An Online Deterministic Algorithm for Non-Metric Facility Leasing

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Abstract: Leasing has become one of the most widely spread business models in almost all markets. The online algorithmic study of leasing was initiated in 2005. Unlike classical algorithms, online algorithms are not given the entire input sequence at once. A portion of the input sequence is revealed in each step and the online algorithm is required to react to each step while targeting the given optimization goal against the entire input sequence. In a leasing setting, resources are leased and expire once their lease duration is over. Many well-known optimization problems are defined and studied in the leasing setting. In this paper, we continue the online algorithmic study of leasing by addressing the so-called *Online Non-metric Facility Leasing* problem (ONFL), the leasing variant of the non-metric *Online Facility Location* problem (non-metric OFL). Given a collection of facility and client locations. Facilities can be leased using a fixed number of lease types, each characterized by length and price. Lease types respect the economy of scale, such that longer leases cost more but are cheaper per unit of time. In each step, a client appears. The algorithm needs to promptly connect it to a facility that is leased at the current time step. To this end, it needs to decide which facility locations to lease, the start of their lease, and the lease duration. Connecting a client to a facility incurs a cost equal to the distance between the facility and the client. The goal is to minimize the total connecting and facility leasing costs. In this work, we develop the first deterministic algorithm for ONFL and evaluate it using the notion of competitive analysis, a worst-case performance analysis in which the solution of the online algorithm is compared, over all instances of the given problem, to the optimal solution of the offline variant of the problem.

1 INTRODUCTION

Facility Location problems, which ask to place facilities in the best possible way to satisfy a set of constraints, are one of the most well-studied optimization problems in computer science and operations research (Drezner and Hamacher, 2004; Mirchandani and Francis, 1990). These problems appear in applications for warehouses, disaster management, healthcare, public transportation, power plants, among others (Farahani and Hekmatfar, 2009; Adeleke and Olukanni, 2020; Sundarakani et al., 2021; Farahani et al., 2014; Laporte et al., 2019). Their NP-hardness, network design structure, and combinatorial nature have attracted researchers in a variety of fields, including approximation algorithms (Vazirani, 2001), online algorithms (Albers and Leonardi, 1999), and combinatorial optimization (Korte et al., 2011).

The simplest form of a *Facility Location* problem (Shmoys et al., 1997) consists of a collection of facility and client locations. Each facility is associated with an *opening cost* and each client-facility pair is associated with a *connecting cost*, which is the distance between the client and the facility. To open a facility, the corresponding facility cost needs to be paid. To connect a client to a facility, the corresponding connecting cost needs to be paid. The aim is to open a set of facilities and connect each client to an open facility, while minimizing the total opening and connecting costs. There are two known variations of *Facility Location* problems: the *metric*, in which distances satisfy the triangle inequality, and the *non-metric*, in which distances are arbitrary. Moreover, *Facility Location* problems have been studied in the *offline* and *online* settings (Shmoys et al., 1997; Jain et al., 2002; Charikar et al., 2001; Meyerson, 2001; Fotakis, 2008; Alon et al., 2006). In the offline setting, the algorithm is given the entire input sequence, to which it reacts once. In the online setting, the so-called *online algorithm* is given a client in each step, and needs to react to each step by connecting each arriving client, as soon as it arrives, to an open facility. The *Facility Location* problem in the online setting is known as the *Online Facility Location* problem (OFL), first introduced by (Meyerson, 2001) as a metric variant. The goal in both settings is to minimize

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the total opening and connecting costs.

Online algorithms are evaluated using the notion of *competitive analysis* (Borodin and El-Yaniv, 2005), defined as follows.

Definition 1. (Competitive Analysis). *Let I be the collection of all instances of a given problem P. We designate by $C(ALG,i)$ the cost of an online algorithm ALG of P on instance $i \in I$ and by $C(OPT,i)$ the cost of an optimal offline algorithm on instance $i \in I$. ALG has competitive ratio r or is r-competitive if, for all instances $i \in I$, $C(ALG,i) \leq r \cdot C(OPT,i) + c$ for some constant c independent of i.*

In this work, our focus will be the non-metric variant in the online setting. In particular, we address the so-called *Online Non-metric Facility Leasing* problem (ONFL) (Markarian and auf der Heide, 2019), in which facilities are *leased* rather than purchased.

Leasing is now widely used in almost all markets. The flexibility it offers makes it an indispensable business model with numerous benefits (Merrill, 2020). Lease-based applications naturally call for online algorithms, since decisions about leasing are often required to be made without knowing the future in advance. Consider, for instance, the cloud computing market, in which a third-party company leases machines from the cloud to serve its clients. The company may decide to make a yearly lease contract for a machine and then realize that the machine was not as useful as thought throughout the year. Making wise decisions in regards to when to lease which resource and for how long is modeled as complex optimization problems. An online algorithm will make such leasing decisions with a provable guarantee (i.e., a competitive ratio). Without knowing the future clients in advance, it aims to achieve the given optimization goal. This motivates the study of facility location problems, which commonly appear in real-world leasing applications, in an online leasing setting.

The *Online Non-metric Facility Leasing* problem (ONFL) is defined as follows.

Definition 2. (Online Non-metric Facility Leasing or ONFL). *Given a collection of facility and client locations. Facilities can be leased using L different lease types, each characterized by length and price. Lease types respect the economy of scale, such that longer leases cost more but are cheaper per unit of time. In each step, a client appears. The algorithm needs to immediately connect it to a facility that is leased at the current time step. It needs to decide which facility locations to lease, the start of their lease, and the lease duration. Connecting a client to a facility incurs a cost equal to the distance between the facility and the client. The goal is to minimize the total connecting and facility leasing costs.*

ONFL is a generalization of non-metric OFL in which there is one lease type that has infinite length.

To the best of our knowlege, the only online algorithm for ONFL in the literature is randomized, with an $O(\log n \log m + \log L \log n)$-competitive ratio, due to (Markarian and auf der Heide, 2019), where n is the number of clients, m is the number of facilities, and L is the number of lease types. The algorithm's design combines a randomized rounding strategy with a multiplicative incremental technique. We believe converting the latter into a deterministic algorithm does not seem straightforward, if possible at all.

1.1 Our Contribution

In this paper, we design the first online deterministic algorithm for the *Online Non-metric Facility Leasing* problem (ONFL). We prove that the latter has an $O(\log(m(L+\frac{l_{max}}{l_{min}}))(\log l_{max} + \log\log(m(L+\frac{l_{max}}{l_{min}}))$-competitive ratio, where:

- m is the number of facilities
- L is the number of lease types
- l_{max} is the longest lease length
- l_{min} is the shortest lease length

1.2 Roadmap

The rest of the paper is structured as follows. In Section 2, we present the lower bounds associated with the *Online Non-metric Facility Leasing* problem (ONFL). In Section 3, we describe a simplified structure for the leases that helps ease the competitive analysis. In Section 4, we present an overview of works related to ONFL. In Section 5, we give a description of our online algorithm for ONFL and show its competitive analysis in Section 6. In Section 7, we conclude with a discussion about the results and open problems that arise in the context of ONFL.

2 LOWER BOUNDS

The *Online Non-metric Facility Leasing* problem (ONFL) generalizes the *Parking Permit Problem* due to (Meyerson, 2005) and the *Online Set Cover* problem (OSC) due to (Alon et al., 2009). This enables us to conclude the following lower bounds for ONFL. Recall that L denotes the number of lease types, n the number of clients, and m the number of facilities.

- No online deterministic algorithm for ONFL can achieve a competitive ratio smaller than $\Omega(L)$, following the deterministic lower bound for the *Parking Permit Problem* (Meyerson, 2005).

– No online deterministic algorithm for ONFL can achieve a competitive ratio smaller than $\Omega(\frac{\log m \log n}{\log\log m + \log\log n})$, following the deterministic lower bound for the *Online Set Cover* problem (OSC) (Alon et al., 2009).
– A stricter bound of $\Omega(\log m \log n)$ holds for polynomial-time randomized algorithms for OSC, due to (Korman, 2004), assuming $BPP \neq NP$. Hence, no online polynomial-time randomized algorithm for ONFL can achieve a competitive ratio smaller than $\Omega(\log m \log n)$, assuming $BPP \neq NP$.
– No online randomized algorithm for ONFL can achieve a competitive ratio smaller than $\Omega(\log L)$, following the randomized lower bound for the *Parking Permit Problem* (Meyerson, 2005).

Using a simple observation, Bienkowski *et al.* (Bienkowski et al., 2021) showed that any deterministic algorithm for non-metric *Online Facility Location* (non-metric OFL) that does not know the facility-client graph with all the connections in advance can't achieve a competitive ratio better than m. Since ONFL generalizes non-metric OFL, the same holds for any deterministic algorithm for ONFL.

3 SIMPLIFIED LEASE STRUCTURE

The first leasing problem studied from the perspective of online algorithms was introduced by Meyerson (Meyerson, 2005), and was known as the *Parking Permit* problem. The latter is a simple yet algorithmically rich problem, defined as follows: On each day, the algorithm is told if it is raining or not. If it is raining, the algorithm must provide a valid permit for the day, selected from L different permit or lease types, each with a duration and price. The algorithm only knows whether it is raining or not on the same day. A longer permit costs more but is less expensive per day. For example, it would be cheaper to buy a weekly permit for a rainy week than to buy seven daily permits, one for each day. The online algorithm needs to minimize the total permit costs while covering all rainy days.

In order to make the competitive analysis easier, Meyerson (Meyerson, 2005) used a simplified form of the lease structure known as the *Interval model* (Theorem 2.2 in (Meyerson, 2005)), defined as follows.

– Lengths of leases are powers of two.
– No leases of the same type overlap with one another.

By using this structure, he showed that only a factor of 4 is lost in the competitive ratio. For the *Parking Permit* problem, he developed deterministic and randomized algorithms, with competitive ratios of $O(L)$ and $O(\log L)$, respectively, where L is the number of lease types. He also proved matching lower bounds for both. This structure was used in the majority of leasing problems that were studied afterword (Abshoff et al., 2016; Li et al., 2018; Nagarajan and Williamson, 2013). In this work, we assume the same lease structure for the *Online Non-metric Facility Leasing* problem (ONFL).

4 RELATED WORK

Alon *et al.* (Alon et al., 2006) developed an online randomized algorithm for the non-metric *Online Facility Location* problem (non-metric OFL), with competitive ratio $O(\log m \log n)$, where m is the number of facilities and n is the number of clients. Their algorithm is based on first relaxing the problem to its fractional variant and solving the latter using a multiplicative update approach, and then using randomized rounding to compute a feasible integral solution.

A reduction from non-metric *Facility Location* to *Set Cover*, that does not induce an exponential increase in the input size, was given by Kolen *et al.* (Kolen and Tamir, 1984). This reduction accompanied with doubling techniques and the deterministic algorithm for the *Online Set Cover* problem due to (Alon et al., 2009), yields a deterministic solution for non-metric OFL, with competitive ratio $O((\log n + \log m) \cdot (\log n + \log\log m))$.

Recently, Bienkowski *et al.* (Bienkowski et al., 2021) improved this ratio by designing an online deterministic algorithm for non-metric OFL, with competitive ratio $O(\log m \cdot (\log n + \log\log m))$. Their algorithm is based on fractional relaxation of the problem with clustered facilities and a combination of dual fitting and multiplicative weight-update approaches.

Other extensions of non-metric OFL were later introduced in the context of service installation costs (Markarian and Khallouf, 2021) and service quality costs (Markarian, 2022).

Since the introduction of the leasing setting by Meyerson (Meyerson, 2005), many optimization problems (Bienkowski et al., 2017; De Lima et al., 2018; Markarian and Kassar, 2022; Markarian, 2021; Markarian and Khallouf, 2021; Markarian and Kassar, 2020; Markarian, 2015), including Facility Location problems (Nagarajan and Williamson, 2013;

Clients	# of time steps = 2 # of clients = 3	
	I: (client)	*I'*: (client, time-step)
t = 1	client (1)	(1, t=1)
	client (2)	(2, t=1)
	client (3)	(3, t=1)
t = 2	client (1)	(1, t=2)
	client (2)	(2, t=2)
	client (3)	(3, t=2)

Figure 1: Client-pair Formulation.

Kling et al., 2012; Markarian and auf der Heide, 2019; Abshoff et al., 2016), were studied in this setting.

5 ONLINE ALGORITHM

Our online algorithm for ONFL makes use of the online deterministic algorithm due to (Bienkowski et al., 2021) for the non-metric *Online Facility Location* problem. Their polynomial-time algorithm is based on dual fitting and multiplicative weight update approaches. It is up to $\log\log$-factor optimal and has an $O(\log m \cdot (\log n + \log\log m))$-competitive ratio, where m is the number of facilities and n is the number of clients.

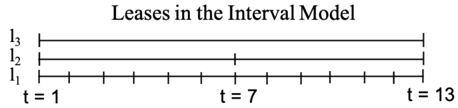
Given an instance I of ONFL, we transform it into an instance I' of non-metric OFL as follows.

1. **The clients of I' will be formed as follows:**
   If the given instance I comprises of one lease type $L = 1$ that has an infinite length, then I would be exactly an instance of non-metric OFL. Thus, we just run the deterministic algorithm of (Bienkowski et al., 2021) for non-metric OFL on I. Otherwise, given client j of instance I. For each time step t of instance I, we construct client (j,t). We do this for all clients of instance I. The clients constructed will form the clients of instance I'. We refer the reader to Figure 1 for an example. We let $\mathcal{N}$ be the collection of all these clients.

2. **The facilities of I' will be formed as follows:**
   Each facility of instance I can be leased with L lease types. Given facility i of instance I. We denote facility i of lease length l and lease start t by a triplet, (i,t,l). These triplets will form the facilities of instance I'. We refer the reader to Figure 2 for an example. We let $\mathcal{M}$ be the collection of all these facilities.

3. **The facility-client connecting costs will be formed as follows:**
   Given client $(j,t') \in \mathcal{N}$ of instance I'. For each triplet $(i,t,l) \in \mathcal{M}$ of instance I', we set the connecting cost between (j,t') and (i,t,l) to infinity if $t' \notin [t+l]$ or otherwise, equal to the connecting cost between j and i as per instance I. We refer the reader to Figure 3 for an example.

Facilities	# of facilities = 1 # of lease types = 3	
	I: (facility)	*I'*: (facility *i*, lease type, lease start)



Facility Triplets	$(i, l_2, t = 1)$	$(i, l_2, t = 1)$	$(i, l_2, t = 1)$
	$(i, l_2, t = 2)$	$(i, l_{2,} t = 7)$	
	. . .		
	$(i, l_2, t = 13)$		

Figure 2: Facility-triplet Formulation.

Connecting Costs	Client pair (*j*, t') Facility triplet (*i*, t, l)

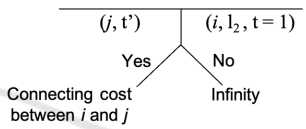


Figure 3: Connecting Costs.

The clients, facilities, and connecting costs of instance I' will be formed *before* the arrival of the first client of instance I. Upon the arrival of a new client j of instance I at time step t, the algorithm will consider the client (j,t) corresponding to j and time step t as an input to the algorithm for non-metric OFL. All the other pairs corresponding to j and the future time steps will be ignored.

Running the online deterministic algorithm ALG_{OFL} of (Bienkowski et al., 2021) for non-metric OFL on I' would yield to a feasible solution for I. Whenever ALG_{OFL} opens a facility $(i,t,l) \in \mathcal{M}$, we immediately purchase the corresponding lease for that facility. Whenever ALG_{OFL} connects a pair $(j,t) \in \mathcal{N}$ to a facility $(i,t,l) \in \mathcal{M}$, we also do the same and pay the corresponding connecting cost. Notice that, for a client arriving at time step t, the algorithm will never connect it to a facility whose lease does not cover time step t, since the connecting cost would be infinity as per our construction.

6 COMPETITIVE ANALYSIS

Following the simplified lease structure defined earlier, we divide the timeline into intervals of length l_{max}. We fix any of these intervals, I, and evaluate the performance of the algorithm over this interval. As per the simplified lease structure, all leases on this interval have a start time and end time within this interval. This means that the optimal leases too. Hence, by proving the competitive ratio of the algorithm over this interval, we can conclude the competitive ratio of the algorithm over the entire timeline.

Since in each time step a single client arrives, we have in total l_{max} clients of instance I appearing on the interval I. According to the formation of our clients of I' described earlier, we construct for each of these l_{max} clients, $(l_{max})^2$ clients of instance I'. Therefore, we have that $|\mathcal{N}| = (l_{max})^2$.

As for the facilities, we show next an upper bound on the cardinality of $\mathcal{M}$ following the formation of facilities of I' described earlier. We order the lease types in increasing order of length, denoted as $\{l_1, l_2, l_3, ..., l_L\}$, such that $l_{j+1} > l_j$ for $1 \leq j \leq L-1$. We can now upper bound $|\mathcal{M}|$ as follows:

$$|\mathcal{M}| \leq m \cdot (\sum_{j=1}^{L} \left\lceil \frac{l_L}{l_j} \right\rceil)$$

As per the simplified lease structure, lease lengths l_js are increasing and powers of two. Hence, the sum above can be upper bounded by the sum of a geometric series with ratio $1/2$. Thus, we have that:

$$\sum_{j=1}^{L} \left\lceil \frac{l_L}{l_j} \right\rceil \leq L + l_L \left[\frac{1}{l_1} \left(\frac{1-(1/2)^l_L}{1-1/2} \right) \right] =$$
$$L + l_L \left[\frac{2}{l_1} \left(1 - (1/2)^L \right) \right]$$

Since $L \geq 1$, we have:

$$L + l_L \left[\frac{2}{l_1} \left(1 - (1/2)^L \right) \right] \leq L + \frac{2_L}{l_1}.$$

Therefore, $|\mathcal{M}| \leq m \cdot (L + \frac{2l_{max}}{l_{min}})$, where $l_{min} = l_1$ is the shortest lease length and $l_{max} = l_L$ is the longest lease length.

Since the algorithm for non-metric *Online Facility Location* which we run on instance I' is $O(\log \mathcal{M} \cdot (\log \mathcal{N} + \log\log \mathcal{M}))$-competitive, the theorem below follows.

Theorem 1. *There is an online* $O(\log(m(L + \frac{l_{max}}{l_{min}}))(\log l_{max} + \log\log(m(L + \frac{l_{max}}{l_{min}}))$*-competitive deterministic algorithm for the Online Non-metric Facility Leasing problem, where m is the number of facilities, L is the number of lease types,* l_{max} *is the longest lease length, and* l_{min} *is the shortest lease length.*

7 CONCLUSION

We have presented in this paper the first deterministic online algorithm for ONFL, with competitive ratio depending on the parameters l_{max} and l_{min}. It would be useful to achieve a competitive ratio that does not depend on these parameters or prove a lower bound in terms of these parameters.

We believe the design structure of our algorithm for ONFL could be extended to other leasing problems for which there is no deterministic algorithm in the literature. Even more interesting would be to achieve a similar structure for a generalized transformation between the leasing and non-leasing variations of any network design problem.

Considering other adverserial models, such as uniform distribution (as in (Meyerson, 2001; Kaplan et al., 2023)), for the input sequence that would probably be less harsh on the algorithm is always worth the investigation. Given that both leasing and Facility Location scenarios appear as sub-problems in many real-world applications, these investigations could play a vital role in closing the gap between the theoretical results and the practical world.

Finally, incremental (Arulselvan et al., 2015; Dai and Zeng, 2010; Divéki and Imreh, 2011; Fotakis, 2006; Fotakis, 2011). and dynamic algorithms (Cygan et al., 2018; Feldkord and Meyer auf der Heide, 2018; Fotakis et al., 2021) have been used to address many online Facility Location variants in the metric setting. It would be interesting to design such algorithms in the non-metric setting too.

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