Logic of Awareness in Agent's Reasoning

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Abstract: The aim of this study is to formally express awareness for modeling practical agent communication. The notion of awareness has been proposed as a set of propositions for each agent, to which he/she pays attention, and has contributed to avoiding *logical omniscience*. However, when an agent guesses another agent's knowledge states, what matters are not propositions but are accessible possible worlds. Therefore, we introduce a partition of possible worlds connected to awareness, that is an equivalence relation, to denote *indistinguishable* worlds. Our logic is called Awareness Logic with Partition ($\mathcal{ALP}$). In this paper, we first show a running example to illustrate a practical social game. Thereafter, we introduce syntax and Kripke semantics of the logic and prove its completeness. Finally, we outline an idea to incorporate some epistemic actions with dynamic operators that change the state of awareness.

1 INTRODUCTION

In a society of rational agents, communication among them can be defined by means of message's exchanges in which each message is represented by a logical formula. In this context, a recipient agent may change or revise his/her belief according to the received message to maintain the logical consistency of knowledge.

First, we denote a unit of knowledge by $\varphi, \psi, \cdots$ and write $K_a\varphi$ for 'agent a knows φ,' or we may write anonymously $K\varphi$. In such a formalization, *logical omniscience* matters; in ordinary logic, we employ *Modus Ponens* (MP)[1] for logical inference, and when one knows φ and $\varphi \rightarrow \psi$, i.e., $K\varphi$ and $K(\varphi \rightarrow \psi)$, respectively, $K\psi$ would necessarily be inferred in his/her knowledge if we adopt the axiom K.[2] However, such exhaustive reasoning is unrealistic for human model. The logic of an agent's knowledge/belief is called *epistemic logic*, and its semantics is given by a Kriple model that consists of a set of possible worlds where each world has different valuation for propositions, to which each agent may or may not be accessible. When an agent can access two worlds of different valuations, e.g., each of which includes φ or $\neg\varphi$, he/she does not know whether φ is true or not, that is, $\neg K\varphi$. On the contrary, when an agent can find φ in all his/her accessible worlds, $K\varphi$ holds. To avoid logical omniscience, we need to restrict propositions to be employed in reasoning, apart from those not to be employed.

(Fagin and Halpern, 1988) proposed components that represent agents' state of awareness called an *awareness set* and incorporated it into epistemic logic. This logic distinguishes the knowledge that the agents cannot use for their reasoning, called *implicit knowledge*, from that they can, called *explicit knowledge*. The former, implicit knowledge, represents unaware information. The idea of (Fagin and Halpern, 1988) is

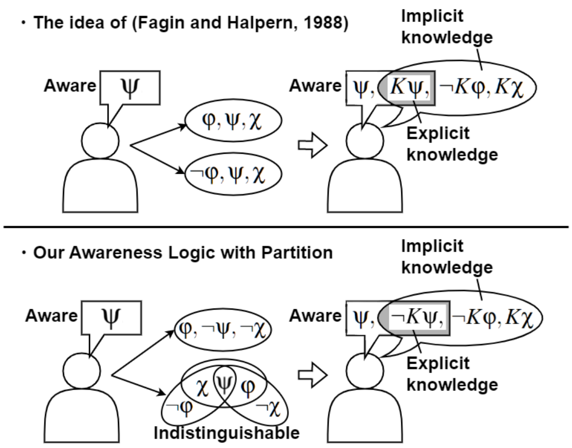

Figure 1: A comparison on the intuitions of the previous study and this paper.

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[1] From φ and $\varphi \rightarrow \psi$, we conclude ψ.
[2] K: $K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$.

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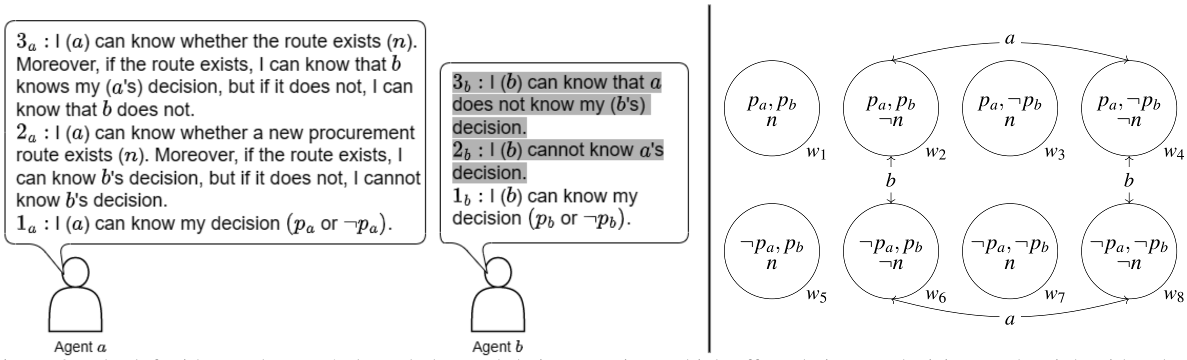

Figure 2: The left side: each agent's knowledge and their reasoning, which affect their own decisions. The right side: the Kripke model from a's viewpoint.

to classify knowledge into implicit or explicit knowledge according to whether an agent is aware of the proposition (Figure 1). It is a simple, intuitive definition and the main approach in logic of awareness.

However, we argue that awareness should also affect the distinction of possible worlds in addition to propositions. In the previous study, awareness only concerns the propositions. However, when an agent is unaware of a certain proposition, he/she must not also be aware of the distinction of two possible worlds. In figure 1, the agent accesses only the possible worlds where χ is true despite unawareness of χ. This indistinction plays an important role when agents make inferences about other agents' knowledge states, as shown in our illustrative example in Section 2. Therefore, in this paper, we propose a framework to mention that two possible worlds are indistinguishable from the viewpoint of an agent. Our logic can allow us to handle reasoning correctly about what knowledge other agents have and also enables to formalize practical agent communications.

Besides, in game theory, players make their own decisions by guessing other players' reasoning. It is based on specific decision criteria, such as the best strategy to a dominant strategy. Whether player b is aware of actions that player a can take affects b's strategy to find an equilibrium. In this sense, our logic is supposed to be useful in its application to game theory.

The paper is structured as follows. In Section 2, we introduce an example about inferences among multi-agents that shows the necessity for introducing our logic. In Section 3, we introduce *Awareness Logic with Partition* ($\mathcal{ALP}$), which is based on Awareness Logic (Fagin and Halpern, 1988). Its semantics was given in the Kripke-style. We add a new equivalence relation, which is connected to the states of an agent's awareness from another agent's viewpoint, to the standard Kripke model. Besides, we show how our logic works using the example presented in Section 2. In Section 4, we give a proof system **ALP** of our logic $\mathcal{ALP}$ in Hilbert-style. As for proving the completeness theorem, we use techniques of logic of the modality for transitive closure (van Ditmarsch et al., 2007). Section 5 discusses two epistemic actions: *becoming aware of* and *becoming unaware of*, and gives an extension of $\mathcal{ALP}$. In Section 6, we introduce some related work. Section 7 concludes.

2 EXAMPLE: CONVENIENCE STORE'S EXPANSION

This section gives an example at convenience stores. It describes a situation where agents have different states of awareness.

Example. *Let agent a be the owner of the convenience store A and agent b be the owner of the convenience store B considering to open his/her own new store. The cost of products has risen due to poor harvests, and a reckless expansion leads to a significant loss. Agent a is aware of a new procurement route that allows the owners to purchase products at half the current price. Moreover, a is unaware that b is unaware of the existence of the new route.*

In this example, if owner a can know b's decision, that decision can be a helpful factor in a's decision. For example, if owner b decides to open a new store, owner a is also likely to decide to expand a new store because otherwise, it may be disadvantageous.

We denote p_a and p_b for the propositions that a and b expand their stores, respectively, and n for the proposition that there is a new procurement route. As for the agents' knowledge, a can know p_a and p_b in the case that n is true. On the other hand, b can know only p_b in any case because he/she does not have a clue about a's knowledge state. Then, we give the Kripke model in Figure 2 to see how each agent guesses the opponent's knowledge. Note that we omit reflexivity on accessibility relations from a figure for visibility. Basically, each agent does not know the decision of the opponent, i.e., b's decision for a and a's decision for b are unknown to each other. However, in possible worlds where n holds, a can know p_b, and b can know p_a. Thus, there is no accessibility relation between possible worlds such that it has a different valuation for each p_b and p_a and n is true. As for states of awareness, a is aware of p_a, p_b, and n. However, b is unaware of n. We can summarize it as awareness sets $\mathscr{A}_a = \{p_a, p_b, n\}$ for a and $\mathscr{A}_b = \{p_a, p_b\}$ for b.

Note that 2_b and 3_b in Figure 2 cannot be correctly represented by the existing method. Let K_a and K_b be operators expressing each explicit knowledge. We consider the truth value of $K_b K_a p_b$ at w_1, which means we assume w_1 as the actual world. In the method of (Fagin and Halpern, 1988), a K_i operator for each agent i is defined to be satisfied when the proposition is included in an awareness set and holds in all the accessible worlds (Figure 1). Thus, $K_b K_a p_b$ at w_1 implies that p_b is contained in b's awareness set, and $K_a p_b$ holds at w_1 that is accessible from w_1 on the edge labeled b. Also, $K_a p_b$ implies that p_b is contained in a's awareness set, and p_b holds at w_1 that is accessible from w_1 on the edge labeled a. Since p_b holds in w_1, the formula holds at w_1.

However, $K_b K_a p_b$ contradicts 3_b in Figure 2, and our intuition considers that this formula should not hold at w_1. It is because b is unaware of n and cannot think that a has a way of knowing p_b at all. This unawareness means that some worlds, such as w_1 and w_2, are indistinguishable from b's viewpoint. Therefore, the Kripke model from b's viewpoint should be the form of Figure 3, in which $K_b K_a p_b$ must be evaluated by whether p_b holds in all possible worlds that are w'_1, w'_2, w'_3 and w'_4. Thus, $K_b K_a p_b$ does not hold. Figure 2 is a Kripke model from a's viewpoint who is aware of all atomic propositions discussed, which is different from that of b, because atomic propositions that a is aware of are different from of those b is aware. Besides, a formula $K_b p_a$ also holds at w_1 and contradicts 2_b for the same reason.

As we have seen, the existing method cannot treat some information used in the decision-making. In order to correctly represent inference of agents, it is necessary not only to classify knowledge using states of awareness about propositions, but also to consider the distinction of possible worlds corresponding to agents' viewpoints.

Models that can distinguish between *awareness of* and *awareness that* has been proposed in the field of philosophy. The models allow us to represent a more accurate description of situations (Grossi and Velázquez-Quesada, 2015; Fernández-Fernández and Velázquez-Quesada, 2021). The former is awareness in the sense of being able to refer to the information. The latter is awareness in the sense of acknowledging that the information is true through reasoning or observation. Although both concepts are similar, they have different properties. Previously proposed logic defines explicit knowledge by combining these two concepts. In this paper, we focus on 'awareness of.' This is because the concept is more relevant to the example, and we do not need to consider the other one.

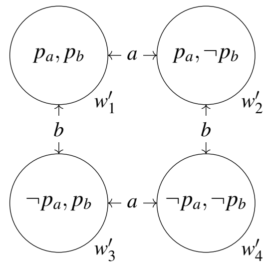

Figure 3: The Kripke model from b's viewpoint.

3 AWARENESS LOGIC WITH PARTITION

3.1 Language

First of all, we define the syntax of $\mathcal{ALP}$. Let $\mathcal{P}$ be a countable set of atomic propositions and $\mathcal{G}$ be a countable set of agents. The language $\mathcal{L}_{\mathcal{P}}$ is the set of formulas generated by the following grammar:

$$\mathcal{L}_{\mathcal{P}} \ni \varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid A^i_j \varphi \mid L_j \varphi \mid [\equiv]^i_j \varphi \mid C^i_j \varphi \mid K^i_j \varphi,$$

where $p \in \mathcal{P}$ and $i, j \in \mathcal{G}$. Other logical connectives $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined in the usual manner.

We call A^i_j, L_j, and K^i_j as an awareness operator, an implicit knowledge operator, and an explicit knowledge operator, respectively. Notationally,

- $A^i_j \varphi$ means φ is information that j is aware of from i's viewpoint.
- $L_j \varphi$ means that φ is j's implicit knowledge.
- $K^i_j \varphi$ means that φ is j's explicit knowledge from agent i's viewpoint.

$[\equiv]^i_j$ and C^i_j are special operators introduced to define explicit knowledge and used as the basis for proofs studied in this paper. The former operator means that the information is true at j's state of awareness from

i's viewpoint. The latter means that φ is a kind of j's implicit knowledge from agent i's viewpoint. Note that implicit knowledge referred to by $C^i_j\varphi$ is stronger than that referred to by $L_j\varphi$. It might be interesting to explore the relationship of these two operators; however, it is outside our scope and remains as our future task.

3.2 Semantics

Now, we move on to the semantics of $\mathcal{ALP}$.

Definition 1. *An epistemic model with awareness M is a tuple $\langle W, \{R_i\}_{i\in\mathcal{G}}, V, \{\mathscr{A}^i_j\}_{i,j\in\mathcal{G}}, \{\equiv^i_j\}_{i,j\in\mathcal{G}}\rangle$ consists of a domain W, a set of accessibility relations R_i, a valuation function V, a set of awareness sets $\mathscr{A}^i_j$, and a set of relations $\equiv^i_j$, where:*

- *W is a non-empty set of possible worlds;*
- *$R_i \subseteq W \times W$ is an equivalence relation on W;*
- *$V : \mathcal{P} \to 2^W$;*
- *$\mathscr{A}^i_j$ is a non-empty set of atomic propositions satisfying that $\mathscr{A}^i_j \subseteq \mathscr{A}^i_i$;*
- *$(w,v) \in \equiv^i_j$ iff $(w \in V(p)$ iff $v \in V(p)$ for every $p \in \mathscr{A}^i_j)$.*

The pair (M,w) with M and $w \in W$ in it is called a pointed model. We can say that i's viewpoint is formally an epistemic model with awareness where the superscript index is restricted to i, that is $\langle W, \{R_j\}_{j\in\mathcal{G}}, V, \{\mathscr{A}^i_j\}_{j\in\mathcal{G}}, \{\equiv^i_j\}_{j\in\mathcal{G}}\rangle$. We call the restricted model i's epistemic model with awareness.

The condition $\mathscr{A}^i_j \subseteq \mathscr{A}^i_i$ means that atomic propositions of which j is aware from i's viewpoint do not contain a proposition of which i himself/herself is unaware. We call $\equiv^i_j$ *indistinguishable relations* for j from i's viewpoint. An indistinguishable relation $\equiv^i_j$ is a relation between possible worlds with a different valuation for atomic propositions that j is unaware of from i's viewpoint. This represents that, from i's viewpoint, j cannot distinguish such possible worlds. By partitioning W using an indistinguishable relation, we can formalize knowledge according to the propositions of which the agent is aware. Possible worlds that are indistinguishable because of being unaware are collapsed with an equivalence class.

Note that there are local and global definitions of an awareness set. The former defines $\mathscr{A}_i$ as a function that takes a possible world as an argument and changes elements of an awareness set for each possible world. The latter defines an awareness set as the same in all possible worlds. Generally, a state of awareness is fixed within an agent's scope, which is a set of the agent's accessible possible worlds. Thus, a global definition is used in the logic that does not consider the outside of a specific agent's scope, such as a single-agent case. On the other hand, a local definition can represent a state of awareness in possible worlds outside the agent's scope. It is possible to express the possibility that there is a difference between the state of an agent's awareness in his/her scope and that in other agents' scope.

This logic adopts the global one, because even with the global definition, it is possible to express the possibility that the state of an agent's awareness from his/her viewpoint is different from that from other agents' viewpoints, which is an advantage of the local definition. It follows easily from the definition that every state of awareness is uniquely set for each agent.

We move on to the satisfaction relation. At first, we introduce some notations for the definition: $At(\varphi)$ is denoted as the set of atomic propositions that appear in φ; $R_j \circ \equiv^i_j$ is denoted as a sequential composition of $\equiv^i_j$ and R_j; R^+ is denoted as the transitive closure of R. This R^+ is the smallest set such that $R \subseteq R^+$, and for all x,y,z, if $(x,y) \in R^+$ and $(y,z) \in R^+$, then $(x,z) \in R^+$.

Definition 2. *For any epistemic models with awareness M and possible worlds $w \in W$, the satisfaction relation $\vDash$ is given as follows:*

$$
\begin{aligned}
&M,w \vDash p \text{ iff } w \in V(p);\\
&M,w \vDash \neg\varphi \text{ iff } M,w \nvDash \varphi;\\
&M,w \vDash \varphi \wedge \psi \text{ iff } M,w \vDash \varphi, \text{ and } M,w \vDash \psi;\\
&M,w \vDash A^i_j\varphi \text{ iff } At(\varphi) \subseteq \mathscr{A}^i_j;\\
&M,w \vDash L_j\varphi \text{ iff } M,v \vDash \varphi \text{ for all } v \text{ such that } (w,v) \in R_j;\\
&M,w \vDash [\equiv]^i_j\varphi \text{ iff } M,v \vDash \varphi \text{ for all } v \text{ such that } (w,v) \in \equiv^i_j;\\
&M,w \vDash C^i_j\varphi \text{ iff } M,v \vDash \varphi \text{ for all } v \text{ such that } (w,v) \in (R_j \circ \equiv^i_j)^+;\\
&M,w \vDash K^i_j\varphi \text{ iff } M,w \vDash A^i_j\varphi, \text{ and } M,w \vDash C^i_j\varphi.
\end{aligned}
$$

From Definition 1, it spells out that if both indistinguishable relations $\equiv^i_j$ and accessibility relations R_j is equivalent, then $(R_j \circ \equiv^i_j)^+$ is equivalent. Since both relations are equivalence relations, the reverse direction on the composition is also reachable, although it consumes a few extra steps. Thus, $(R_j \circ \equiv^i_j)^+$ gives a new partition of possible worlds. From the definitions, we can also find that $[\equiv]^i_j L_j\varphi$ corresponds to $R_j \circ \equiv^i_j$. However, this relation is not equivalent, unlike its transitive closure.

Next, we define the validity in the usual way.

Definition 3. *A formula φ is valid at M, if φ holds at every pointed model M,w in M, which is denoted by*

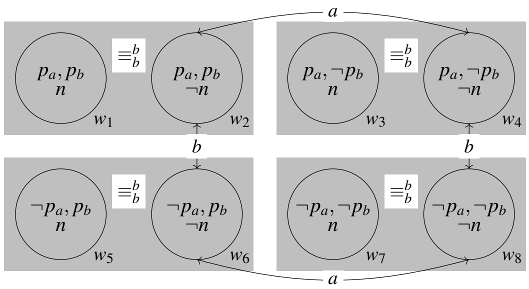

Figure 4: The Kripke model depicted by $\mathcal{ALP}$.

$M \vDash \varphi$. A formula φ is valid if φ holds at every pointed model M, w, which is denoted by $\vDash \varphi$.

3.3 Formalization of the Example

We formalize the running example using our logic and consider the truth values of $K_b K_a p_b$ again. Agent b is unaware of n. Then there are indistinguishable relations between possible worlds where n holds or not, such as w_1 and w_2. Formally, we write $\equiv_a^a$ and $\equiv_b^a$ as $\emptyset$. We also formalize $\equiv_a^b$ and $\equiv_b^b$ as $\{(w_1, w_2), (w_3, w_4), (w_5, w_6), (w_7, w_8), (w_2, w_1), (w_4, w_3), (w_6, w_5), (w_8, w_7), (w_1, w_1), \cdots, (w_8, w_8)\}$. In our logic, $K_b K_a p_b$ is rewritten as the form $K_b^b K_a^b p_b$ by introducing agents' viewpoint. This formula is evaluated by whether p_b holds in all the reachable worlds on $\equiv_b^b$, R_b, $\equiv_a^b$, and R_a. As seen in Figure 4, since it is false in some worlds, the formula does not hold at w_1. On the other hand, $K_b^b \neg K_a^b p_b$, which is consistent with 3_b in Figure 2, is true.

As for other formulas, such as $\neg K_b^b p_a$, which is consistent with 2_b, and $K_a^b K_b^b \neg K_a^b p_b$, these are also consistent with our intuition. The latter means that from b's viewpoint, a knows that b knows that a does not know p_b. A formula $K_a^a K_b^a \neg K_a^a p_b$, which has the same meaning from a's viewpoint, does not hold at w_1. Moreover, we can consider a situation where a is aware that b is unaware of n and express the situation by formalizing $\mathscr{A}_b^a$ as $\{p_a, p_b\}$. $K_a^a K_b^a \neg K_a^a p_b$ is true at w_1, which is a knows b's incorrect knowledge.

In Figure 4, the equivalence classes of the indistinguishable relation $\equiv_b^b$ are represented by the light gray background. By interpreting the equivalence class as one possible world from b's viewpoint, Figure 4 represents the same graph as the Kripke model from b's viewpoint in Figure 3 in terms of possible worlds and accessibility relations. Thus, we can say that our logic represents the distinction of possible worlds according to states of awareness for each agent.

4 HILBERT-SYSTEM FOR $\mathcal{ALP}$

We now move on to the proof theory for $\mathcal{ALP}$. The Hilbert-system of our logic is given in Table 1. AN, AC, AA, AL, A[$\equiv$], and AK mean that if an agent is aware of atomic propositions, he/she is aware of more complex formulas produced by the atomic propositions and correspond to the meaning of 'awareness of.' For $\mathrm{K_L}, \mathrm{T_L}, 5_\mathrm{L}, \mathrm{K}_{[\equiv]}, \mathrm{T}_{[\equiv]}$, and $5_{[\equiv]}$, we adopt K,T, and 5 axioms in modal logic. 5_L called negative introspection in epistemic logic means that an agent always knows what he/she does not know. This axiom also characterizes logical omniscience. In our logic, as with most logics of awareness, this formula does not hold for K_j^i operators. Instead, $\neg K_j^i \varphi \wedge A_j^i \neg K_j^i \varphi \to K_j^i \neg K_j^i \varphi$ is valid. $\mathrm{K_C}$, IND, and MIX are based on axioms of logic with common knowledge (Fagin et al., 1995), because the idea of transitive closure is the same as that one. KAC corresponds to the definition of satisfaction relation of K_j^i. It means that explicit knowledge is the things that meet implicit knowledge referred to by C_j^i and aware propositions.

Definition 4. *A system* **ALP** *is a set of formulas that contains the axioms in Table 1 and is closed under inference rules in it. We write $\vdash \varphi$ if $\varphi \in$* **ALP**. *Let Γ be a set of formulas in* **ALP** *and $\bigwedge \Gamma$ be an abbreviation of $\bigwedge_{\varphi \in \Gamma} \varphi$. If there is a finite subset Γ' of Γ such that $\vdash \bigwedge \Gamma' \to \varphi$, we write $\Gamma \vdash \varphi$ and call φ derivation from Γ.*

4.1 Soundness

Theorem 1. *If $\vdash \varphi$, then $\vDash \varphi$.*

Proof. By induction on the construction of **ALP**, we prove it for any formulas. First, we prove that all axioms are valid. For logical connectives, L_j, and $[\equiv]_j^i$ can be proven similarly to those used in **S5**. A_j^i and K_j^i are also straightforward. We show the proof of only C_j^i here.

- For $\mathrm{K_C}$, suppose that $M, w \vDash C_j^i(\varphi \to \psi)$, and $M, w \vDash C_j^i \varphi$. Since $M, v \vDash \varphi \to \psi$, and $M, v \vDash \varphi$ for all v such that $(w, v) \in (R_j \circ \equiv_j^i)^+$, $M, v \vDash \psi$. Thus $M, w \vDash C_j^i \psi$.
- For MIX, suppose that $M, w \vDash C_j^i \varphi$. Since $(R_j \circ \equiv_j^i)^+$ is equivalent and the transitive closure, $M, w \vDash \varphi \wedge [\equiv]_j^i L_j C_j^i \varphi$.
- For IND, suppose that $M, w \vDash C_j^i(\varphi \to [\equiv]_j^i L_j \varphi)$, and $M, w \vDash \varphi$, then for all v such that $(w, v) \in (R_j \circ \equiv_j^i)^+$, $M, v \vDash \varphi \to [\equiv]_j^i L_j \varphi$. Thus $M, w \vDash [\equiv]_j^i L_j \varphi$. It means φ holds at all possible worlds

Table 1: Axiom schemas and inference rules of **ALP**.

Axioms	
TAUT	The set of propositional tautologies
AN	$\vdash A^i_j\varphi \leftrightarrow A^i_j\neg\varphi$
AC	$\vdash A^i_j(\varphi\wedge\psi) \leftrightarrow A^i_j\varphi \wedge A^i_j\psi$
AA	$\vdash A^i_j\varphi \leftrightarrow A^i_jA^k_l\varphi$
A[≡]	$\vdash A^i_j\varphi \leftrightarrow A^i_j[\equiv]^k_l\varphi$
AL	$\vdash A^i_j\varphi \leftrightarrow A^i_jL_k\varphi$
AK	$\vdash A^i_j\varphi \leftrightarrow A^i_jK^k_l\varphi$
AN[≡]	$\vdash A^i_j\varphi \wedge \varphi \rightarrow [\equiv]^i_j\varphi$
K_L	$\vdash L_j(\varphi\rightarrow\psi) \rightarrow (L_j\varphi \rightarrow L_j\psi)$
T_L	$\vdash L_j\varphi \rightarrow \varphi$
5_L	$\vdash \neg L_j\varphi \rightarrow L_j\neg L_j\varphi$
$K_{[\equiv]}$	$\vdash [\equiv]^i_j(\varphi\rightarrow\psi) \rightarrow ([\equiv]^i_j\varphi \rightarrow [\equiv]^i_j\psi)$
$T_{[\equiv]}$	$\vdash [\equiv]^i_j\varphi \rightarrow \varphi$
$5_{[\equiv]}$	$\vdash \neg[\equiv]^i_j\varphi \rightarrow [\equiv]^i_j\neg[\equiv]^i_j\varphi$
K_C	$\vdash C^i_j(\varphi\rightarrow\psi) \rightarrow (C^i_j\varphi \rightarrow C^i_j\psi)$
MIX	$\vdash C^i_j\varphi \rightarrow \varphi \wedge [\equiv]^i_jL_jC^i_j\varphi$
IND	$\vdash C^i_j(\varphi \rightarrow [\equiv]^i_jL_j\varphi) \rightarrow (\varphi \rightarrow C^i_j\varphi)$
KAC	$\vdash K^i_j\varphi \leftrightarrow A^i_j\varphi \wedge C^i_j\varphi$
Inference Rules	
MP	If $\vdash\varphi$ and $\vdash \varphi\rightarrow\psi$, then $\vdash\psi$
LG	If $\vdash\varphi$ then $\vdash L_j\varphi$
[≡]G	If $\vdash\varphi$ then $\vdash [\equiv]^i_j\varphi$
CG	If $\vdash\varphi$ then $\vdash C^i_j\varphi$

from w on $R_j \circ \equiv^i_j$, and $[\equiv]^i_jL_j\varphi$ holds even at that world. Therefore, $M,w \models \varphi \rightarrow C^i_j\varphi$.

Then, it is enough to prove that if the assumptions are valid, they are also valid for all inference rules. All of them are straightforward. □

4.2 Completeness

In proof of the completeness theorem, we use the canonical model used in the proof on modal logic (Chellas, 1980). However, in **ALP**, we can take a set of formulas, such as $\Phi = \{([\equiv]^i_jL_j)^n\varphi \mid n\in\mathbb{N}\} \cup \{\neg C^i_j\varphi\}$ for each $i,j\in\mathcal{G}$, where $([\equiv]^i_jL_j)^n$ is n iterations of $[\equiv]^i_jL_j$. Therefore, our logic is no longer compact. It is necessary to restrict canonical models to a finite set of formulas. This technique is used in proof on logic with common knowledge defined by the reflexive-transitive closure of relations. We customize the tools and techniques in (van Ditmarsch et al., 2007) for our logic and use them.

First, we define *closure* as a restricted set of formulas.

Definition 5. *Let $cl : \mathcal{L} \rightarrow 2^{\mathcal{L}}$ be the function such that for every $\varphi \in \mathcal{L}$ and each $i,j\in\mathcal{G}$, $cl(\varphi)$ is the smallest set satisfying that:*

1. *$\varphi \in cl(\varphi)$;*
2. *If $\psi\in cl(\varphi)$ then $sub(\psi)\subseteq cl(\varphi)$ where $sub(\psi)$ is the set of subformulas of ψ;*
3. *If $\psi \in cl(\varphi)$ and ψ is not a form of negation, then $\neg\psi \in cl(\varphi)$;*
4. *If $C^i_j\psi \in cl(\varphi)$, then $[\equiv]^i_jL_jC^i_j\psi \in cl(\varphi)$.*

We call it the closure of φ.

Lemma 1. *For every φ, $cl(\varphi)$ is finite.*

Proof. We prove it by induction on the structure of φ. This proof is straightforward. □

Definition 6. *Let Φ be the closure of a formula. Γ is a maximal consistent set in Φ iff*

1. *$\Gamma\subseteq\Phi$;*
2. *$\Gamma \nvdash \bot$;*
3. *There is no Γ' such that $\Gamma\subset\Gamma'$ and $\Gamma'\nvdash\bot$.*

Lemma 2. *Let Φ be the closure of a formula. If Γ is a consistent set in Φ, then there exists a maximal consistent set Δ in Φ such that $\Gamma\subseteq\Delta$.*

Proof. It follows immediately from the property that Φ is finite. □

Then, a maximal consistent set can be generated at any time from a consistent set.

Next, the canonical model for a restricted set of formulas is defined as follows.

Definition 7. *Let Φ be the closure of a formula. The canonical model C(ALP) for Φ is a tuple $\langle C(W), \{C(R_j)\}_{j\in\mathcal{G}}, C(V), \{C(\mathscr{A}^i_j)\}_{i,j\in\mathcal{G}}, \{C(\equiv^i_j)\}_{i,j\in\mathcal{G}}\rangle$, where:*

- *$C(W) := \{\Gamma \mid \Gamma$ is a maximal consistent set in $\Phi\}$;*
- *$(w,v)\in C(R_j)$ iff $\{\varphi \mid L_j\varphi\in w\}\subseteq v$;*
- *$C(V)(p) := \{\Gamma \mid p\in\Gamma\}$;*
- *$C(\mathscr{A}^i_j) := \{p \mid$ for all $w\in W$, $A^i_ip\wedge A^i_jp\in w\}$;*
- *$(w,v)\in C(\equiv^i_j)$ iff $\{\varphi \mid [\equiv]^i_j\varphi \in w\}\subseteq v$.*

Lemma 3. *For every φ, the canonical model for the closure of φ is an epistemic model with awareness.*

Proof. We prove that the canonical model for the closure of φ satisfies the definition of an epistemic model with awareness.

- For $C(R_i)$, we can prove it in the same proof strategy as **S5**.
- For $C(\mathscr{A}^i_j)$, it is enough to prove that if $p\in C(\mathscr{A}^i_j)$ then $p\in C(\mathscr{A}^i_i)$ for every $p\in\mathcal{P}$. Suppose $p\in C(\mathscr{A}^i_j)$, then for all $w\in C(W)$, $A^i_ip\wedge A^i_jp\in w$. Thus, $A^i_ip\wedge A^i_ip\in w$ for all $w\in C(W)$.

- For $C(\equiv^i_j)$, it is enough to prove that for all $(w,v) \in C(\equiv^i_j)$, for every $p \in C(\mathscr{A}^i_j)$, if $w \in C(V)(p)$, then $v \in C(V)(p)$, and vice versa. From left to right, suppose that $w \in C(V)(p)$ for every $p \in C(\mathscr{A}^i_j)$. Then, $p \in w$ and $A^i_j p \in w$. $p \in v$ follows from AN[$\equiv$]. The reverse direction is proved similarly. □

We introduce C^i_j-paths on the canonical models.

Definition 8. *Let Φ be the closure of a formula. A C^i_j-path from Γ is a sequence $\Gamma_0, \cdots, \Gamma_n$ of maximal consistent sets in Φ such that $(\Gamma_k, \Gamma_{k+1}) \in C(R_j) \circ C(\equiv^i_j)$ for all k, where $0 \leq k \leq n$, and $\Gamma_0 = \Gamma$. The length of $\Gamma_0, \cdots, \Gamma_n$ is n. A φ-path is a sequence $\Gamma_0, \cdots, \Gamma_n$ of maximal consistent sets in Φ such that $\varphi \in \Gamma_k$ for all k, where $0 \leq k \leq n$.*

Lemma 4. *Let Φ be the closure of a formula and Γ, Δ be a maximal consistent set in Φ. If $\bigwedge\Gamma \wedge \neg[\equiv]^i_j L_j \neg \bigwedge\Delta$ is consistent, then $(\Gamma, \Delta) \in (C(R_j) \circ C(\equiv^i_j))$.*

Proof. Suppose that $\bigwedge\Gamma \wedge \neg[\equiv]^i_j L_j \neg\bigwedge\Delta$ is consistent, and $[\equiv]^i_j L_j \varphi \in \Gamma$ for every φ. Then, $[\equiv]^i_j L_j \varphi \wedge \neg[\equiv]^i_j L_j \neg\bigwedge\Delta$ is consistent. If $\varphi \notin \Delta$, $\neg\varphi \in \Delta$. It follows that $[\equiv]^i_j L_j \varphi \wedge \neg[\equiv]^i_j L_j \varphi$ is consistent, but this formula is a contradiction. Thus $\varphi \in \Delta$. □

Lemma 5. *Let Φ be the closure of a formula and Γ, Δ be maximal consistent sets in Φ. If $C^i_j\varphi \in \Phi$, then $C^i_j\varphi \in \Gamma$ iff every C^i_j-path from Γ is a φ-path.*

Proof. ($\Rightarrow$) We prove it by induction on the length of a C^i_j-path.

- For the base case, suppose that the length of a C^i_j-path is 0, $C^i_j\varphi \in \Phi$, and $C^i_j\varphi \in \Gamma$. Then $\Gamma = \Gamma_0 = \Gamma_n$. By MIX, $\varphi \in \Gamma$.
- For induction steps, suppose that the length of a C^i_j-path is $k+1$, $C^i_j\varphi \in \Phi$, and $C^i_j\varphi \in \Gamma$. By the induction hypothesis, $C^i_j\varphi \in \Gamma_k$. Since MIX and the definition of $C(R_j)$ and $C(\equiv^i_j)$, $\varphi \in \Gamma_{k+1}$.

($\Leftarrow$) Let $S(C^i_j, \varphi)$ be a set of maximal consistent sets Δ in Φ such that every C^i_j-path from Δ is a φ-path. We introduce a special formula:

$$\chi = \bigvee_{\Delta \in S(C^i_j, \varphi)} \bigwedge \Delta.$$

Suppose that every C^i_j-path from Γ is a φ-path. First, we need to prove these three:

(1) $\vdash \bigwedge\Gamma \to \chi$; (2) $\vdash \chi \to \varphi$; (3) $\vdash \chi \to [\equiv]^i_j L_j \chi$.

- For (1), $\Gamma \in S(C^i_j, \varphi)$ by the assumption. Thus, $\vdash \bigwedge\Gamma \to \chi$.
- For (2), since every C^i_j-path from Δ is a φ-path, $\varphi \in \Delta$ for every $\Delta \in S(C^i_j, \varphi)$. Thus, φ is derived from χ.
- For (3), we prove it by contradiction. Suppose $\chi \wedge \neg[\equiv]^i_j L_j \chi$ is consistent. By the construction of χ, there exists Δ such that $\bigwedge\Delta \wedge \neg[\equiv]^i_j L_j \chi$ is consistent. The set $\neg\bigvee_{\Theta \in C(W) \setminus S(C^i_j, \varphi)} \bigwedge\Theta$ is equivalent to χ because the disjunction of the complement of the other combinations can express a particular set of combinations represented by χ. Therefore, $\bigwedge\Delta \wedge \neg[\equiv]^i_j L_j \neg\bigvee_{\Theta \in C(W) \setminus S(C^i_j, \varphi)} \bigwedge\Theta$ is consistent. There is Θ such that $\bigwedge\Delta \wedge \neg[\equiv]^i_j L_j \neg\bigwedge\Theta$ is consistent. By Lemma 4, $(\Delta, \Theta) \in C(R_j) \circ C(\equiv^i_j)$. There exists a C^i_j-path from Δ that is not a φ-path. This is a contradiction. Thus, $\vdash \chi \to [\equiv]^i_j L_j \chi$.

By (3) and CG, $\vdash C^i_j(\chi \to [\equiv]^i_j L_j \chi)$. It follows that $\vdash \chi \to C^i_j \chi$ from IND. By (1) and (2), $\vdash \bigwedge\Gamma \to C^i_j\varphi$. Thus, $C^i_j\varphi \in \Gamma$. □

Lemma 6. *Let Φ be the closure of a formula and $C(\mathrm{ALP})$ be the canonical model for Φ. For all $w \in C(W)$ and every $\varphi \in \Phi$, if $C(\mathrm{ALP}), w \vDash \varphi$ then $\varphi \in w$.*

Proof. We prove it by induction on the structure of formulas. The cases other than A^i_j, L_j, and C^i_j are trivial, including the base case.

- For the case of $A^i_j\varphi$, we prove $A^i_j\varphi \in w$ by induction on the structure of φ. Suppose that $C(\mathrm{ALP}), w \vDash A^i_j\varphi$, then $At(\varphi) \subseteq C(\mathscr{A}^i_j)$. It means for every $p \in \mathcal{P}$, and for all $v \in C(W)$, if $p \in At(\varphi)$ then $A^i_i p \wedge A^i_j p \in v$.
  - For the base case, $A^i_j p \in v$ for all $v \in C(W)$, since $p = At(p)$. Thus, $A^i_j p \in w$.
  - For the other cases, we obtain it by induction hypothesis and decomposing the formula with corresponding axioms: AN, AC, AA, A[$\equiv$], AL, and AK.
- For the case of $L_j\varphi$, we prove it by contraposition. Suppose that $L_j\varphi \notin w$. By the definition of $C(R_j)$, $\varphi \notin v$ for all v such that $(w,v) \in C(R_j)$. Thus, $C(\mathrm{ALP}), w \nvDash L_j\varphi$.
- For the case of $[\equiv]^i_j$, we prove it by contraposition. Suppose that $[\equiv]^i_j\varphi \notin w$. By the definition of $C(\equiv^i_j)$, $\varphi \notin v$ for all v such that $(w,v) \in C(\equiv^i_j)$. Thus, $C(\mathrm{ALP}), w \nvDash [\equiv]^i_j\varphi$.
- For the case of C^i_j, suppose that $C(\mathrm{ALP}), w \vDash C^i_j\varphi$. Then, $C(\mathrm{ALP}), v \vDash \varphi$ for all v such that $(w,v) \in$

$(C(R_j)\ \circ C(\equiv^i_j))^+$. It means every C^i_j-path from w is a φ-path. By Lemma 5, $C^i_j\varphi \in w$. □

Lemma 7. *Let Φ be the closure of a formula and Γ be a maximal consistent set in Φ. For every $\varphi \in \Phi$ and every maximal consistent set Γ, if $\varphi \in \Gamma$ then $\vdash \varphi$.*

Proof. We prove it by contraposition. Suppose $\nvdash \varphi$. It means any maximal consistent sets of **ALP** do not contain φ. Thus, $\varphi \notin \Gamma$. □

Theorem 2. *For every $\varphi \in \mathcal{L_P}$, if $\vDash \varphi$, then $\vdash \varphi$.*

Proof. Suppose that $\vDash \varphi$, then $C(M), w \vDash \varphi$ for the closure of φ by Lemma 3. $\varphi \in w$ by Lemma 6. Thus, $\vdash \varphi$ by Lemma 7. □

5 EPISTEMIC ACTIONS

In epistemic logic, including logic of awareness, we formalize how the information held by agents changes for applications and understanding of concepts. In this paper, we introduce two actions that are 'becoming aware of' and 'becoming unaware of' as preparation for incorporating agent communication. These are the basic actions relevant to 'awareness of' (van Benthem and Velázquez-Quesada, 2010).

First, we add two new operators to syntax, which are $[+\varphi]^i_j$ and $[-\varphi]^i_j$ for each $i, j \in \mathcal{G}$. $[+\varphi]^i_j\psi$ reads 'j become aware of φ in i's viewpoint'. $[-\varphi]^i_j\psi$ reads 'j become unaware of φ in i's viewpoint.' We extend the satisfaction relation of $\mathcal{ALP}$ as follows:

$$M, w \vDash [+\varphi]^i_j\varphi \textit{ iff } M[+\varphi]^i_j, w \vDash \varphi;$$
$$M, w \vDash [-\varphi]^i_j\varphi \textit{ iff } M[-\varphi]^i_j, w \vDash \varphi.$$

Formulas with dynamic operators are evaluated in the updated models, which are $M[+\varphi]^i_j$ and $M[-\varphi]^i_j$. We define these as follows:

Definition 9. *$M[+\varphi]^i_j$ is a tuple $\langle W, \{R_k\}_{k\in\mathcal{G}}, V, \{\mathscr{A}[+\varphi]^k_l\}_{k,l\in\mathcal{G}}, \{\equiv^k_l\}_{k,l\in\mathcal{G}}\rangle$, where:*

$$\mathscr{A}[+\varphi]^k_l := \begin{cases} \mathscr{A}^k_l \cup \{At(\varphi)\} & k = i \textit{ and } l = j, \\ \mathscr{A}^k_l & \textit{otherwise}. \end{cases}$$

$M[-\varphi]^i_j$ is a tuple $\langle W, \{R_k\}_{k\in\mathcal{G}}, V, \{\mathscr{A}[-\varphi]^k_l\}_{k,l\in\mathcal{G}}, \{\equiv^k_l\}_{k,l\in\mathcal{G}}\rangle$, where:

$$\mathscr{A}[-\varphi]^k_l := \begin{cases} \mathscr{A}^k_l \setminus \{At(\varphi)\} & k = i \textit{ and } l = j, \\ \mathscr{A}^k_l & \textit{otherwise}. \end{cases}$$

For example, $[+n]^b_b K^b_b K^b_a p_b$ is true at w_1 in the example in Section 3.

In order to provide the corresponding Hilbert-system, there is a technique to prove the completeness theorem of logic with dynamic operators, such as PAL (Public Announcement Logic) (Plaza, 1989). The technique replaces a formula with a dynamic operator of a simple formula that is logically equivalent. For example, $[+\varphi]L_j\psi \leftrightarrow L_j[+\varphi]\psi$ holds for L_j operator. For $[\equiv]^i_j$ and C^i_j operators, we refer to (Grossi et al., 2015) for identifying possible worlds with the same valuation for atomic propositions in the changed awareness set. The logic in (Grossi et al., 2015) has the operator that represents a proposition holds at the possible world whose the same valuation for all the elements of a particular set of atomic propositions. We leave this part as our future work.

6 RELATED WORK

We introduce some logic or ideas relevant to our logic. (van Ditmarsch and French, 2009; van Ditmarsch and French, 2011) is based on a similar idea as this paper, which is to connect agents' state of awareness with the distinctions of possible worlds. The main difference is that our logic can represent not only the distinctions of possible worlds but also possible worlds searched according to the distinction. Unlike our logic, these logics can search even for worlds with a different valuation for propositions of which an agent is unaware but do not adjust accessibility.

Team semantics used in dependence logic (Sano and Virtema, 2015) also has a similar idea that formulas are true in a specific group of possible worlds. This semantics has a structure that a subset of possible worlds called a *term* supports a formula.

As for epistemic actions for awareness, several papers are using the idea of PAL (Plaza, 1989), including this paper. In particular, (Grossi and Velázquez-Quesada, 2015; Fernández-Fernández and Velázquez-Quesada, 2021) proposed a realistic formalization of epistemic actions, such as updating awareness by inference. The idea of an *action model* (Baltag et al., 1998) might help formalize agent communication in our logic. Action models control communicative actions separately from a Kripke model that decides knowledge. It allows us to formalize complex actions, such as a misleading private announcement.

Semantic approaches to awareness are active in the field of economics. It is also called the event-based approach, in which the concept of *events* that are a set of possible worlds is introduced, and knowledge is expressed as an operator on events. The logic system proposed in (Modica and Rustichini, 1994) is the early work of the approach. (Halpern, 2001) found it to be equivalent to a part of the logic in (Fagin and Halpern, 1988). Since the work of (Modica and Rustichini, 1999), the focus has been on a formalization of the concept of unawareness (Heifetz et al., 2006; Heifetz et al., 2008).

7 CONCLUSION

In this paper, we have introduced Awareness Logic with Partition ($\mathcal{ALP}$), where we incorporated the notion of partition among possible worlds and have extended the distinction of aware/unaware propositions to indistinguishable possible worlds. With this, we have properly reflected the agent's awareness to other agents' knowledge. Employing this framework, we have shown an example where the behavior of each agent could be logically explained.

Our contributions of this logic are two-fold. From the logical viewpoint, we introduced the syntax and the semantics of $\mathcal{ALP}$ and have shown its completeness. From the viewpoint of applicability to real world, we have shown the architecture to explain the strategic behavior of rational agents in a society or game theory. We expect that the logic offers a foundation for formal expressions of human minds and practical agent communication.

There are several directions in the future. On the conceptual side, we consider incorporating more epistemic actions and concepts, such as common knowledge, to represent practical agent communication. On the technical side, the axiomatic system of the dynamic extension, discussed in Section 5, remains. In addition, our logic is applicable to the studies dealing with multiple agents' reasoning, such as description and analysis of games that take into account players' awareness of possible strategies (Feinberg, 2005; Kaneko and Suzuki, 2002). Specifically, we plan to use the logic to analyze rationality to reach an equilibrium in games with awareness.

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