

Performance Metrics Based on Data Envelopment Analysis for Evaluating Multi-Objective Linear Programming Solution Methods

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Keywords: Multi-Objective Linear Programming, MOLP Performance Metrics, MOLP Solution Methods, Data Envelopment Analysis, Slack Based Measure Model, Super-efficiency DEA Model.

Abstract: Three performance metrics based on data envelopment models are proposed for evaluating MOLP solution methods. Every proposed metric is associated to a one category, being these categories the cardinality, accuracy, and diversity. In addition, the proposed metrics are classified as unary or binary. The cardinality and accuracy metrics are estimated using a DEA model based on the slack based measure model, while the diversity metric is calculated using the super-efficiency DEA model. The proposed metrics were applied to compare two sets of solutions for a MOLP tactical harvest planning model, that were obtained using two strategies of a MO-GRASP algorithm. The results show that the metrics allow discriminating between the MOLP solution methods and, moreover, to select one.

1 INTRODUCTION

The multi-objective linear programming (MOLP) is an area of operations research where many practical problems have been addressed, as transport (Demir et al., 2014), agriculture (Varas et al., 2020), manufacturing (Mirzapour Al-E-Hashem et al., 2011), location (Karatas & Yakıcı, 2018), among others. Usually, these MOLP models are difficult to solve (Deb, 2014). Therefore, exact and heuristic methods have been proposed for solving them. However, selecting a suitable solution method is not a simple task. In this way, different performance metrics have been proposed for analysing the solutions obtained by these methods. Regarding this issue, Riquelme et al. (2015) carried out a literature review about the performance metrics for evaluating MOLP solution methods, classifying them in three categories: cardinality, accuracy, and diversity. Cardinality represents the number of non-dominated solutions found by a MOLP solution method.

Accuracy refers to the convergence of the non-dominated solution to the Pareto frontier. Thus, it represents the distance between every non-dominated solution with the theoretical Pareto frontier (Riquelme et al., 2015). Diversity considers the distribution and spread of the non-dominated solutions. The distribution considers the relative distance among the non-dominated solutions, and the spread corresponds to the range of the objective function values covered by the non-dominated solutions. It is important to mention that, in every category, different metrics have been proposed (Audet et al., 2021; Riquelme et al., 2015). Furthermore, Riquelme et al. (2015) also classified the performance metrics into unary and binary. A metric is unary if the non-dominated solutions are obtained by only one solution method. On the other hand, a metric is binary if the non-dominated solutions are obtained by two solution methods.

In the literature, data envelopment analysis (DEA) models have been used for estimating MOLP

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performance metrics. In Bal & Satoglu (2019), the BCC model (Banker et al., 1984) was used as a metric for evaluating the performance of Pareto optimal solutions obtained by the augmented epsilon constraint method 2. This solution method was applied to a MOLP model with four objective functions, aiming to improve the coordination of an appliance supply chain. Hong & Jeong (2019) used a CCR (Charnes et al., 1978) for evaluating the solutions obtained by the weighting method. This method was used for solving a MOLP model with five objective functions, which sought to determine strategic decisions for a facility location–allocation problem.

In this study, three performance metrics based on DEA model are proposed for evaluating MOLP solution methods. Every metric is associated to a category of cardinality, accuracy, and diversity, respectively, and can be classified as unary or binary.

This article is divided as follows: Section 2 describes the applied DEA models and the procedure for calculating every metric. Section 3 presents the results of this study, while Section 4 summarizes the conclusions.

2 MATERIAL AND METHODS

The DEA models used for assessing different performance metrics of MOLP solution methods, as the associated procedure for applying them, are presented in this section. As mentioned previously, the categories considered in this analysis are: cardinality, accuracy, and diversity. The proposed cardinality and accuracy metrics are estimated using a DEA model based on the slacks-based measure proposed by Tone (2001). On the other hand, the proposed diversity metric is calculated using the super efficiency DEA model developed by Andersen & Petersen (1993). The characteristics for selecting these DEA models and the way that they can be used for estimating every metric are detailed in the following sub-sections.

2.1 Applied DEA Models

The common nomenclature of parameters and decision variables used in the applied DEA models are defined in Table 1 and Table 2, respectively. In this definition, the MOLP nature of the DEA assessment is considered.

Table 1: Parameters of the DEA models.

Parameter	Definition
m	Number of objective functions to be minimized in the MOLP model.
s	Number of objective functions to be maximized in the MOLP model.
n	Number of solutions obtained by a MOLP solution method.
x_{ij}	Value of the minimized objective function i obtained by solution j , where $i = 1, \dots, m, j = 1, \dots, n$.
y_{rj}	Value of the maximized objective function r obtained by solution j , where $r = 1, \dots, s, j = 1, \dots, n$.
j_0	Evaluated solution in every execution of the applied DEA models.

The different decision variables used in the applied DEA models are described in Table 2.

Table 2: Decision variables of the DEA models.

Variable	Definition
S_i^-	Slack of the minimized objective function i , where $i = 1, \dots, m$
S_r^+	Slack of the maximized objective function r , where $r = 1, \dots, s$
λ_j	Intensity of the solution j for establishing the target in the Pareto frontier of the evaluated solution.
t	Auxiliary variable that represents a positive scalar used for the model linearization.
A_j	Auxiliary variable for the model linearization. It is a binary variable, where $A_j = 1$ if $\lambda_j = t$; $A_j = 0$ otherwise, $j = 1, \dots, n$.
θ	Proportional reduction of the minimized objective functions obtained by solution j_0 .

2.1.1 DEA Model Based on the Slacks-based Measure of Efficiency (INT-SBM)

The proposed DEA model is based on the slacks-based measure model (SBM) developed by Tone (2001). This author presented a non-linear model, which was linearized using the linear transformation proposed by Charnes & Cooper (1962). The SBM model was selected because it does not require inputs in an output-oriented model, or it does not require outputs in an input-oriented model. In this way, it can evaluate solution methods that solve MOLP problems that have only maximization objective functions or only minimization objective functions. Moreover, the

non-dominated solutions that belong to the Pareto frontier will obtain a score equals to one, while the dominated solutions will obtain a score greater than zero and lower than one. This score represents the closeness to the Pareto frontier.

The DEA model formulated in this study differs from SBM linear model because it considers binary variables. For this reason, it is called Integer-SBM (INT-SBM), and it evaluates solutions located in the non-convex region of the Pareto frontier. These solutions will obtain a score equals to one.

The proposed DEA model must be executed for every solution j , where j_0 represents the evaluated solution in a specific model execution.

The INT-SBM formulation is:

$$\text{Minimize } \xi_{j_0} = t - \frac{1}{m} \sum_{i=1}^m S_i^- / x_{ij_0} \quad (1)$$

Subject to

$$1 = t + \frac{1}{s} \sum_{r=1}^s S_r^+ / y_{rj_0} \quad (2)$$

$$tx_{ij_0} = \sum_{j=1}^n x_{ij} \lambda_j + S_i^-, \quad i = 1, \dots, m, \quad (3)$$

$$ty_{rj_0} = \sum_{j=1}^n y_{rj} \lambda_j - S_r^+, \quad r = 1, \dots, s, \quad (4)$$

$$\sum_{j=1}^n \lambda_j = t, \quad (5)$$

$$\sum_{j=1}^n A_j = 1, \quad (6)$$

$$\lambda_j \geq t - (1 - A_j), \quad j = 1, \dots, n, \quad (7)$$

$$\lambda_j \leq t + (1 - A_j), \quad j = 1, \dots, n, \quad (8)$$

$$A_j \in \{0,1\}, \quad j = 1, \dots, n, \quad (9)$$

$$\lambda_j \geq 0, \quad j = 1, \dots, n, \quad (10)$$

$$S_i^- \geq 0, \quad i = 1, \dots, m, \quad (11)$$

$$S_r^+ \geq 0, \quad r = 1, \dots, s, \quad (12)$$

$$t \geq 0. \quad (13)$$

The objective function (1) estimates the efficiency score based on the minimization of slacks associated to the minimized objective functions obtained by

solution j_0 . Constraint (2) allows linearizing the objective function, making the expression related to the maximized objective function slacks equals to a constant value. Constraints (3) and (4) calculate the slacks of minimized and maximized objective functions, respectively. Constraint (5) establishes the convexity of the efficient frontier associated to the variable returns to scale Tone (2001). Constraints (6) to (8) allow linearizing $\lambda_j \times A_j$, which are decision variables. Finally, constraints (9) to (13) establish the nature of the decision variables.

In the INT-SBM model, ξ_{j_0} corresponds to the efficiency score of the evaluated solution j_0 , which varies between zero and one. In this case, $\xi_{j_0} = 1$ represents a non-dominated solution.

2.1.2 Super-Efficiency DEA Model

Andersen & Petersen (1993) proposed the super-efficiency DEA model for ranking all the evaluated units according to their efficiency score. This efficiency score could be greater than one, for an input-oriented model, or lower than one, for an output-oriented model. These values are possible because the data of every evaluated solution j_0 are not considered in the observed data of the DEA model for determining the DEA efficient frontier.

In this study, an input-oriented super-efficiency DEA model (SE-DEA) was used, aiming to improve the discrimination among the non-dominated solutions. It is important to mention that the model orientation does not vary the identification of the super-efficient solutions' set. In addition, the SE-DEA model must be executed for every solution j , where j_0 corresponds to the evaluated solution in a specific model execution.

The SE-DEA formulation is:

$$\text{Minimize } \delta_{j_0} = \theta \quad (14)$$

Subject to

$$\sum_{\substack{j=1 \\ j \neq j_0}}^n \lambda_j x_{ij} \leq \theta x_{ij_0}, \quad i = 1, \dots, m, \quad (15)$$

$$\sum_{\substack{j=1 \\ j \neq j_0}}^n \lambda_j y_{rj} \geq y_{rj_0}, \quad r = 1, \dots, s, \quad (16)$$

$$\sum_{\substack{j=1 \\ j \neq j_0}}^n \lambda_j = 1, \quad (17)$$

$$\lambda_j \geq 0, \quad j = 1, \dots, n, \quad (18)$$

$$\theta \text{ free} \tag{19}$$

The objective function (14) minimizes the proportional reduction of the minimized objective functions obtained by solution j_0 . Constraint (15) establishes that the proportional reduction of the minimized objective functions obtained by solution j_0 must be greater or equal than the composed target in the efficient frontier (left hand of the constraint). Constraints (16) estimates that the maximized objective functions obtained by solution j_0 must be lower or equal than the composed target in the efficient frontier (left hand of the constraint). Constraint (17) imposes the convexity of the efficient frontier, which is associated to the variable returns to scale (Banker et al., 1984). Constraints (18) and (19) establish the nature of the decision variables.

In the SE-DEA model, δ_{j_0} corresponds to the efficiency score of the evaluated solution j_0 . This efficiency score, differently from a traditional BCC input-oriented model (Banker et al., 1984), could achieve values greater than one or even the model could be infeasible. The necessary and sufficient conditions for infeasibility of SE-DEA models when variable returns to scale are considered (constraint 17), are presented in the study of Seiford & Zhu (1999). Consequently, a solution j_0 that is an extreme point of the Pareto efficient frontier will have a δ_{j_0} value greater than one or the associated model could be infeasible.

In the following sub-section, the performance metrics for evaluating MOLP solution methods and the steps for implementing them using the formulated DEA models are described.

2.2 Performance Metrics for Evaluating MOLP Solution Methods

As mentioned previously, the considered categories for evaluating MOLP solution methods are cardinality, accuracy, and diversity. The proposed metrics in every category and the steps for calculating them are presented as follows.

2.2.1 Cardinality Metric - CM

The cardinality metric (*CM*) represents the domination degree of the solutions obtained by a MOLP method. For this reason, it is a unary metric, using the information of a unique solution set. In this study, it is calculated using the INT-SBM model, where data of all the solutions (S) obtained by a solution method are evaluated. It is important to

highlight that the dominated and non-dominated solutions obtained by a solution method are considered as observed data of the model. The following steps must be carried out for obtaining the cardinality metric *CM*.

Step 1: Execute the INT-SBM model for every solution of set S . In this step, a vector Ξ is obtained, which corresponds to the vector of ξ_i , the efficient measure of the INT-SBM model for every solution i of the set S .

Step 2: Calculate the efficiency average of vector Ξ . This value will correspond to the cardinality metric *CM*.

The cardinality metric *CM* is greater than zero, and lower than or equal to one. A value equal to one means that it does not exist any solution dominated by other in the set S . On the other hand, a value close to zero means that few non-dominated solutions exist in the set S .

2.2.2 Accuracy Metric - AC

The accuracy metric (*AC*) represents the domination degree of one MOLP solution method over other MOLP solution method. Furthermore, it is a binary metric because it needs two sets of non-dominated solutions for making the comparison. In this study, for estimating the accuracy metric *AC*, the INT-SBM model and the metafrontier approach, proposed by O'Donnell et al. (2008), are used together. The metafrontier approach allows classifying the non-dominated solutions into different groups. In this way, two sets of non-dominated solutions, S_1 and S_2 , obtained by two different solution methods, are compared. The following steps must be carried out for estimating the proposed accuracy metric *AC*.

Step 1: Execute the INT-SBM model for every non-dominated solution of set S_1 . In this step, a vector Ξ^1 is obtained, which corresponds to the vector of ξ_i , the efficient measure of the INT-SBM model for every non-dominated solution i of the set S_1 .

Step 2: Execute the INT-SBM model for every non-dominated solution of set S_2 . In this step, a vector Ξ^2 is obtained, which corresponds to the vector of ξ_i , the efficient measure of the INT-SBM model for every non-dominated solution i of the set S_2 .

Step 3: Execute the INT-SBM model for every solution belonging to the union of sets S_1 and S_2 . In this step, a vector Ξ^3 is obtained, which corresponds to the vector of ξ_i , the efficient measure of the INT-

SBM model for solution i belonging to the union of sets S_1 and S_2 .

Step 4: Separate the efficiency vector Ξ^3 in two sets, efficiencies scores of solutions from set S_1 (Ξ_1^3), and efficiencies scores of solutions from set S_2 (Ξ_2^3).

Step 5: Calculate the efficiency averages of vectors Ξ^1 , Ξ^2 , Ξ_1^3 , and Ξ_2^3 , individually.

Step 6: Make the difference between the efficiency averages of vectors Ξ^1 and Ξ_1^3 , which corresponds to AC_1 , and between Ξ^2 and Ξ_2^3 , which corresponds to AC_2 .

Step 7: Calculate the minimum value between AC_1 and AC_2 . This value will correspond to the accuracy metric AC .

It is important to notice that AC is greater than or equal to zero, and lower than one. Moreover, the solution method with the minimum value AC will be the best method, meaning that this method obtains a lower number of dominated solutions than the other solution method.

2.2.3 Diversity Metric - DM

The diversity metric DM evaluates a change in the Pareto frontier when a new solution is added. This is a unary metric because it uses the information of a unique solution set. In this study, the diversity metric DM is calculated using the SE-DEA model. In this model, the non-dominated solutions (NS) obtained by a MOLP method are evaluated. The following steps must be carried out for obtaining DM .

Step 1: Execute the SE-DEA model for every solution of the set NS . A vector Δ is obtained, which corresponds to the vector of δ_i , that is, a vector of the efficiency score obtained by the SE-DEA model for every solution i of the set NS .

Step 2: Identify the subset of NS that corresponds to extreme solutions. These solutions are those that in the step 1 obtained a δ_i value greater than one or the respective SE-DEA model is infeasible. This subset, denominated ES , defines the Pareto frontier.

Step 3: Calculate DM using equation (20).

$$DM = \frac{|ES|}{|NS|} \tag{20}$$

The diversity metric DM is greater than zero, and lower than or equal to one. A value close to zero means that most of the solutions are a linear combination of extreme solutions in ES . A value equals to one means that all the solutions are not a

linear combination of other extreme solutions in ES . In this way, the best value for DM is one.

3 RESULTS

In this section, for calculating the proposed metrics, the solutions obtained by two MOLP methods are used. The solutions were obtained for a MOLP model based on the tactical harvest planning model proposed by Gómez-Lagos et al. (2021), where the same case study used in this article was analysed. In this MOLP model, the first objective corresponds to the harvest costs' minimization (Z_1); the second objective corresponds to the harvest days' minimization (Z_2); and the third objective corresponds to the harvest fruit in the optimal conditions' maximization (Z_3). The two applied MOLP methods are two solution strategies of the MO-GRASP algorithm (algorithms a and b) (Martí et al., 2015).

Executing 1000 times every MO-GRASP algorithm for solving the MOLP model, two sets of solutions were obtained, S_a and S_b ; one set of 1000 solutions for every algorithm. In Figure 1, the trade-off between the objective function values obtained by the set S_a are represented. The first trade-off corresponds to Z_1 and Z_2 ; the second, Z_1 and Z_3 ; and the third, Z_2 and Z_3 .

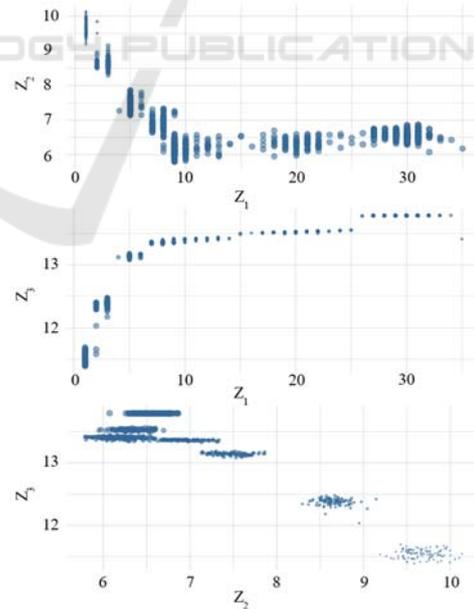


Figure 1: Objective function values of set S_a .

A conflict between the objective functions can be observed in Figure 1 because when an objective function improves, the other deteriorates.

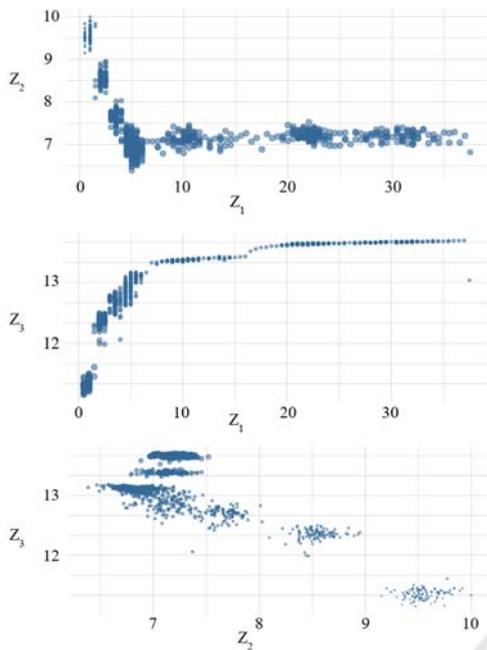


Figure 2: Objective function values of set S_b .

Figure 2 represents the trade-off between the objective function values obtained by the set S_b .

The conflict between the objective functions is also observed in Figure 2.

Table 3 summarizes metrics calculated for both sets of solutions, S_a and S_b . In this way, it can be observed that the set S_a obtains the best value for the cardinality metric CM (0.980). Regarding the accuracy metric AC , the set S_a again achieve the best value (0.999), meaning that main of solutions of S_a are not dominated by the solutions of S_b . Finally, for the diversity metric DM , both sets obtain low values. However, the set S_b obtains the best value (0.131), meaning that around 13% are extreme-efficient solutions, that is, define the Pareto frontier.

Table 3: Values of the performance metrics for S_a and S_b .

Set of Solutions	CM	AC	DM
S_a	0.980	0.999	0.101
S_b	0.977	0.962	0.131

From the values presented in Table 3, it could be suggested to select the algorithm a for solving the MOLP model because it has a best performance in the binary metric AC , and in the unary metric CM . Furthermore, for the unary metric DM , around 10% the solutions obtained by the algorithm a are extreme-efficient, close to DM obtained by the algorithm b .

4 CONCLUSIONS

In this study, three performance metrics based on DEA models for evaluating MOLP solution methods were proposed. The considered metrics are associated to cardinality, accuracy, and diversity categories. A procedure for calculating every metric is presented and applied to a real case study, where two MOGRASP algorithms were compared. Therefore, the two sets of solutions obtained by every algorithm were used for estimating the metrics. For the cardinality and accuracy metrics based on the INT-SBM efficiency score, it was possible to discriminate among the non-dominated solutions, independently of the frontier region where they were located. For the diversity metric based on the SE-DEA efficiency score, it was possible to identify the non-dominated solutions that determine the Pareto frontier. In this way, these metrics allow discriminating between the MOLP solution methods and even to select one.

For future research, it could be interesting to explore DEA models where zero or negative values can be incorporated in the set of solutions. In addition, new DEA models could be explored in order to identify the non-dominated solutions located in non-convex regions of the Pareto frontier.

ACKNOWLEDGEMENTS

DSc Marcela González-Araya would like to thank FONDECYT Project 1191764 (Chile) for its financial support. Ms Javier Gómez-Lagos would like to thank CONICYT PFCHA/BECA DE DOCTORADO NACIONAL/2019 under Grant 21191364 for its financial support.

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