Rough Real Functions and Intuitionistic L-fuzziness

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Abstract: This work has been motivated by developing tools to manage rough real functions. Rough real function is a real function attached to a special Cartesian coordinate system. Its values are categorized via the x and y axes. Some papers establish a connection between the rough real functions and the intuitionistic fuzzy sets to achieve the set goal. Until now, rough real functions could only take values from the unit interval. This paper presents the possible extension of the previous methods to more realistic rough real functions. However, care must be taken to ensure that the selected tools are semantically consistent with the nature of the rough real functions.

1 INTRODUCTION

In the mid-1990s, Z. Pawlak, based on the rough set theory (RST) (Pawlak, 1982; Pawlak, 1991; Pawlak and Skowron, 2007), initiated studying the rough calculus of real functions (Pawlak, 1994; Pawlak, 1995a; Pawlak, 1996; Pawlak, 1997).

Relying on the representations of rough real functions, some papers established a connection between the rough real functions and the intuitionistic fuzzy sets (Csajbók, 2020; Csajbók, 2022). So far, these methods have only dealt with rough real functions that take values in the *unit interval*.

Very early, in 1967, the idea was born that the codomain of fuzzy sets should be a lattice, usually a complete distributive lattice L ((Pradera et al., 2007), p.17). That was the L-fuzzy set which was proposed by J. Goguen (Goguen, 1967). Soon, in 1984, K. Atanassov and S. Stoeva further generalized L-fuzzy set to intuitionistic L-fuzzy set, see (Atanassov and Stoeva, 1984) and (Atanassov, 1999).

Many papers combine rough set theory with fuzzy sets (Dubois and Prade, 1980; Dubois and Prade, 1987; Dubois and Prade, 1990; Dubois and Prade, 1992), and with intuitionistic fuzzy sets (Abdunabi and Shletiet, 2021; Tripathy, 2006; Zhou and Wu, 2011; Zhou et al., 2009).

Additional articles study dependency relationships between generalized fuzzy set theories. In several cases, equivalences between two theories have been demonstrated (Deschrijver and Kerre, 2003; Cornelis et al., 2003; Hatzimichailidis and Papadopoulos, 2007). An exhaustive list of possible connections, with appropriate literature references, can be found in ((Pradera et al., 2007), Ch. 3). In (Deschrijver and Kerre, 2003), Figure 1 graphically depicts some important relationships.

This paper is not about combining rough set theory and intuitionistic L-fuzzy sets. It starts from a nonnegative rough real function f and finds one or more special intuitionistic L-fuzzy sets that closely relate to f semantically. Since the paper deals with rough *real* functions, for the sake of simplicity, let L be a closed real interval, i.e. a linearly (totally) ordered complete lattice with the usual $\leq$ ordering.

Section 2 summarizes some basic notations. Section 3 defines the rough real functions. Section 4 outlines the representations of rough real functions. Section 5 discusses the notions of roughness and fuzziness, whereas Section 6 shows the roughness and L-fuzziness. Section 7 presents some basic facts about the roughly derived intuitionistic L-fuzzy sets.

2 BASIC NOTATIONS

To avoid misunderstandings, we summarize the main notations used in this paper.

Let U and V be two nonempty sets.

A function f is denoted by $f : U \to V, u \mapsto f(u)$ with domain U and codomain V; $u \mapsto f(u)$ is the assignment or mapping rule of f. V^U denotes the set of all functions from U into V.

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For any $S \subseteq U$, $f(S) = \{f(u) \mid u \in S\} \subseteq V$ is the direct image of S. $f(U)$ is the range of f.

Let $S \subseteq U$. S^c is the complement of S with respect to U. If $f \in V^U$, the complement of $f(S)$ with respect to V is denoted by $f^c(S)$ instead of $(f(S))^c$.

$\mathcal{P}(U)$ is the set of subsets of U, that is the power set of U.

$U \times V$ is the Cartesian product of U and V.

If $U, V \subseteq \mathbb{R}$ and $f, g \in V^U$, the operation $f \odot g$, $\odot \in \{+, -, \cdot, /\}$, the relation $f \boxdot g, \boxdot \in \{=, \leq, <, \geq, >\}$, the intervals $[f,g]$, $]f,g]$, $[f,g[$, $]f,g[$, and the ordered pair (f,g) are understood pointwise.

Let $\mathbb{R}$, $\mathbb{R}^{\geq 0}$, $\mathbb{R}^+$ represent the real numbers, non-negative real numbers, and positive real numbers, respectively.

If $a, b \in \mathbb{R}$ and $a \leq b$, $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ and $]a,b[= \{x \in \mathbb{R} \mid a < x < b\}$ are closed and open intervals. The degenerate interval $[a,a] = \{a\}$ is identified with the real number $a \in \mathbb{R}$. It is easy to interpret the open-closed $]a,b]$ and closed-open $[a,b[$ intervals.

By $\overline{[a,b]}$, $\overline{]a,b]}$, $\overline{[a,b[}$, and $\overline{]a,b[}$ we mean the closed interval $[a,b]$.

$|[a,b]|$ is the length of $[a,b]$.

$(\cdot,\cdot)$ denotes an ordered pair.

Let $[n] = \{0, 1, \ldots, n\} \subseteq \mathbb{N}$ be a finite set of natural numbers. Accordingly, $]n] = \{1, \ldots, n\}$, $[n[= \{0, 1, \ldots, n-1\}$, and $]n[= \{1, \ldots, n-1\}$.

3 ROUGH REAL FUNCTIONS

Pawlak's rough real calculus relies on the notion of Pawlak's approximation space.

Let U be a nonempty set. Then $\mathrm{PAS}(U) = (U, \mathcal{B}, \mathcal{D}_{\mathcal{B}}, \ell, u)$ is a *Pawlak's approximation space* if

- $\mathcal{B} = \Pi(U)$ is a partition of U; its equivalence classes are called *base sets*.
- $\mathcal{D}_{\mathcal{B}}$ is defined with inductive definition: $\emptyset \in \mathcal{D}_{\mathcal{B}}$, $\mathcal{B} \subseteq \mathcal{D}_{\mathcal{B}}$; if $D_1, D_2 \in \mathcal{D}_{\mathcal{B}}$, then $D_1 \cup D_2 \in \mathcal{D}_{\mathcal{B}}$.
  The members of $\mathcal{D}_{\mathcal{B}}$ are called *definable sets*.
- *Lower* and *upper approximation operators* ℓ and u are defined as
  - $\ell : \mathcal{P}(U) \to \mathcal{D}_{\mathcal{B}}, S \mapsto \cup\{B \in \mathcal{B} \mid B \subseteq S\}$;
  - $u : \mathcal{P}(U) \to \mathcal{D}_{\mathcal{B}}, S \mapsto \cup\{B \in \mathcal{B} \mid B \cap S \neq \emptyset\}$.

The main features of Pawlak's approximation space are characterized with the following notions:

- For any $S \in \mathcal{P}(U)$, the *boundary* of S is $bnd(S) = u(S) \setminus \ell(S)$.
  It is easy to check that $bnd(S)$ is also definable.
- $S \in \mathcal{P}(U)$ is *crisp* (*exact*) if $\ell(S) = u(S)$, i.e., $bnd(S) = \emptyset$.
- $S \in \mathcal{P}(U)$ is *rough* (*inexact*) if it is not exact, i.e., $bnd(S) \neq \emptyset$.

The distinguishing feature of Pawlak's approximations spaces is that the notions of exactness and definability coincide. Formally, $D \in \mathcal{D}_{\mathcal{B}}$ iff $\ell(D) = D = u(D)$.

For each $S \in \mathcal{P}(U)$, the approximation operators ℓ and u divide U into three mutual disjoint regions:

- $\mathrm{POS}(S) = \ell(S)$, *positive region* of S;
- $\mathrm{NEG}(S) = U \setminus u(S) = u^c(S)$, *negative region* of S;
- $\mathrm{BND}(S) = bnd(S)$, *boundary region* of S.

Common semantic interpretations of these regions are the following:

- positive region $\mathrm{POS}(S) = \ell(S)$ consists of all elements of U which *certainly* belong to S;
- upper approximation $u(S) = \mathrm{POS}(S) \cup \mathrm{BND}(S)$ consists of all elements of U which *possibly* belong to S;
- negative region $\mathrm{NEG}(S) = U \setminus u(S) = u^c(S)$ consists of of all elements of U which certainly do not belong to S.

After these, the starting notions of the rough calculus are the categorized interval and the rough coordinate system. Then the rough real function is a real function attached to a rough coordinate system (Pawlak, 1996; Pawlak, 1997; Pawlak, 1994; Pawlak, 1995b; Pawlak, 1995a).

Let I denote a closed interval of the form $I = [0,a]$ ($a \in \mathbb{R}^+$).

Definition 1. A *categorization* of I is a strictly monotone, finite sequence $I_S = \{x_i\}_{i \in [n]} \subset \mathbb{R}^{\geq 0}$, where $n \geq 1$ and $0 = x_0 < x_1 < \cdots < x_n = a$.

Elements of I_S are called the *categorization points* of I; I equipped with a categorization I_S is called the *I_S-categorized interval*. □

Let E_S denote the equivalence relation generated by I_S. That is for every $x, y \in I$, xE_Sy if $x = y = x_i \in I_S$ for some $i \in [n]$, or $x, y \in]x_i, x_{i+1}[$ for some $i \in [n[$. Then the partition generated by E_S is

$$I/E_S = \{\{x_0\},]x_0, x_1[, \{x_1\}, \ldots,]x_{n-1}, x_n[, \{x_n\}\},$$

where $[x_i, x_i] = \{x_i\}$ ($i \in [n]$).

The members of I/E_S are called the *rough numbers*; within the rough numbers, the singletons formed by the categorization points are also called *rough integers* or *roughly isolated points*.

The block containing $x \in I$ is denoted by $[\![x]\!]_{E_S}$.

To make the blocks of the partition I/E_S easier to handle technically, its members are enumerated as follows:

$B_0 = \{x_0\}, B_1 =]x_0, x_1[,$
$B_2 = \{x_1\}, B_3 =]x_1, x_2[, \ldots, B_{2n} = \{x_n\}.$

The blocks formed by the categorization points have *even indices*, i.e., $i \equiv 0 \,(\mathrm{mod}\, 2)$ $(i \in [2n])$, while the open intervals have *odd indices*, i.e., $i \equiv 1 \,(\mathrm{mod}\, 2)$ $(i \in [2n])$.

Blocks of I/E_S and B_i's are used interchangeably:

- If $x_i \in I_S$ for some $i \in [n]$, $[\![x_i]\!]_{E_S} = B_{2i} = \{x_i\}$; furthermore $\overline{[\![x_i]\!]}_{E_S} = \overline{B}_i = \overline{[x_i, x_i]} = [x_i, x_i] = \{x_i\}$.
- If $x \in I$ but $x \notin I_S$, $x \in [\![x]\!]_{E_S} = B_{2i+1} =]x_i, x_{i+1}[$ for some $i \in [n[$; furthermore $\overline{[\![x]\!]}_{E_S} = \overline{B}_{2i+1} = \overline{]x_i, x_{i+1}[} = [x_i, x_{i+1}]$.

In terms of rough set theory, E_S is an *indiscernibility relation* on I. The members of I/E_S are called the *base sets*, and any union of them are the *definable sets*. By definition, $\emptyset$ is definable. Their collection is denoted by $\mathcal{D}_{I/E_S}$. In rough calculus, intervals of the form $[0,x]\,(x \in I)$ are approximated. Their Pawlak's lower and upper approximation operators ℓ_S and u_S on I are defined as

- $\ell_S([0,x]) = \cup\{[\![x']\!]_{E_S} \in I/E_S \mid [\![x']\!]_{E_S} \subseteq [0,x]\}$,
- $u_S([0,x]) = \cup\{[\![x']\!]_{E_S} \in I/E_S \mid [\![x']\!]_{E_S} \cap [0,x] \neq \emptyset\}$.

In sum, $\mathsf{PAS}\,(I)_S = (I, I/E_S, \mathcal{D}_{I/E_S}, \ell_S, u_S)$ is a *Pawlak's approximation space* on I.

Let $\mathsf{PAS}(I)$ and $\mathsf{PAS}(J)$ be two Pawlak's approximation spaces defined on the intervals $I = [0,a]$ and $J = [0,b]$ on the x-axis and y-axis, respectively. Their categorizations are $I_S = \{x_0 = 0, x_1, \ldots, x_n = a\}$ and $J_P = \{y_0 = 0, y_1, \ldots, y_m = b\}$. The equivalence classes of J/E_P can be enumerated in the same way as those of I/E_S. Latter classes are referred to as C_j's $(j \in [2m])$.

A Cartesian coordinate system equipped its x and y axes with Pawlak's approximation spaces $\mathsf{PAS}\,(I)_S$ and $\mathsf{PAS}\,(J)_P$ is called the (I_S, J_P)*–rough coordinate system*.

A function $f \in J^I$ attached to a rough coordinate system is called the (I_S, J_P)*–rough real function*.

Throughout the paper, if there is no confusion, the ordered pair (I_S, J_P) will be omitted.

4 REPRESENTATIONS OF ROUGH REAL FUNCTIONS

To manage rough real functions, they must be represented. There are two types of representations, discrete and non-discrete ones. For our purposes, the non-discrete representations are the right ones. These representations may happen either pointwise or blockwise. The former comes from Pawlak.

In pointwise representation, lower and upper approximations are assigned to all function values point by point on the interval I.

Definition 2 (Pawlak, 1994). (Pawlak, 1994) Let $f \in J^I$ be an (I_S, J_P)–rough real function.

The *pointwise* (I_S, J_P)*–lower* and (I_S, J_P)*–upper approximations* of f are the functions

$$\underline{f} : I \to J_P,\ x \mapsto \max\{y \in J_P \mid y \leq f(x)\},$$
$$\overline{f} : I \to J_P,\ x \mapsto \min\{y \in J_P \mid y \geq f(x)\}.$$

f is *pointwise exact* at x if $\underline{f}(x) = \overline{f}(x)$, otherwise f is *pointwise inexact* or *rough* at x.

f is *pointwise exact on* $I' \subseteq I$ if $\underline{f}(x) = \overline{f}(x)$ for all $x \in I'$, otherwise f is *pointwise inexact* (*rough*) *on* I'. □

Definition 3. Let $f \in J^I$ be an (I_S, J_P)–rough real function.

The *blockwise lower* and *upper approximations* of $f \in J^I$ are the functions

$$\underleftrightarrow{f} : I \to J_P,\ x \mapsto \max\{y \in J_P \mid y \leq \inf f([\![x]\!]_{E_S})\},$$
$$\overleftrightarrow{f} : I \to J_P,\ x \mapsto \min\{y \in J_P \mid y \geq \sup f([\![x]\!]_{E_S}\}.$$

f is *blockwise exact on* B_i if $\underleftrightarrow{f}(B_i) = \overleftrightarrow{f}(B_i)$; otherwise f is *blockwise inexact* (*rough*) on B_i.

f is *blockwise exact on a definable set* $D \in \mathcal{D}_{I/E_S}$ if f is blockwise exact on its all constituents $B_i \subseteq D$; otherwise f is *blockwise inexact* (*rough*) *on* D.

In particular, f is blockwise exact on I if f is blockwise exact on all base sets. □

Remark 1. Definition 2 and Definition 3 do not use the I_S categorization directly. However, these definitions deal with rough real functions, which are treated in rough coordinate systems by definition, so I_S is not ignorable. □

Remark 2. Although the approximation functions $\underleftrightarrow{f}$ and $\overleftrightarrow{f}$ are formally defined point by point, they are constant on every block because $\inf\ f([\![x]\!]_{E_S})$ and $\sup\ f([\![x]\!]_{E_S})$ are constant on every B_i. Thus, the use of the term "blockwise" is justifiable. □

Proposition 1. *Let* $f \in J^I$ *be an* (I_S, J_P)*-real function.*

(a) f *is pointwise exact at* $x \in I$ *if and only if* $f(x) = y_j \in J_P$ *for some* $j \in [m]$.

(b) f *is pointwise inexact at* $x \in I$ *if and only if* $f(x) \in \,]\underline{f}(x), \overline{f}(x)[\, = \,]y_j, y_{j+1}[$ *with* $y_j, y_{j+1} \in J_P$ *for some* $j \in [m[$.

Proof. (a) It is straightforward. (Csajbók, 2022), Proposition 1.

(b) It is the contrapositive of *(a)*. □

Proposition 1 (a) geometrically means that f is pointwise exact at a point in I iff at this point f touches or intersects a horizontal line segment $y = y_j$ for some $y_j \in J_P$.

Proposition 2 ((Csajbók, 2022), Proposition 2). *Let $f \in J^I$ be an (I_S, J_P)-real function.*

f is blockwise exact on $B_i \in I/E_S$ if and only if $f(x) = y_j \in J_P$ on B_i for some $j \in [m]$.

Proposition 2 geometrically means that f is blockwise exact on B_i iff f coincides on B_i with a horizontal line segment $y = y_j$ for some $y_j \in J_P$.

It is easy to check that the blockwise exactness on a block implies the pointwise exactness on the same block. The reverse statement, however, is not true. Indeed, if f is pointwise exact on an open B_i, it may occur that $f(x) = y_j$ on $B' \subsetneq B_i$ and $f(x) = y_{j'}$ on $B_i \setminus B'$ for some $y_j \neq y_{j'} \in J_P$. Then, on B_i, f is pointwise exact but not blockwise.

It is easy to check the following simple but important statement.

Lemma 1. *Let $f \in J^I$ be an (I_S, J_P)–rough real function. Then*

$$\underset{\leftrightarrow}{f} \leq \underline{f} \leq f \leq \overline{f} \leq \overset{\leftrightarrow}{f}.$$

Let $f \in J^J$. Then, for every $f(x) \in J$, positive, negative, and boundary regions can be defined based on the pointwise approximation of f as follows:

$$\begin{aligned}
\mathrm{POS}^-(f(x)) &= \ell_S([0, f(x)]) \\
&= \cup\{[\![x']\!]_{E_S} \in I/E_S \mid [\![x']\!]_{E_S} \subseteq [0, f(x)]\} \\
&= \left[0, \underline{f}(x)\right]
\end{aligned}$$

$$\begin{aligned}
\mathrm{NEG}^-(f(x)) &= J \setminus u_S([0, f(x)]) = [0, b] \\
&\setminus \cup\{[\![x']\!]_{E_S} \in I/E_S \mid [\![x']\!]_{E_S} \cap [0, f(x)] \neq \emptyset\} \\
&= \begin{cases} [0,b] \setminus \left[0, \overline{f}(x)\right] = \left]\overline{f}(x), b\right], & \text{if } f(x) \in J_P \\ [0,b] \setminus \left[0, \overline{f}(x)\right[= \left[\overline{f}(x), b\right], & \text{if } f(x) \notin J_P \end{cases}
\end{aligned}$$

$$\begin{aligned}
\mathrm{BND}^-(f(x)) &= u_S([0, f(x)]) \setminus \ell_S([0, f(x)]) \\
&= \begin{cases} \left[0, \overline{f}(x)\right] \setminus \left[0, \underline{f}(x)\right] = \emptyset, & \text{if } f(x) \in J_P \\ \left[0, \overline{f}(x)\right[\setminus \left[0, \underline{f}(x)\right] = \left]\overline{f}(x), \underline{f}(x)\right[, & \\ & \text{if } f(x) \notin J_P \end{cases}
\end{aligned}$$

Similar formulas can be derived based on blockwise approximations of f:

$$\mathrm{POS}^{\leftrightarrow}(f(x)) = \ell_S\left(\left[0, f(x)\right]\right) = \left[0, \underset{\leftrightarrow}{f}(x)\right]$$

$$\begin{aligned}
&\mathrm{NEG}^{\leftrightarrow}(f(x)) = \\
&= \begin{cases} \left[0,b\right] \setminus \left[0, \overset{\leftrightarrow}{f}(x)\right] = \left]\overset{\leftrightarrow}{f}(x), b\right], & \text{if } f(x) \in J_P \\ \left[0,b\right] \setminus \left[0, \overset{\leftrightarrow}{f}(x)\right[= \left[\overset{\leftrightarrow}{f}(x), b\right], & \text{if } f(x) \notin J_P \end{cases}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{BND}^{\leftrightarrow}(f(x)) = \\
&= \begin{cases} \left[0, \overline{f}(x)\right] \setminus \left[0, \underline{f}(x)\right] = \emptyset, & \text{if } f(x) \in J_P \\ \left[0, \overset{\leftrightarrow}{f}(x)\right[\setminus \left[0, \underset{\leftrightarrow}{f}(x)\right] = \left]\overset{\leftrightarrow}{f}(x), \underset{\leftrightarrow}{f}(x)\right[, & \\ & \text{if } f(x) \notin J_P \end{cases}
\end{aligned}$$

5 ROUGHNESS AND FUZZINESS

A *fuzzy set* (FS) on U is a function $\mu \in [0,1]^U$ (Zadeh, 1965). It is called the membership function. $\mathcal{FS}(U)$ denotes the family of all fuzzy sets on U.

Let $\mathbb{I} = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$.

If $\mu_A, \nu_A \in \mathcal{FS}(U)$ with $\mu_A \leq \nu_A$, the function $\mu_A^{IVFS} : U \to \mathbb{I}$, $u \mapsto [\mu_A(u), \nu_A(u)]$ forms an *interval-valued fuzzy set* (IVFS) on U (Gorzałczany, 1987). An IVFS is also denoted by $\mu_A^{IVFS} = [\mu_A, \nu_A]$.

Let $\mu_A, \nu_A \in \mathcal{FS}(U)$ with $0 \leq \mu_A + \nu_A \leq 1$. *Intuitionistic fuzzy set* (IFS) on U is the function pair $\mu_A^{IFS} = (\mu_A, \nu_A)$ (Atanassov, 1986; Atanassov, 1999; Atanassov, 2012). μ_A is the membership, ν_A is the nonmembership, and $\pi_A = 1 - \mu_A - \nu_A \in \mathcal{FS}(U)$ is the hesitancy or indeterminacy function.

The family of all intuitionistic fuzzy sets on U is denoted by $\mathcal{IFS}(U)$.

Any $\mu \in \mathcal{FS}(U)$ may be viewed as a special IFS μ_A^{IFS} with the membership function $\mu_A = \mu$ and the derived nonmembership function $\nu_A = 1 - \mu$, i.e., $\mu_A^{IFS} = (\mu, 1 - \mu)$. It is clear that an $\mu_A^{IFS} \in \mathcal{IFS}(U)$ is a fuzzy set iff $\pi_A = 0$ or, what is the same, $\mu_A + \nu_A = 1$.

It is well known that every IVFS $[\mu_A, \nu_A]$ corresponds to an IFS $(\mu_A, 1 - \nu_A)$. On the contrary, every IFS (μ_A, ν_A) corresponds to an IVFS $[\mu_A, 1 - \nu_A]$ (Atanassov and Gargov, 1989; Bustince and Burillo, 1996).

In order to establish a connection between the rough real functions and the intuitionistic fuzzy sets, let us change the $\mathsf{PAS}(J)$ Pawlak's approximation space on the y-axis for $\mathsf{PAS}([0,1])$. That is, on the y-axis, let us take the closed interval $[0,1]$ with the categorizations $P_{[0,1]} = \{y_0 = 0, y_1, \ldots, y_m = 1\}$. $\mathsf{PAS}(J)$ remains unchanged. Last, let $f \in [0,1]^I$ be a rough real function attached to the $(S_I, P_{[0,1]})$–coordinate system.

The pointwise lower and upper approximations $\underline{f}, \overline{f} \in [0,1]^I$ of f are fuzzy sets, and $\underline{f} \leq \overline{f}$ holds. Hence, $f_{pw}^{IVFS} = \left[\underline{f}, \overline{f}\right]$ forms an interval-valued fuzzy set. Then $f_{pw}^{IFS} = \left(\underline{f}, 1 - \overline{f}\right)$ forms a pointwise intuitionistic fuzzy set on I.

Similarly, the blockwise lower and upper approximations $\underset{\leftrightarrow}{f}, \overset{\leftrightarrow}{f} \in [0,1]^I$ of f are fuzzy sets, and $\underset{\leftrightarrow}{f} \leq \overset{\leftrightarrow}{f}$ holds. Thus $f_{bw}^{IVFS} = \left[\underset{\leftrightarrow}{f}, \overset{\leftrightarrow}{f}\right]$ is an interval-valued fuzzy set, and $f_{bw}^{IFS} = \left(\underset{\leftrightarrow}{f}, 1 - \overset{\leftrightarrow}{f}\right)$ is a blockwise intuitionistic fuzzy set on I.

In terms of intuitionistic fuzzy set theory, $\underline{f}$, $\underset{\leftrightarrow}{f}$ are IFS membership, $1 - \overline{f}$, $1 - \underset{\leftrightarrow}{f}$ are nonmembership, and

- $\pi_{\overline{f}}^- = 1 - \underline{f} - \left(1 - \overline{f}\right) = \overline{f} - \underline{f}$,

- $\pi_f^{\leftrightarrow} = 1 - \underset{\leftrightarrow}{f} - \left(1 - \overset{\leftrightarrow}{f}\right) = \overset{\leftrightarrow}{f} - \underset{\leftrightarrow}{f}$

are IFS hesitancy functions.

If IFS $\left(\underline{f}, 1 - \overline{f}\right)$ is regarded as IVFS $\left[\underline{f}, \overline{f}\right]$, it can be observed that for every $x \in I$ where $f(x)$ is inexact,

$$\begin{aligned} |\mathrm{BND}^{-}(f(x))| &= \left| \left] \underline{f}(x), \overline{f}(x) \right[\right| = \left| \left[\underline{f}(x), \overline{f}(x) \right] \right| \\ &= \overline{f}(x) - \underline{f}(x) = \pi_f^{-}(x). \end{aligned} \tag{1}$$

That is, the length of the boundary interval of an inexact value $f(x)$ is exactly equal to the hesitancy degree $\pi_f^{-}(x)$. Thus the hesitancy function establishes a relationship between the rough real functions and their derived intuitionistic fuzzy sets.

6 ROUGHNESS AND *L*–FUZZINESS

6.1 *L*-fuzziness

The essence of *L*-fuzziness is that the membership and many other fuzziness functions take their values from a lattice. This lattice most commonly is a complete one. In the literature, many proposals expand or narrow the features of the complete lattice; however, we use the classical approaches. For the details of lattice theory, see (Grätzer, 2011).

$(L, \leq_L)$ is a *lattice* if

- L is a nonempty set;
- $\leq_L \subseteq L \times L$ is a partially ordered relation on L;
- and any two elements $a, b \in L$ have supremum $\sup\{a, b\}$ and infimum $\inf\{a, b\}$.

If L is a lattice, any finite subset of L has supremum and infimum.

A partially ordered set in which every subset has supremum and infimum is called a *complete lattice*.

Definition 4. Let L be a complete lattice and $N : L \to L$ be an unary operator on L with the following properties:

- $a \leq b$ if $N(b) \leq N(a)$ $(a, b \in L)$ (order-reversing);
- $N(N(a)) = a$ $(a \in L)$ (involutive).

N is called the *involutive negation*.

Such a lattice is denoted by $(L, \leq_L, N)$. If no confusion arises, $(L, \leq_L, N)$ is simplified as L. □

Definition 5 (Goguen, 1967). Let $(L, \leq_L, N)$ be a complete lattice provided with an involutive negation N. An *L-fuzzy set* (LFS) on U is a lattice valued function $\mu_L \in L^U$.

μ_L is called the membership function. The subscript L is dropped if no confusion arises. □

$\mathcal{FS}_L(U)$ denotes the family of all L-fuzzy sets on U.

Definition 6 ((Atanassov, 1999), Definition 3.1). Let $(L, \leq_L, N)$ be a complete lattice provided with an involutive negation N.

Let $\mu_A, \nu_A \in \mathcal{FS}_L(U)$ be two L-fuzzy sets on U satisfying the following condition:

$$\mu_A(u) \leq_L N(\nu_A(u)) \quad (u \in U). \tag{2}$$

Then *intuitionistic L-fuzzy set* (ILFS) on U is a function pair $\mu_A^{ILFS} = (\mu_A, \nu_A)$. μ_A is the *L-membership*, and ν_A is the *L-nonmembership* function. □

The family of all intuitionistic L-fuzzy sets on U is denoted by $I\mathcal{FS}_L(U)$.

Definition 7. Let $\mu_A^{ILFS}, \mu_B^{ILFS} \in I\mathcal{FS}_L(U)$. Then

$$\mu_A^{ILFS} = \mu_B^{ILFS} \text{ if } \mu_A = \mu_B \text{ and } \nu_A = \nu_B; \tag{3}$$

$$\mu_A^{ILFS} \subseteq \mu_B^{ILFS} \text{ if } \mu_A \leq_L \mu_B \text{ and } \nu_A \geq_L \nu_B; \tag{4}$$

$$\begin{aligned} \mu^{ILFS}_{\mu_A^{ILFS} \cap \mu_B^{ILFS}} &= \mu_A^{ILFS} \cap \mu_B^{ILFS} \\ &= \left(\inf\{\mu_A, \mu_B\}, \sup\{\nu_A, \nu_B\}\right); \end{aligned} \tag{5}$$

$$\begin{aligned} \mu^{ILFS}_{\mu_A^{ILFS} \cup \mu_B^{ILFS}} &= \mu_A^{ILFS} \cup \mu_B^{ILFS} \\ &= \left(\sup\{\mu_A, \mu_B\}, \inf\{\nu_A, \nu_B\}\right). \end{aligned} \tag{6}$$

6.2 Rough Real Functions and *L*–fuzziness

Let $f \in J^I$ be an (I_S, J_P)–rough real function with its pointwise and blockwise representations as defined in Section 4.

$J = [0, b]$ is a linearly (totally) ordered complete lattice with the usual $\leq$ ordering. On $(J, \leq)$, let us define the standard involutive negation as

$$N_S : J \to J, \quad x \mapsto b - x.$$

Proposition 3. *N_S is a unary operation, and the involution and order-reserving conditions hold for it.*

Proof. It is easy to see that

- N_S is a unary operation on J because if $x \in J$, $b - x \in J$;
- involution holds because

  $$N_S(N_S(x)) = N_S(b - x) = b - (b - x) = x \ (x \in J);$$

- order-reserving holds because if $x \leq y$,

  $$N_S(x) = b - x \geq N_S(y) = b - y \ (x, y \in J).$$

□

In the following the co-domain of the J-fuzzy and intuitionistic J-fuzzy sets will be the lattice $(J, \leq, N_S)$.

Proposition 4. *Let $f \in J^I$ be an (I_S, J_P)–rough real function.*

(i) By the pointwise lower and upper approximations $\underline{f}, \overline{f}$ of f, the function pair $\left(\underline{f}, b - \overline{f}\right)$ forms an intuitionistic J-fuzzy set on I.

(ii) By the blockwise lower and upper approximations $\underset{\leftrightarrow}{f}, \overset{\leftrightarrow}{f}$ of f, the function pair $\left(\underset{\leftrightarrow}{f}, b - \overset{\leftrightarrow}{f}\right)$ forms an intuitionistic J-fuzzy set on I.

Proof. *(i)* $\underline{f}, \overline{f} \in J^I$ are J-fuzzy sets, and $b - \overline{f} \in J^I$ is also a J-fuzzy set. Furthermore, for every $x \in J$,

$$\underline{f}(x) \leq \overline{f}(x) = b - (b - \overline{f}(x)) = N_S(b - \overline{f}(x)).$$

That is Equation (2) holds. Hence, $\left(\underline{f}, b - \overline{f}\right)$ forms an intuitionistic J-fuzzy set on I.

(ii) With the same argument as in *(i)*, it can be proved that $\left(\underset{\leftrightarrow}{f}, b - \overset{\leftrightarrow}{f}\right)$ also forms an intuitionistic J-fuzzy set on I. □

$f_{pw}^{ILFS} = \left(\underline{f}, b - \overline{f}\right) \in I\mathcal{FS}_J(I)$ is called the *pointwise intuitionistic L-fuzzy set* of f.

$f_{bw}^{ILFS} = \left(\underset{\leftrightarrow}{f}, b - \overset{\leftrightarrow}{f}\right) \in I\mathcal{FS}_J(I)$ is called the *blockwise intuitionistic J-fuzzy set* of f.

They are together called the *roughly derived intuitionistic J-fuzzy sets* of f.

In intuitionistic fuzzy set theory, the hesitancy function has an important role. However, in the L-fuzziness context, its definition and role is an interesting question.

In (Atanassov, 1999), Equation (3.3), p. 180), the following L-hesitancy function is defined:

$$\pi_A^{sup} : U \to L,\ u \mapsto N(\sup(\mu_A(u), \nu_A(u)) \tag{7}$$

where L is a complete lattice provided with the involutive negation N.

Let us examine the following L-hesitancy function too, which is analogous with Equation (7):

$$\pi_A^{inf} : U \to L,\ u \mapsto N(\inf(\mu_A(u), \nu_A(u)) \tag{8}$$

According to Equations (7) and (8), with the co-domain $(J, \leq, N_S)$, the pointwise J-hesitancy functions $\pi_{f,pw}^{sup}$ and $\pi_{f,pw}^{inf}$ of f are the following (since J is linearly ordered, "max" and "min" is applicable instead of "sup" and "inf"):

$$\begin{aligned}\pi_{f,pw}^{sup} &= N_S(\sup\{\underline{f}, b - \overline{f}\}) = N_S(\max\{\underline{f}, b - \overline{f}\}) \\ &= \begin{cases} b - \underline{f}, & \text{if } \underline{f} \geq b - \overline{f} \\ b - (b - \overline{f}) = \overline{f} & \text{if } \underline{f} < b - \overline{f} \end{cases}.\end{aligned}$$

$$\begin{aligned}\pi_{f,pw}^{inf} &= N_S(\inf\{\underline{f}, b - \overline{f}\}) = N_S(\min\{\underline{f}, b - \overline{f}\}) \\ &= \begin{cases} b - \underline{f}, & \text{if } \underline{f} \leq b - \overline{f} \\ b - (b - \overline{f}) = \overline{f}, & \text{if } \underline{f} > b - \overline{f} \end{cases}.\end{aligned}$$

Neither $\pi_{f,pw}^{sup}$ nor $\pi_{f,pw}^{inf}$ establish a connection between the rough real functions and their derived intuitionistic fuzzy sets in the sense of Equation 1. Therefore, in the next definition, a new hesitancy function is suggested.

Definition 8. Let $\mu_A^{ILFS} = (\mu_A, \nu_A)$ be an intuitionistic $(J, \leq, N_S)$-fuzzy set on I.

Then its *J-hesitancy function* π_A^+ is

$$\pi_A^+ : I \to J,\ x \mapsto N_S(\mu_A(x) + \nu_A(x)).$$

□

First, it is needed to make sure that N_S in Definition 8 is an involutive negation. Since, for every $x \in I$,

$$\begin{aligned} 0 &\leq \mu_A(x) + \nu_A(x) \\ &\leq N_S(\nu_A) + \nu_A = b - \nu_A + \nu_A = b, \end{aligned}$$

so $(\mu_A + \nu_A)(I) = J$. Then, by Proposition 3, N_S is indeed an involutive negation.

Let $f \in J^I$ be a rough real function. The following proposition shows that, according to Definition 8, the J-hesitancy functions $\pi_{f,pw}^+$ and $\pi_{f,bw}^+$ of the pointwise and blockwise intuitionistic fuzzy sets of f establish the required connections between f and its derived ILFS f_{pw}^{ILFS} and f_{bw}^{ILFS}.

Proposition 5. *Let $f \in J^I$ be an (I_S, J_P)–rough real function. Its pointwise and blockwise intuitionistic J-fuzzy sets are*

$$f_{pw}^{ILFS} = \left(\underline{f}, b - \overline{f}\right) \text{ and } f_{bw}^{ILFS} = \left(\underset{\leftrightarrow}{f}, b - \overset{\leftrightarrow}{f}\right).$$

Then, for every $x \in I$ where $f(x)$ is inexact,

(i) $\pi_{f,pw}^+(x) = |\mathrm{BND}^-(f(x))|$,

(ii) $\pi_{f,bw}^+(x) = |\mathrm{BND}^{\leftrightarrow}(f(x))|$.

Proof. For every $x \in I$ where $f(x)$ is inexact,

$$\begin{aligned}\pi_{f,pw}^+(x) &= N_S\left(\underline{f}(x) + b - \overline{f}(x)\right) \\ &= b - \left(\underline{f}(x) + b - \overline{f}(x)\right) \\ &= \overline{f}(x) - \underline{f}(x) \\ &= |]\overline{f}(x), \underline{f}(x)[| = |\mathrm{BND}^-(f(x))|.\end{aligned}$$

The case $\pi_{f,bw}^+(x)$ can be proved similarly. □

7 SOME BASIC PROPERTIES OF ROUGHLY DERIVED INTUITIONISTIC *L*-FUZZY SETS

Let $f \in J^I$ be an (I_S, J_P)–rough real function. Its roughly derived intuitionistic J-fuzzy sets are:

$$f_{pw}^{ILFS} = \big(\underline{f}, b - \overline{f}\big) \text{ and } f_{bw}^{ILFS} = \big(\underset{\leftrightarrow}{f}, b - \overset{\leftrightarrow}{f}\big).$$

Proposition 6. *The inclusion relations $f_{bw}^{ILFS} \subseteq f_{pw}^{ILFS}$ and $f_{pw}^{ILFS} \subseteq f_{bw}^{ILFS}$ fail in general.*

Proof. Due to Lemma 1,

- in the case of $f_{bw}^{ILFS} \subseteq f_{pw}^{ILFS}$, $\underline{f} \leq \underset{\leftrightarrow}{f}$ does not hold in general;
- in the case of $f_{pw}^{ILFS} \subseteq f_{bw}^{ILFS}$, $1 - \overset{\leftrightarrow}{f} \geq 1 - \overline{f}$ does not hold in general. □

With the help of common intuitionistic fuzzy set operations, intersection and union (cf. (Atanassov, 1999), Definition 1.4), in addition by Lemma 1 , we obtain

Proposition 7. *Intersection and union of f_{pw}^{ILFS} and f_{bw}^{ILFS} are:*

(i) $f_{pw \cap bw}^{ILFS} = f_{pw}^{ILFS} \cap f_{bw}^{ILFS} = \big(\underset{\leftrightarrow}{f}, 1 - \overline{f}\big)$;

(ii) $f_{pw \cup bw}^{ILFS} = f_{pw}^{ILFS} \cup f_{bw}^{ILFS} = \big(\underline{f}, 1 - \overset{\leftrightarrow}{f}\big)$.

Proof. According to Lemma (1), and Equations (5) and (6), respectively, we get

$$\begin{aligned}
\text{(i) } f_{pw \cap bw}^{ILFS} &= f_{pw}^{ILFS} \cap f_{bw}^{ILFS} \\
&= \big(\min(\underline{f}, \underset{\leftrightarrow}{f}), \max(1 - \overline{f}, 1 - \overset{\leftrightarrow}{f})\big) \\
&= \big(\underset{\leftrightarrow}{f}, 1 - \overline{f}\big); \\
\text{(ii) } f_{pw \cup bw}^{ILFS} &= f_{pw}^{ILFS} \cup f_{bw}^{IFS} \\
&= \big(\max(\underline{f}, \underset{\leftrightarrow}{f}), \min(1 - \overline{f}, 1 - \overset{\leftrightarrow}{f})\big) \\
&= \big(\underline{f}, 1 - \overset{\leftrightarrow}{f}\big).
\end{aligned}$$

□

Proposition 8. *The following inclusion relations hold:*

(i) $f_{bw}^{ILFS} \cap f_{pw}^{ILFS} \subseteq f_{bw}^{ILFS} \cup f_{pw}^{ILFS}$.

(ii) $f_{bw}^{ILFS} \cap f_{pw}^{ILFS} \subseteq (\overline{f}, 1 - \overline{f})$.
It means that the ILFS $f_{bw}^{ILFS} \cap f_{pw}^{ILFS}$ is included in the LFS $\overline{f}$ if $\overline{f}$ is considered as ILFS.

(iii) $f_{bw}^{IFS} \cup f_{pw}^{IFS} \subseteq \big(\overset{\leftrightarrow}{f}, 1 - \overset{\leftrightarrow}{f}\big)$.
It means that the ILFS $f_{bw}^{ILFS} \cup f_{pw}^{ILFS}$ is included in the LFS $\overset{\leftrightarrow}{f}$ if $\overset{\leftrightarrow}{f}$ is considered as ILFS.

(iv) $(\underline{f}, 1 - \underline{f}) \subseteq f_{bw}^{IFS} \cup f_{pw}^{IFS}$.
It means that the LFS $\underline{f}$, if it is considered as ILFS, is included in the ILFS $f_{bw}^{ILFS} \cup f_{pw}^{ILFS}$.

(v) $\big(\underset{\leftrightarrow}{f}, 1 - \underset{\leftrightarrow}{f}\big) \subseteq f_{bw}^{IFS} \cap f_{pw}^{IFS}$.
It means that the LFS $\underset{\leftrightarrow}{f}$, if it is considered as ILFS, is included in the ILFS $f_{bw}^{ILFS} \cap f_{pw}^{ILFS}$.

Proof.

$$\begin{aligned}
\text{(i) } f_{bw}^{ILFS} \cap f_{pw}^{ILFS} &= \big(\underset{\leftrightarrow}{f}, 1 - \overline{f}\big) \\
&\subseteq \big(\underline{f}, 1 - \overset{\leftrightarrow}{f}\big) = f_{pw}^{ILFS} \cup f_{bw}^{ILFS},
\end{aligned}$$

because $\underset{\leftrightarrow}{f} \leq \underline{f}$ and $1 - \overline{f} \geq 1 - \overset{\leftrightarrow}{f}$.

$$\text{(ii) } f_{bw}^{ILFS} \cap f_{pw}^{ILFS} = \big(\underset{\leftrightarrow}{f}, 1 - \overline{f}\big) \subseteq (\overline{f}, 1 - \overline{f}),$$

because $\underset{\leftrightarrow}{f} \leq \overline{f}$ and $1 - \overline{f} \geq 1 - \overline{f}$.

$$\text{(iii) } f_{bw}^{ILFS} \cup f_{pw}^{ILFS} = \big(\underline{f}, 1 - \overset{\leftrightarrow}{f}\big) \subseteq \big(\overset{\leftrightarrow}{f}, 1 - \overset{\leftrightarrow}{f}\big),$$

because $\underline{f} \leq \underset{\leftrightarrow}{f}$ and $1 - \overset{\leftrightarrow}{f} \geq 1 - \overset{\leftrightarrow}{f}$.

$$\text{(iv) } (\underline{f}, 1 - \underline{f}) \subseteq \big(\underline{f}, 1 - \overset{\leftrightarrow}{f}\big) = f_{bw}^{IFS} \cup f_{pw}^{IFS},$$

because $\underline{f} \leq \underline{f}$ and $1 - \underline{f} \geq 1 - \overset{\leftrightarrow}{f}$.

$$\text{(v) } \big(\underset{\leftrightarrow}{f}, 1 - \underset{\leftrightarrow}{f}\big) \subseteq \big(\underset{\leftrightarrow}{f}, 1 - \overline{f}\big) = f_{bw}^{ILFS} \cap f_{pw}^{ILFS},$$

because $\underset{\leftrightarrow}{f} \leq \underset{\leftrightarrow}{f}$ and $1 - \underset{\leftrightarrow}{f} \geq (1 - \overline{f})$.

8 CONCLUSION

The paper has shown how the more realistic real functions can connect with intuitionistic *L*-fuzzy sets. The boundary region on the rough real function side, and the hesitancy function on the intuitionistic *L*-fuzzy set side, are extremely important for rough real functions. These two regions are a connecting link, both semantically and syntactically, between the two otherwise distant areas.

Several goals can be formulated for the future. The rich tool set of intuitionistic *L*-fuzzy sets can be applied to studying rough real functions. Care must be taken, however, that the tools chosen are consistent with the semantics of rough real functions. Deciding on this is not always easy, so the tools must be carefully considered. The intuitionistic fuzzy tools applied in rough set theory may also imply that the rough calculus can have a fertilizing effect on the intuitionistic fuzzy set theory.

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