Topology-Preserving Reductions on (18,12) Pictures of the Face-Centered Cubic Grid

Gábor Karai[a], Péter Kardos[b] and Kálmán Palágyi[c]
Department of Image Processing and Computer Graphics, University of Szeged, Szeged, Hungary

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Abstract: Reductions transform binary pictures only by changing some black points to white ones. Topology preservation is a major concern of thinning algorithms that are composed of reductions. For $(18,12)$ binary pictures on the 3D face-centered cubic (FCC) grid, we propose four sufficient conditions for topology-preserving parallel reductions that can change a set of black points simultaneously. The first two conditions examine some configurations of changed points, and they provide methods of verifying that formerly constructed parallel reductions preserve the topology. The further two conditions focus on individual points, directly provide deletion rules of topology-preserving parallel reductions, and make us possible to establish topologically correct parallel thinning algorithms.

1 INTRODUCTION

It is the common practice that 3D digital pictures are sampled on the *cubic grid* $\mathbb{Z}^3$, since it is the only regular grid in 3D, it has a fairly simple structure, and digital pictures on the cubic grid can be naturally stored in usual 3D arrays. Among non-standard grids our attention is focused on the *face-centered cubic (FCC) grid* (Kong and Rosenfeld, 1989). The points $(x,y,z) \in \mathbb{Z}^3$ such that $x+y+z$ is even are the grid points of the FCC grid denoted by $\mathbb{F}$.

Herman touched upon some disadvantages of the cubic grid (Herman, 1998), Gau and Kong reported three advantages of the FCC grid over the cubic grid (Gau and Kong, 1999), and Edelsbrunner et al. showed that the FCC grid provides the densest sphere packing (Edelsbrunner et al., 2015). That is why the importance of the FCC grid shows an upward tendency (Čomić and Magillo, 2020; Čomić and Nagy, 2016; Gastineau and Togni, 2021; Koshti et al., 2018; Rácz and Csébfalvi, 2018; Strand and Stelldinger, 2007).

A *binary digital picture* (or *picture*, for short) on a discrete grid is composed of *black* or *white* points (Kong and Rosenfeld, 1989). *Reduction* operators transform a picture by changing some black points to white ones that is referred to as *deletion*, while all white points remain unchanged (Hall, 1996). *Parallel reductions* can delete a set of points simultaneously, while *sequential reductions* traverse the black points of a picture, and focus on the actually visited single point for possible deletion (Hall, 1996).

Thinning algorithms iteratively apply reductions (Saha et al., 2016), and a crucial issue in thinning is to ensure *topology preservation* (Kong and Rosenfeld, 1989). The problems of verifying that existing 3D parallel thinning algorithms always preserve the topology (Kong, 1995) and how to construct such topologically correct algorithms (Palágyi et al., 2012) have been solved for pictures on the traditional cubic grid. Since it cannot be absolutely said in the case of the FCC grid, this paper establishes four sufficient conditions for topology-preserving parallel reductions acting on this unconventional 3D grid. The first two conditions focus on some configurations of deleted points. Thus they state *configuration-based* results, and they are suitable for verifying that a formerly constructed 3D reduction is topology-preserving for all possible pictures on the FCC grid. Since the remaining two conditions examine the deletability of individual points, they are said to be *point-based* sufficient conditions for topology-preserving reductions on the FCC grid. They directly provide deletion rules of parallel reductions, and make us possible to generate various topologically correct parallel thinning algorithms.

The rest of this paper is organized as follows: Section 2 gives an overview of the basic notions and re-

[a] https://orcid.org/0000-0001-9609-8628
[b] https://orcid.org/0000-0001-8857-4102
[c] https://orcid.org/0000-0002-3274-7315

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sults. Then, in Section 3 the two configuration-based sufficient condition for topology-preserving parallel reductions on the FCC are proposed. Section 4 presents our point-based sufficient conditions, and we generate directly two topology-preserving parallel reductions in Section 5. Finally, we round off this work with some concluding remarks.

2 BASIC NOTIONS AND EXISTING RESULTS

Next, we define the key concepts of digital topology as reviewed in (Kong and Rosenfeld, 1989), and recall the previously stated results that we need later on.

The *Voronoi neighborhood* of a point $p \in \mathbb{F}$ is the set of all points in the 3D Euclidean space that are at least as close to p as to any other point in the FCC grid (Kong and Rosenfeld, 1989). It is the *rhombic dodecahedron* with twelve faces centered on p, and it is called an *FCC-voxel* (*voxel*, for short). As it is illustrated by Fig. 1, there are exactly two *Voronoi adjacency relation*s on the FCC grid: 12-*adjacency* and 18-*adjacency*. The set $N_{12}(p)$ contains the twelve grid points that are at a distance of $\sqrt{2}$ from p. The set $N_{18}(p)$ is formed of the six grid points that are at a distance of 2 from p together with the twelve grid points in $N_{12}(p)$. Elements of $N_{12}(p)$ and $N_{18}(p)$ are 12-*adjacent* and 18-*adjacent* to p, respectively. Note that each voxel meets eighteen others, see Fig. 1. The voxel associated with p shares a face with the twelve voxels corresponding to the grid points in $N_{12}(p)$, and it shares just a vertex with the six voxels of $N_{18}(p) \setminus N_{12}(p)$.

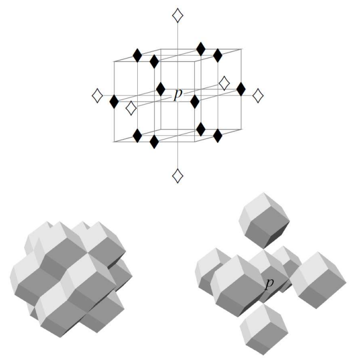


Figure 1: The studied adjacency relations on $\mathbb{F}$. The twelve points marked '♦' form the set $N_{12}(p)$, and $N_{18}(p)$ contains six more points marked '◇' (top). (Note that unmarked elements of $\mathbb{Z}^3$ are not grid points in $\mathbb{F}$.) The voxel-representations of $N_{12}(p)$ (bottom left) and $N_{18}(p) \setminus N_{12}(p)$ (bottom right), where each voxel is a rhombic dodecahedron.

An (m,n) *picture* on the FCC grid is a quadruple $(\mathbb{F}, m, n, B)$ (Kong and Rosenfeld, 1989), where $B \subseteq \mathbb{F}$ denotes the set of *black points*; each point in $\mathbb{F} \setminus B$ is said to be a *white point*; adjacency relations m and n are assigned to B and $\mathbb{F} \setminus B$, respectively. In their seminal work, Gau and Kong examined three types of pictures on the FCC grid: $(m,n) = (18,12), (12,18)$ and $(12,12)$ (Gau and Kong, 1999). Since our attention is focused on the $(18,12)$ pictures, in the rest of this paper, B denotes the set of black points in the picture $(\mathbb{F}, 18, 12, B)$.

Since both of the studied relations are symmetric, their transitive closures form equivalence relations, and their equivalence classes are called *component*s. A *black component* or an *object* is an 18-component of B, while a *white component* is a 12-component of the set of white points $\mathbb{F} \setminus B$.

A point $p \in B$ is a *border point* if $N_{12}(p) \cap (\mathbb{F} \setminus B) \neq \emptyset$ (i.e., p is 12-adjacent to at least one white point), it is an *interior point* if $N_{12}(p) \subset B$ (i.e., is not a border point), and it is an *isolated point* if $N_{18}(p) \cap B = \emptyset$ (i.e., p forms a singleton object).

Gau and Kong introduced a further concept (Gau and Kong, 1999): A set of mutually 18-adjacent points is called a *small set*. It can be readily seen that a small set may consist of at most six points, see Fig. 2. Note that a small set is an 18-component, but it does not need to be a 12-component.

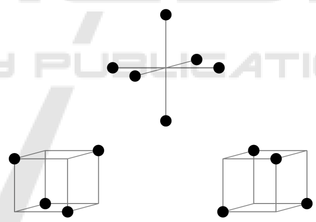


Figure 2: The three possible maximal small sets. All small sets are (nonempty) subsets of them.

A crucial issue in thinning algorithms, composed of reductions, is to ensure topology preservation (Kong and Rosenfeld, 1989; Kong, 1995). A 2D reduction is topology-preserving if and only if any object in the input picture contains exactly one object in the output picture, and any white component in the output picture contains exactly one white component in the input picture. There is an additional concept called *hole* (or *tunnel*) in 3D pictures. A hole (which donuts have) is formed from white points, but it is not a white component (Kong and Rosenfeld, 1989). To preserve topology, a 3D reduction must not create nor eliminate any hole.

There is a key concept in digital topology called a *simple point*. A simple point in a picture is a black point whose deletion is a topology-preserving reduction (Kong and Rosenfeld, 1989). Gau and Kong gave the following characterization of simple points in $(18,12)$ pictures:

Theorem 1. (Gau and Kong, 1999) *A point $p \in B$ is* simple *for B if and only if the following conditions hold:*

1. *$N_{18}(p) \cap B$ contains exactly one 18-component.*
2. *$N_{12}(p) \setminus B$ contains exactly one 12-component.*

Theorem 1 implies that only non-isolated border points may be simple, and simple points can be locally characterized (i.e., the simpleness of a point p can be decided by examining the points in $N_{18}(p)$). Figure 3 gives four illustrative examples of simple and non-simple points.

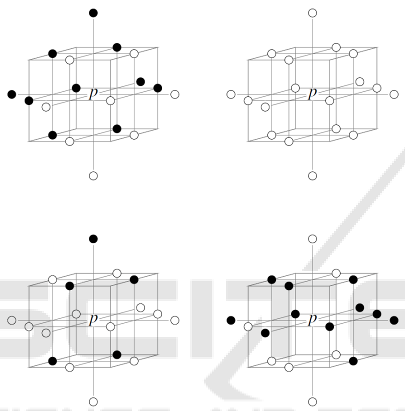


Figure 3: Examples of simple and non-simple points in $(18,12)$ pictures. The positions denoted by '•' and '∘' refer to black and white points, respectively. Black point p is simple only in the top left configuration. In the top right case, p is an isolated black point, while in the bottom left example, $N_{18}(p) \cap B$ contains two 18-components, hence both cases violate condition 1 of Theorem 1. In the bottom right figure, there are two 12-components in $N_{12}(p) \setminus B$, thus condition 2 of Theorem 1 does not hold.

It is obvious that a sequential reduction (or a thinning algorithm composed of sequential reductions) preserves the topology if and only if it deletes only simple points. Unlike the sequential case, parallel reductions can delete a set of points simultaneously. Thus we need to consider what is meant by topology preservation when more than one point is deleted at a time.

We are to define the concepts of a simple set, a simple sequence, and a minimal non-simple set.

Definition 1. (Kong, 1995) *Let* $\mathbf{P}$ *be an arbitrary picture. A set of k black points Q is a* simple set *in* $\mathbf{P}$ *if it is possible to arrange the elements of Q in a sequence $\langle q_1,\ldots,q_k\rangle$ such that q_1 is simple in* $\mathbf{P}$ *and each q_i is simple after the set of points $\{q_1,\ldots,q_{i-1}\}$ is deleted* $(i=2,\ldots,k)$. *Such a sequence is called a* simple sequence. *(And let the empty set be simple.)*

Definition 2. (Ronse, 1988) *A set of black points is a* minimal non-simple (MNS) set *in an arbitrary picture if it is not simple, but any of its proper subsets is simple.*

Figure 4 presents examples of simple, non-simple, and MNS sets in an $(18,12)$ picture.

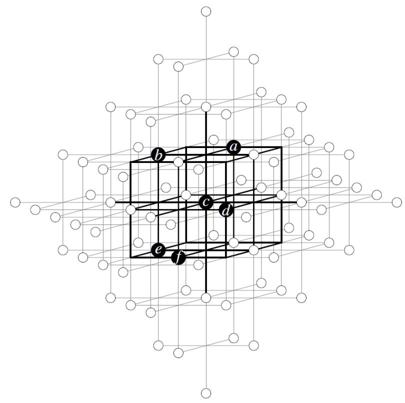


Figure 4: Examples of simple and non-simple sets. The set of black points $\{a,b,c,d\}$ is simple since the 16 sequences (of the possible 24 ones) $\langle a,b,c,d\rangle$, $\langle a,b,d,c\rangle$, $\langle a,c,b,d\rangle$, $\langle a,c,d,b\rangle$, $\langle a,d,b,c\rangle$, $\langle a,d,c,b\rangle$, $\langle b,a,c,d\rangle$, $\langle b,a,d,c\rangle$, $\langle b,d,a,c\rangle$, $\langle c,a,b,d\rangle$, $\langle c,a,d,b\rangle$, $\langle c,d,a,b\rangle$, $\langle d,a,b,c\rangle$, $\langle d,a,c,b\rangle$, $\langle d,b,a,c\rangle$, $\langle d,c,a,b\rangle$ are simple. The set $\{b,c\}$ is minimal non-simple, since both sequences $\langle b,c\rangle$ and $\langle c,b\rangle$ are non-simple and both proper subsets $\{b\}$ and $\{c\}$ of $\{b,c\}$ are simple. The set $\{b,c,d\}$ is non-simple but not minimal non-simple, since $\{b,c\}$ is its proper non-simple subset. Note that points a, b, d, e, and f are all 18-adjacent to point c.

We state the following proposition that is a straightforward consequence of Definition 1:

Proposition 1. *Let $Q \subset B$ be a simple set for B. If $p \in (B \setminus Q)$ is a simple point for $B \setminus Q$, $Q \cup \{p\}$ is also a simple set for B.*

Here, we recall a general lemma stated by Kardos and Palágyi:

Lemma 1. (Kardos and Palágyi, 2015) *Let p and q be two black simple points in an arbitrary picture. If p remains simple after the deletion of q, q remains simple after the deletion of p.*

In other words, the simpleness of a set of two simple points can be decided by examining just one sequence of its elements.

The following theorem gives a universal sufficient condition for topology-preserving parallel reductions:

Theorem 2. (Ronse, 1988) *A reduction preserves the topology for an arbitrary picture if it does not delete any MNS set.*

Gau and Kong stated the following specific property of MNS sets for $(18,12)$ pictures:

Proposition 2. (Gau and Kong, 1999) *If $Q \subseteq B$ is an MNS set, Q is a small set.*

They also identified the set of points that can be MNS sets:

Theorem 3. (Gau and Kong, 1999) *Let $Q \subseteq B$ be a nonempty small set of black points. Then Q satisfies exactly one of the following conditions:*

1. *Q is a subset of three mutually 12-adjacent points.*
2. *Q is a set of four mutually 12-adjacent points.*
3. *The points in Q are not mutually 12-adjacent (i.e., there are two points $p, q \in Q$ such that $q \notin N_{12}(p)$).*

If Q satisfies condition 1 then Q may be an MNS set without being an object. If condition 2 or condition 3 holds then Q is an MNS set if and only if Q is an object.

3 CONFIGURATION-BASED SUFFICIENT CONDITIONS

As a consequence of Theorem 3, we derived the following sufficient condition for topology-preserving parallel reductions:

Theorem 4. *A parallel reduction $\mathcal{R}$ is topology-preserving for $B \subset \mathbb{F}$ if the following conditions hold:*

1. *Any set of at most three mutually 12-adjacent points $Q \subseteq B$ deleted by $\mathcal{R}$ is simple.*
2. *$\mathcal{R}$ does not delete completely any object of B composed of four mutually 12-adjacent points (see the last two small objects in Fig. 2).*
3. *$\mathcal{R}$ does not delete completely any small object in which the points are not mutually 12-adjacent (see Fig. 5).*

Proof. To prove this theorem, we must show that $\mathcal{R}$ does not delete any MNS set (as it is required by Theorem 2). Note that all possible MNS sets are characterized by Theorem 3.

It can be readily seen that if condition i ($i = 1,2,3$) of this theorem holds, $\mathcal{R}$ does not delete any MNS set that is specified by condition i of Theorem 3. □

We can state that Theorem 4 takes some configurations of at most six points into consideration. Thus our theorem states a *configuration-based* sufficient condition for topology-preserving parallel reductions (acting on $(18,12)$ pictures of the FCC grid). Notice that, by condition 1 of Theorem 4, it is difficult to verify that an existing parallel reduction (or thinning algorithm) preserves the topology for all possible pictures. That is why, with the help of Proposition 1 and Lemma 1 we propose a simplified version of the very first configuration-based sufficient condition:

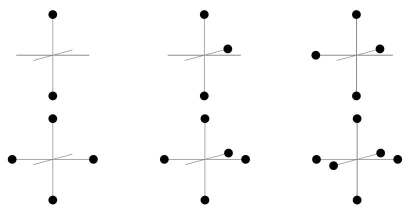


Figure 5: Six of the possible 37 small objects in which the points are not mutually 12-adjacent (see condition 3 of Theorem 4). The remaining ones are rotated and reflected versions of these six base objects.

Theorem 5. *A parallel reduction $\mathcal{R}$ is topology-preserving for $B \subset \mathbb{F}$ if the following conditions hold:*

1. *Only simple points for B are deleted by $\mathcal{R}$.*
2. *If two 12-adjacent points p and q are deleted by $\mathcal{R}$, p is simple for $B \setminus \{q\}$.*
3. *If three mutually 12-adjacent points p, q, and r are deleted by $\mathcal{R}$,*
   p is simple for $B \setminus \{q,r\}$, or
   q is simple for $B \setminus \{p,r\}$, or
   r is simple for $B \setminus \{p,q\}$.
4. *$\mathcal{R}$ does not delete completely any object of B composed of four mutually 12-adjacent points.*
5. *$\mathcal{R}$ does not delete completely any small object in which the points are not mutually 12-adjacent.*

Proof. Since the last two conditions of this theorem are the same as conditions 2 and 3 of Theorem 4, it is sufficient to show that the first three conditions of this theorem together imply condition 1 of Theorem 4.

Let us suppose that $\mathcal{R}$ satisfies all conditions of this theorem, and it deletes the set of points $D \subset B$. Let $Q \subseteq D$ be a set of at most three mutually 12-adjacent black points. Then the following three points are to be investigated:

- $Q = \{p\}$:
  By condition 1 of this theorem, point p is simple for B. Thus the singleton set Q is a simple set.
- $Q = \{p,q\}$:
  By condition 1 of this theorem, both points p and q are simple for B. By condition 2 of this theorem, p is simple for $B \setminus \{q\}$. Consequently the set of two points Q is a simple set. (Note that, by Lemma 1, q is also simple for $B \setminus \{p\}$. That is why we do not need to distinguish p and q.)
- $Q = \{p,q,r\}$:
  By condition 1 of this theorem, all the three points p, q, and r are simple for B. By condition 2 of

this theorem and Lemma 1, all the three sets of two points $\{p,q\}$, $\{p,r\}$, and $\{q,r\}$ are simple sets, and all the six sequences $\langle p,q\rangle$, $\langle q,p\rangle$, $\langle p,r\rangle$, $\langle r,p\rangle$, $\langle q,r\rangle$, and $\langle r,q\rangle$ are simple sequences. By condition 3 of this theorem and Proposition 1, at least two of the six sequences $\langle p,q,r\rangle$, $\langle q,p,r\rangle$, $\langle p,r,q\rangle$, $\langle r,p,q\rangle$, $\langle q,r,p\rangle$, and $\langle r,q,p\rangle$ are simple. Thus the set of three points Q is a simple set.

Since any set of at most three mutually 12-adjacent deleted points is a simple set, condition 1 of Theorem 4 also holds. □

4 POINT-BASED SUFFICIENT CONDITIONS

The configuration-based sufficient conditions stated in Theorems 4 and 5 are capable of verifying the topological correctness of existing parallel reductions, however, they do not serve as a methodology for designing topology-preserving parallel reductions. For this reason, here we propose *point-based* sufficient conditions that directly yield deletion rules of topology-preserving parallel reductions. The following two theorems examine the deletability of individual points:

Theorem 6. *A parallel reduction is topology-preserving for B if each point $p \in B$ deleted by this reduction satisfies the following conditions:*

1. *Point p is simple for B.*
2. *For any point $q \in N_{12}(p) \cap B$ if q is simple for B then p is simple for $B \setminus \{q\}$.*
3. *For any two points $q \in N_{12}(p) \cap B$ and $r \in N_{12}(p) \cap N_{12}(q) \cap B$ if q and r are simple for B, and q is simple for $B \setminus \{r\}$ then p is simple for $B \setminus \{q,r\}$.*
4. *Point p is not an element of an object consisting of four mutually 12-adjacent points.*
5. *Point p is not an element of a small object in which the points are not mutually 12-adjacent.*

Proof. Let us suppose that a parallel reduction satisfies all conditions of this theorem, it deletes the set of points $D \subset B$, and a black point p is in D. To prove this theorem, we must show that all conditions of Theorem 5 hold.

- Let $Q \subseteq D$ be a set of at most three mutually 12-adjacent black points. Then the following three points need to be investigated:
  - $Q = \{p\}$:
    By condition 1 of this theorem, point p is simple for B. Thus condition 1 of Theorem 5 holds.
  - $Q = \{p,q\}$.
    By condition 1 of this theorem, q is a simple point for B. By condition 2 of this theorem, p is simple for $B \setminus \{q\}$. Thus condition 2 of Theorem 5 is satisfied.
  - $Q = \{p,q,r\}$.
    By condition 1 of this theorem, q and r are both simple points for B. By condition 2 of this theorem, q is simple for $B \setminus \{r\}$. By condition 3 of this theorem, p is simple for $B \setminus \{q,r\}$. Hence condition 3 of Theorem 5 holds.
- By condition 4 of this theorem, none of the elements of an object consisting of four mutually 12-adjacent points may be deleted. Since such an object cannot be deleted completely, condition 4 of Theorem 5 is satisfied.
- By condition 5 of this theorem, none of the elements of a small object in which the points are not mutually 12-adjacent may be deleted. Since such objects cannot be deleted completely, condition 5 of Theorem 5 is also satisfied.

Since all the five conditions of Theorem 5 hold, this theorem is true. □

Conditions of Theorem 6 may be viewed as *symmetric* since elements in the examined sets points are not distinguished.

Let us focus on the addressing scheme shown in Fig. 6, which maps every point in $\mathbb{F}$ to a triplet of integer coordinates. The *lexicographical order* relation '$\prec$' between two distinct points $p = (p_x, p_y, p_z)$ and $q = (q_x, q_y, q_z)$ is defined as follows:

$$p \prec q \quad \Leftrightarrow \quad (p_z < q_z) \vee (p_z = q_z \wedge p_y < q_y) \vee (p_z = q_z \wedge p_y = q_y \wedge p_x < q_x)$$

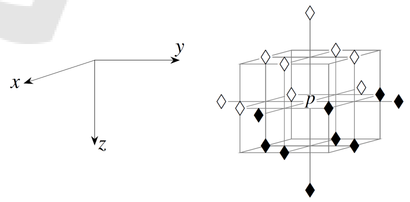


Figure 6: The considered coordinate system (left) and the ordering scheme for the FCC grid (right). The elements of the set of nine points $\{\ q \mid q \in N_{18}(p),\ p \prec q\}$ are marked '♦', and the remaining nine points in the set $\{\ r \mid r \in N_{18}(p),\ r \prec p\}$ are marked '♢'.

Let $Q \subset \mathbb{F}$ be a finite set of points. Point $p \in Q$ is the *smallest element* of Q if for any $q \in Q \setminus \{p\}$, $p \prec q$.

With the help of the proposed ordering (see Fig. 6), we state the following *asymmetric point-based condition* for topology-preserving parallel reductions:

Theorem 7. *A parallel reduction is topology-preserving for B if each point $p \in B$ deleted by that reduction satisfies the following conditions:*

1. *Point p is simple for B.*
2. *For any point $q \in N_{12}(p) \cap B$ if q is simple for B then p is simple for $B \setminus \{q\}$, or $q \prec p$.*
3. *For any two points $q \in N_{12}(p) \cap B$ and $r \in N_{12}(p) \cap N_{12}(q) \cap B$ if q and r are simple for B, and q is simple for $B \setminus \{r\}$ then p is simple for $B \setminus \{q,r\}$, or p is not the smallest element of $\{p,q,r\}$.*
4. *Point p is not the smallest element of an object consisting of four mutually 12-adjacent points.*
5. *Point p is not the smallest element of a small object in which the points are not mutually 12-adjacent.*

Proof. Let us suppose that a parallel reduction satisfies all conditions of this theorem, and it deletes the set of points $D \subset B$. To prove this theorem, we must show that all the five conditions of Theorem 5 hold.

- Let $Q \subseteq D$ be a set of at most three mutually 12-adjacent black points. Then the following three points need to be investigated:
  - $Q = \{u\}$:
    By condition 1 of this theorem, point u is simple for B. Thus condition 1 of Theorem 5 holds.
  - $Q = \{t,u\}$, where $t \prec u$.
    By condition 1 of this theorem, both points t and u are simple for B. Since $t \prec u$, by condition 2 of this theorem, t is simple for $B \setminus \{u\}$. Thus condition 2 of Theorem 5 is satisfied.
  - $Q = \{s,t,u\}$, where $s \prec t \prec u$.
    By condition 1 of this theorem, s, t, and u are simple points for B. By condition 2 of this theorem, t is simple for $B \setminus \{u\}$. Since s is the smallest element of $\{s,t,u\}$, by condition 3 of this theorem, s is simple for $B \setminus \{t,u\}$. Hence condition 3 of Theorem 5 holds.
- By condition 4 of this theorem, the smallest element of an object formed of four mutually 12-adjacent points cannot be deleted. Thus that object cannot be deleted completely, and condition 4 of Theorem 5 is satisfied.
- By condition 5 of this theorem, the smallest element of a small object in which the points are not mutually 12-adjacent cannot be deleted. Thus that object cannot be deleted completely, and condition 5 of Theorem 5 also holds.

Since all the five conditions of Theorem 5 are satisfied, the proof is completed. □

5 TWO GENERATED TOPOLOGY-PRESERVING PARALLEL REDUCTIONS

In this section we show that our point-based sufficient conditions (see Theorems 6 and 7) allow us to construct directly topology-preserving parallel reductions.

Definition 3. *A black point is deleted by the parallel reduction $\mathcal{S}$ if it satisfies all conditions of Theorem 6 (i.e., symmetric point-based condition for topology-preserving parallel reductions).*

Definition 4. *A black point is deleted by the parallel reduction $\mathcal{AS}$ if it satisfies all conditions of Theorem 7 (i.e., asymmetric point-based condition for topology-preserving parallel reductions).*

The *support* of an image operator O is the minimal set of points whose values determine whether a point is changed by O (Hall, 1996). The support of the parallel reduction $\mathcal{S}$ contains 84 points (see Fig. 7), and the count of points is 64 in the support of the parallel reduction $\mathcal{AS}$ (see Fig. 8).

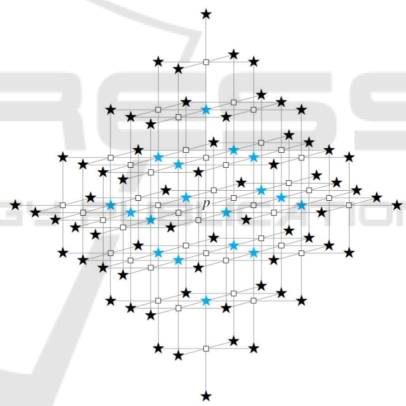


Figure 7: The 84 points marked '★' are in the symmetric support of reduction $\mathcal{S}$. Note that elements of the $N_{18}(p)$ are colored cyan, and points denoted '□' are not elements of the FCC grid.

By Theorems 6 and 7, it is obvious that both derived parallel reductions $\mathcal{S}$ and $\mathcal{AS}$ are topology-preserving. It can be readily seen that reduction $\mathcal{AS}$ can delete more points from a picture than reduction $\mathcal{S}$ does. Figure 9 illustrates the difference between the two derived reductions.

Figures 10–12 give three illustrative examples of the otherness of $\mathcal{S}$ and $\mathcal{AS}$, in which these reductions are repeated until no voxels are deleted. Numbers in parentheses are the counts of voxels in the original objects and the produced residues.

We can state that the iterated asymmetric reduction $\mathcal{AS}$ could extract the *topological kernel* from all

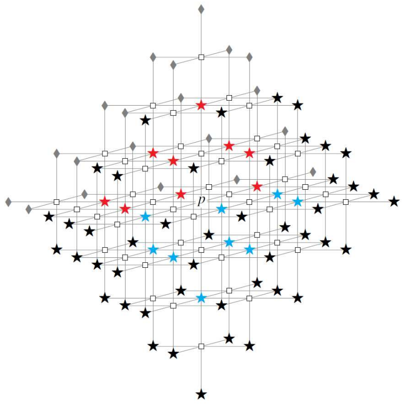
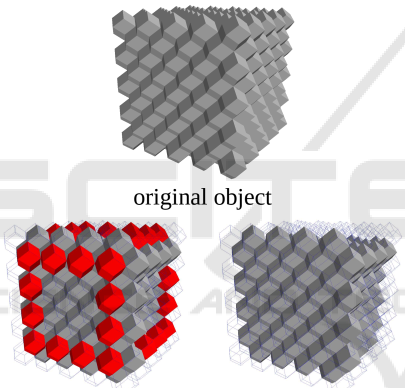


Figure 8: The 64 points in the asymmetric support of reduction $\mathcal{AS}$ marked '★'. The elements of the sets $\{\, q \mid q \in N_{18}(p),\ p \prec q\}$ and $\{\, r \mid r \in N_{18}(p),\ r \prec p\}$ are colored cyan and red, respectively. Note that grid points marked '♦' are just taken into consideration by the symmetric reduction $\mathcal{S}$ (see Fig. 7), and the points denoted '□' are not in the FCC grid.



original object

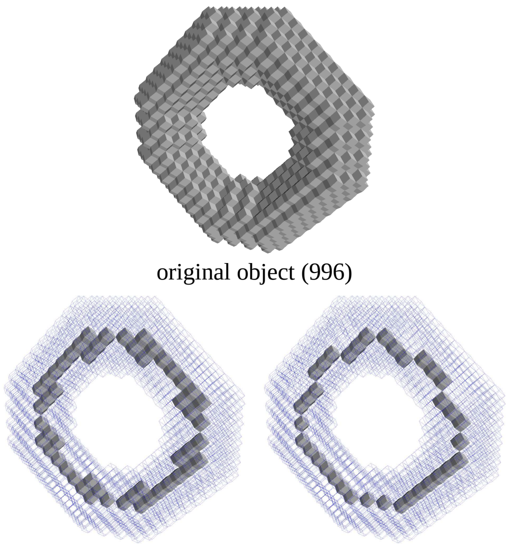
reduced by $\mathcal{S}$ reduced by $\mathcal{AS}$

Figure 9: A $9 \times 9 \times 9$ cube containing 365 voxels and its reduced versions produced by the two derived reductions. The symmetric reduction $\mathcal{S}$ and the asymmetric reduction $\mathcal{AS}$ deleted 122 and 154 voxels, respectively. Note that red voxels are not deleted by $\mathcal{S}$, while $\mathcal{AS}$ managed to remove them.

the three test objects (see Figs. 10–12). A topological kernel of an object is a minimal set of points that is topologically equivalent (Kong and Rosenfeld, 1989) to the original object. It can be readily seen that there is no simple point in a topological kernel.

Due to the conditions of Theorem 6 the iterated symmetric reduction $\mathcal{AS}$ may produce 2-voxel wide line segments with a number or simple points in them. Thus the iterated symmetric reduction $\mathcal{AS}$ cannot extract topological kernels.

It is worthy of note that since the $9 \times 9 \times 9$ cube (see Fig. 9) is symmetric, and it is free from holes and cavities (i.e., internal bubbles), both iterated reductions can shrink that object to an isolated black point (i.e., a singleton object).



original object (996)

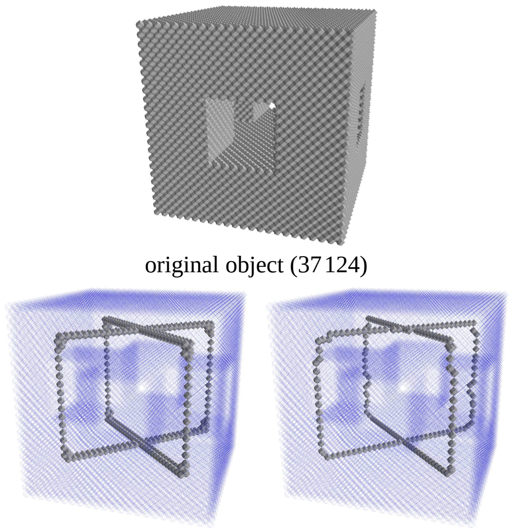
shrunk by $\mathcal{S}$ (52) shrunk by $\mathcal{AS}$ (36)

Figure 10: A $26 \times 25 \times 7$ image of a torus and its shrunk versions produced by the two iterated reductions.

original object (1 505)

shrunk by $\mathcal{S}$ (48) shrunk by $\mathcal{AS}$ (33)

Figure 11: A $23 \times 27 \times 9$ image of a letter "A" and its shrunk versions produced by the two iterated reductions.



original object (37 124)

shrunk by $\mathcal{S}$ (316) shrunk by $\mathcal{AS}$ (188)

Figure 12: A $45 \times 45 \times 45$ image of a cube with two holes and its shrunk versions produced by the two iterated reductions.

6 CONCLUSIONS

In this paper, we gave two configuration-based and two point-based sufficient conditions for topology-preserving parallel reductions acting on $(18,12)$ pictures of the FCC grid. The configuration-based conditions provide methods of verifying that formerly constructed parallel reductions preserve the topology. Our point-based conditions directly provide deletion rules of topology-preserving parallel reductions, and allow us to construct topologically correct parallel thinning algorithms.

As a future work we intend to combine our point-based conditions with parallel thinning strategies and geometric constraints to generate a family of topology-preserving parallel thinning algorithms on the FCC grid.

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REFERENCES

Čomić, L. and Magillo, P. (2020). On Hamiltonian cycles in the FCC grid. *Computers & Graphics*, 89:88–93.

Čomić, L. and Nagy, B. (2016). A topological 4-coordinate system for the face centered cubic grid. *Pattern Recognition Letters*, 83:67–74.

Edelsbrunner, H., Iglesias-Ham, M., and Kurlin, V. (2015). Relaxed disk packing. In *Proc. 27th Canadian Conference on Computational Geometry, CCCG 2015*. Queen's University, Ontario, Canada.

Gastineau, N. and Togni, O. (2021). Coloring of the d^{th} power of the face-centered cubic grid. *Discussiones Mathematicae - Graph Theory*, 41:1001–1020.

Gau, C. J. and Kong, T. Y. (1999). Minimal nonsimple sets of voxels in binary images on a face-centered cubic grid. *Int. J. Pattern Recognition and Artificial Intelligence*, 13:485–502.

Hall, R. W. (1996). Parallel connectivity-preserving thinning algorithms. In Kong, T. Y. and Rosenfeld, A., editors, *Topological algorithms for digital image processing*, pages 145–179. Elsevier Science.

Herman, G. T. (1998). *Geometry of digital spaces*. Birkhäuser.

Kardos, P. and Palágyi, K. (2015). Topology preservation on the triangular grid. *Annals of Mathematics and Artificial Intelligence*, 75:53–68.

Kong, T. Y. (1995). On topology preservation in 2-d and 3-d thinning. *Int. J. Pattern Recognition and Artificial Intelligence*, 9:813–844.

Kong, T. Y. and Rosenfeld, A. (1989). Digital topology: Introduction and survey. *Computer Vision, Graphics, and Image Processing*, 48:357–393.

Koshti, G., Biswas, R., Largeteau-Skapin, G., Zrour, R., Andres, E., and Bhowmick, P. (2018). Sphere construction on the FCC grid interpreted as layered hexagonal grids in 3D. In Barneva, R., Brimkov, V., and Tavares, J., editors, *Combinatorial Image Analysis*, pages 82–96. Springer International Publishing, LNCS, vol 11255.

Palágyi, K., Németh, G., and Kardos, P. (2012). Topology preserving parallel 3d thinning algorithms. In Brimkov, V. E. and Barneva, R. P., editors, *Digital Geometry Algorithms. Theoretical Foundations and Applications to Computational Imaging*, pages 165–188. Springer.

Rácz, G. F. and Csébfalvi, B. (2018). Cosine-weighted B-spline interpolation on the face-centered cubic lattice. *Computer Graphics Forum*, 37:503–511.

Ronse, C. (1988). Minimal test patterns for connectivity preservation in parallel thinning algorithms for binary digital images. *Discrete Applied Mathematics*, 21:67–79.

Saha, P. K., Borgefors, G., and Sanniti di Baja, G. (2016). A survey on skeletonization algorithms and their applications. *Pattern Recognition Letters*, 76:3–12.

Strand, R. and Stelldinger, P. (2007). Topology preserving marching cubes-like algorithms on the face-centered cubic grid. In *14th Int. Conference on Image Analysis & Processing*, pages 781–786.