine the effects of the preconditioning outlined in Section 4 with the following MATLAB commands:

```
Rz = @(th)[cos(th), -sin(th), 0;
           sin(th),  cos(th), 0;
           0,        0,       1];
Ry = @(th)[ cos(th), 0, sin(th);
            0,       1, 0;
           -sin(th), 0, cos(th)];
Rx = @(th)[1, 0,       0;
           0, cos(th), -sin(th);
           0, sin(th),  cos(th)];

m = 3; n = 2; l = 2;
th1 = linspace(.9/m,0.9,m)*pi;
th2 = linspace(0,0.9,n)*pi;
th3 = linspace(0,0.9,l)*pi;
num_sys = m*n*l;
A_3D = zeros(3,3,num_sys);
init = [1;1;1];

E = [-10, 0,  0;
      0, -1,  5;
      0, -5, -1];

for i = 1:num_sys
  r = ceil(i/(n*l));
  t = mod(ceil(i/l)-1,n)+1;
  s = mod(i-1,l)+1;
  V = [Rx(th1(r))*init, ...
       Ry(th1(r)+th2(t))*init, ...
       Rz(th1(r)+th2(t)+th3(s))*init];
  A_3D(:,:,i) = V*E/V;
end
```