A Novel Approach to Weighted Fuzzy Rules for Positive Samples

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Keywords: Fuzzy Relation, Relational Model, Fuzzy Approximation, Implicative Model, Fuzzy IF–THEN Rules.

Abstract: In this contribution, we propose a novel approach to automated fuzzy rule base generation based on underlying observational data. The core of this method lies in adding information to a particular fuzzy rule in the form of attached weight given as a value extracted from a relational data model. In particular, we blend two approaches to receive particular models that overcome their specific drawbacks.

1 INTRODUCTION

Weighted fuzzy rules have been used intensively in fuzzy modeling since the early days of the field. Weights were applied to various parts of fuzzy rules: to input variables, antecedents, consequences, or whole rules; see, e.g., (Nauck, 2000; Ishibuchi and Nakashima, 2001; Alcalá et al., 2003; delaOssa et al., 2009). Learning techniques for weighted fuzzy rules usually involve optimization methods such as neural networks, evolutional computing, or genetic algorithms to find the best possible fuzzy rules together with their associated weights. This typical approach is repeated also in various recent works, e.g. (Bemani-N. and Akbarzadeh-T., 2019; Shiny Irene et al., 2020; Navarro-Almanza et al., 2022).

In this contribution, we focus on weighted fuzzy rules formalized by the so-called normal forms introduced in (Perfiljeva, 2004). Normal forms-based formalization covers the wide spectrum of the weighted fuzzy rules mentioned above. Their relationship was described and studied in (Daňková, 2007).

Furthermore, we deal with observed data that are considered as positive ones, i.e. the given data are the prototypical representatives of a relationship between input and output spaces. In the standard fuzzy sets framework, where the scale for the truth values is $[0,1]$, we assign $F(c,d) = 1$ to the observed data (c,d) and the relationship F.

Having positive sample data at our disposal, we can freely build fuzzy rules for each data using the sample-based generation of fuzzy rules provided in (Hájek, 1998) and called Hájek's approach within this paper.

At this stage, we can face the problem of a huge number of rules that need to be further reduced for a final computation efficiency and a rule-base transparency. We propose to solve this problem by combining the Hájek approach with normal forms, where we can fix a number of fuzzy rules in the fuzzy rule-base to keep a sample-based learned information as small and compact as needed.

The paper is organized as follows. In Section 2, we recall a formalization of the basic fuzzy relational models from (Hájek, 1998) and provide some selected properties. The normal conjunctive and disjunctive norms are presented in Section 3 together with similar results as in the preceding section. Next, in Section 4, we introduce special normal forms based on positive samples that blend the approach of Hájek and the normal forms and provide their basic selected theoretical results. Finally, we summarize the results and propose some future directions in Section 5.

2 SAMPLE-BASED FUZZY RULES

As already stated in the Introduction, there is a vast amount of methods for generating fuzzy rules. Mostly, they can be characterized as the result of expert knowledge, generated from observed data, or their combination. A method based on observed data formalized in (Hájek, 1998) automatically generates a particular fuzzy rule using fuzzy similarity relations for each input data. In the following, let us recall

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Daňková, M.
A Novel Approach to Weighted Fuzzy Rules for Positive Samples.
DOI: 10.5220/0011548800003332
In *Proceedings of the 14th International Joint Conference on Computational Intelligence (IJCCI 2022)*, pages 209-216
ISBN: 978-989-758-611-8; ISSN: 2184-3236

Hájek's approach:

- Consider a complete residuated lattice of the form

$$\mathcal{L} = \langle L, \&, \rightarrow, \wedge, \vee, 0, 1\rangle,$$

with the lattice operations $(\wedge, \vee)$, the residuated pair $(\&, \rightarrow)$, the bottom and top elements 0 and 1, respectively. Additionally, define

$$x \leftrightarrow y = (x \rightarrow y) \,\&\, (y \rightarrow x).$$

- Let $X_1, X_2 \neq \emptyset$ and $\approx_j$ be a similarity relation on X_j, $j = 1, 2$, that is,

$$(x \approx_j x) = 1,$$
$$(x \approx_j y) \leq (y \approx_j x),$$
$$(x \approx_j y) \,\&\, (y \approx_j z) \leq (x \approx_j z),$$

for all $x, y, z \in X_j$.

- Let $(c_i, d_i) \in X_1 \times X_2$, for $i \in I$, where $I = \{1, 2, \ldots, n\}$.
- Moreover, let the following properties be valid for a binary fuzzy relation F on $X_1 \times X_2$:
  - *Extensionality* of F:

$$(x \approx_1 x') \,\&\, (y \approx_2 y') \,\&\, F(x, y) \leq F(x', y'), \quad (1)$$

  for all $x, x' \in X_1, y, y' \in X_2$.
  - *Functionality* of F:

$$(x \approx_1 x') \,\&\, F(x, y) \,\&\, F(x', y') \leq (y \approx_2 y'), \quad (2)$$

  for all $x, x' \in X_1, y, y' \in X_2$.
  - *Positive samples* of F:

$$\bigwedge_{i \in I} F(c_i, d_i) = 1. \quad (3)$$

- Define:

$$\text{Mamd}_F(x, y) = \bigvee_{i \in I} \big((x \approx_1 c_i) \,\&\, (y \approx_2 d_i)\big), \quad (4)$$

$$\text{Rules}_F(x, y) = \bigwedge_{i \in I} \big((x \approx_1 c_i) \rightarrow (y \approx_2 d_i)\big), \quad (5)$$

for all $x \in X_1, y \in X_2$.

We call Mamd_F and Rules_F *relational data models.*

Example 2.1. Consider the data given in Figure 1, where $X_1 = [0, 6]$ and $X_2 = [0, 50]$, moreover, define

$$S_k(x, y) = 0 \vee (1 - k|x - y|),$$

for all $x \in X_1$, $y \in X_2$, $k \in \mathbb{R}$.

Furthermore, let $\approx_1 \equiv S_{1.7}, \approx_2 \equiv S_{0.2}$. Then the relational The models Mamd_F in the standard algebras of Łukasiewicz, Product and Gödel are shown in Figures 2, 3, and 4, respectively. The relational model Rules_F in the standard algebras of Łukasiewicz algebra is in Figure 5. Models Rules_F for the remaining algebras are omitted since their values are 0 nearly everywhere.

But if we change $\approx_2$ as follows: $\approx_2 \equiv S_{0.02}$, we obtain relational models Rules_F in the standard algebras of Łukasiewicz, Product and Gödel as shown in Figures 6, 7, and 8, respectively.

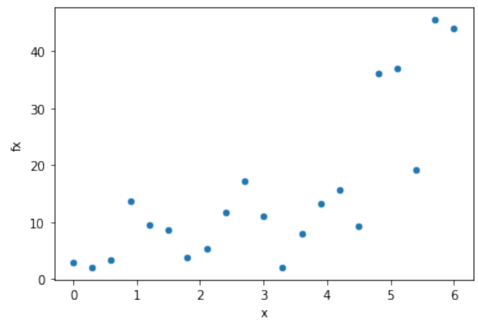

Figure 1: Input data.

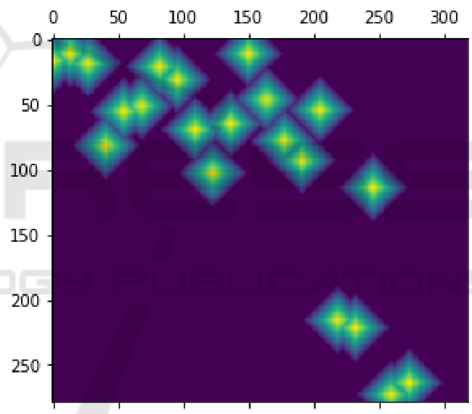

Figure 2: Mamd_F for the input data from Figure 1 in Łkasiewicz standard algebra.

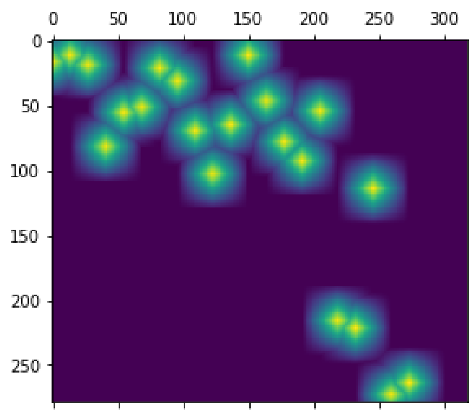

Figure 3: Mamd_F for the input data from Figure 1 in the Product standard algebra.

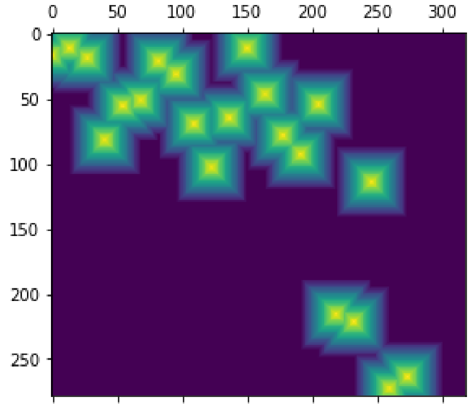

Figure 4: Mamd_F for the input data from Figure 1 in Gödel standard algebra.

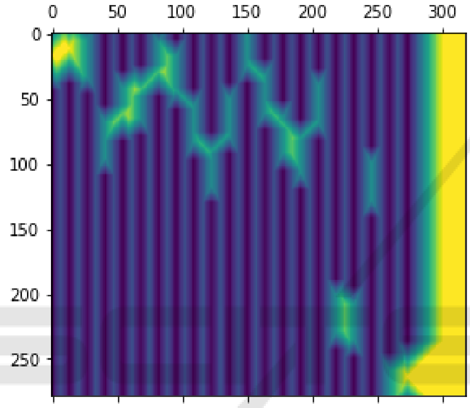

Figure 5: Rules_F for the input data from Figure 1 in Łukasiewicz standard algebra.

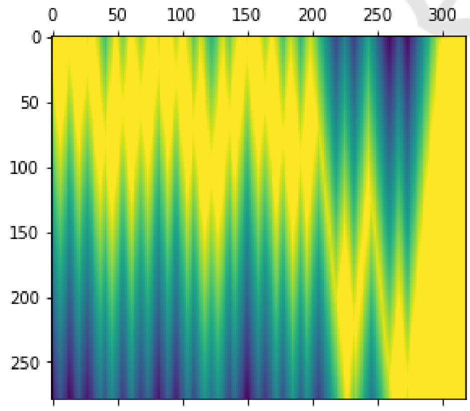

Figure 6: Rules_F for the input data from Figure 1 in Łukasiewicz standard algebra with a finer similarity.

In (Hájek, 1998), we can find many interesting properties, e.g.,

$$\text{Mamd}_F(x,y) \leq F(x,y), \tag{6}$$

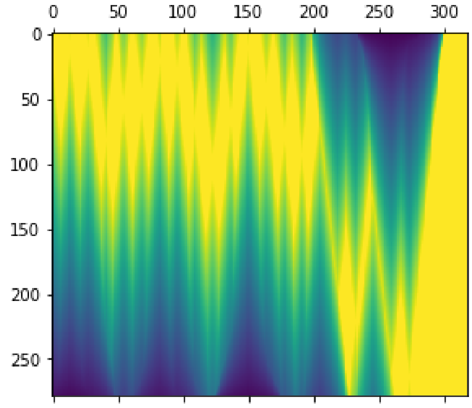

Figure 7: Rules_F for the input data from Figure 1 in the product standard algebra with a finer similarity.

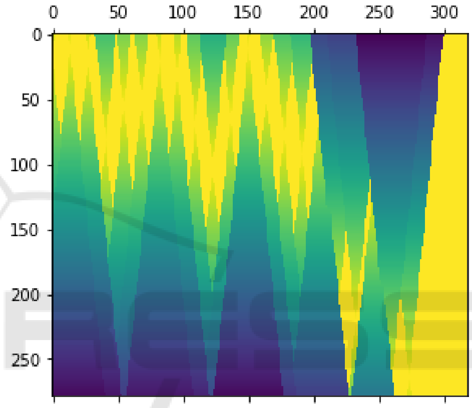

Figure 8: Rules_F for the input data from Figure 1 in Gödel standard algebra with a finer similarity.

$$F(x,y) \leq \text{Rules}_F(x,y), \tag{7}$$

$$\bigvee_{i \in I} (x \approx_1 c_i)^2 \leq \text{Mamd}_F(x,y) \leftrightarrow F(x,y), \tag{8}$$

$$\bigvee_{i \in I} (x \approx_1 c_i)^2 \leq F(x,y) \leftrightarrow \text{Rules}_F(x,y), \tag{9}$$

$$\bigvee_{i \in I} (x \approx_1 c_i)^2 \leq \text{Mamd}_F(x,y) \leftrightarrow \text{Rules}_F(x,y). \tag{10}$$

for all $x \in X_1, y \in X_2$. Hence, Mamd_F is below F and Rules_F is above F. The quality of approximation is expressed here using the equivalence relation $\leftrightarrow$, which is dual to pseudometric.

Due to Lemma 7.2.10, the relational data model Mamd_F is extensional and functional. Contrary to Mamd_F, the relational data model Rules_F is not functional (see Remark 7.2.11 in (Hájek, 1998)). However, it can be proved that Rules_F is extensional.

Proposition 2.2. *Let $F, \approx_1, \approx_2$ be as above.*
Then the fuzzy relation Rules_F *is extensional.*

Proof. We have to prove the following inequality

$$(x \approx_1 x') \,\&\, (y \approx_2 y') \,\&\, \mathrm{Rules}_F(x,y) \le \mathrm{Rules}_F(x',y'),$$

for all $x, x' \in X_1, y, y' \in X_2$.

We start from the left-hand side of the above inequality:

$$(x \approx_1 x') \,\&\, (y \approx_2 y') \,\&\, \bigwedge_{i \in I} ((x \approx_1 c_i) \to (y \approx_2 d_i)) \le$$
$$(x \approx_1 x') \,\&\, (y \approx_2 y') \,\&\, [(x \approx_1 c_i) \to (y \approx_2 d_i)] =$$
$$(x \approx_1 x') \,\&\, [1 \to (y \approx_2 y')] \,\&\, [(x \approx_1 c_i) \to (y \approx_2 d_i)] \le$$
$$(x \approx_1 x') \,\&\, [(x \approx_1 c_i) \to ((y \approx_2 y') \,\&\, (y \approx_2 d_i))] \le$$
$$(x \approx_1 x') \,\&\, [(x \approx_1 c_i) \to ((y' \approx_2 d_i)]$$

From the transitivity of $\approx_1$, it follows

$$x \approx_1 x' \le (x' \approx_1 c_i) \to (x \approx_1 c_i).$$

Thus, we obtain

$$(x \approx_1 x') \,\&\, (y \approx_2 y') \,\&\, \bigwedge_{i \in I} ((x \approx_1 c_i) \to (y \approx_2 d_i))$$
$$\le [(x' \approx_1 c_i) \to (x \approx_1 c_i)] \,\&\, [(x \approx_1 c_i) \to ((y' \approx_2 d_i)]$$
$$\le (x' \approx_1 c_i) \to (y' \approx_2 d_i),$$

for all $x, x' \in X_1, y, y' \in X_2$ and $i \in I$. From this the extensionality of Rules_F follows directly. □

3 WEIGHTED FUZZY RULES FOR EXTENSIONAL FUZZY RELATIONS

Unfortunately, Hájek's approach without any further fuzzy rule reduction method is suitable only for a small number of data samples. One of the possible solutions is to fix the number of rules and use quantifiers of Fuzzy General Unary Hypotheses Automaton (FGUHA) methods (Holeňa, 1998; Ralbovský, 2009) (a generalization of the GUHA methods (Hájek and Havránek, 1978)) to confirm or reject each particular fuzzy rule designed. Examples of successful solutions to real-world problems in various fields of vague nature can be found in (Turunen, 2008).

In the subsequent section, we propose another novel solution to the problem, where we add to each fixed fuzzy rule a special weight based on the given sample data. Since both relational models are extensional, we can use the so-called normal forms introduced in (Perfiljeva, 2004). First, recall the definition of normal forms.

Definition 3.1. Let G be a fuzzy binary relation on $X_1 \times X_2$, $X_1, X_2, \approx_1, \approx_2$ be as above, and $p_j \in X_1, q_j \in X_2$, for $j \in J$, where $J = \{1, 2. \ldots, k\}$. Then

- *The disjunctive normal form* for G is defined as

$$\mathrm{DNF}_G(x,y) = \bigvee_{j \in J} ((x \approx_1 p_j) \,\&\, (y \approx_2 q_j) \,\&\, G(p_j,q_j)), \quad (11)$$

for all $x \in X_1, y \in X_2$.

- *The conjunctive normal form* for G is defined as

$$\mathrm{CNF}_G(x,y) = \bigwedge_{j \in J} (((x \approx_1 p_j) \,\&\, (y \approx_2 q_j)) \to G(p_j,q_j)), \quad (12)$$

for all $x \in X_1, y \in X_2$.

Observe that the relational model Mamd_F can be identified with DNF_G, provided $G(x,y) = F(x,y)$, for all $x \in X_1, y \in X_2$, $I = J$, and $(c_i, d_i) = (p_i, q_i)$, for all $i \in I$. An analogous observation cannot be made for Rules_F and CNF_G. Because the relational model Rules_F is based on functionality, while CNF_G is based on extensionality.

The values $G(p_j,q_j)$, $j \in J$ serve as weights of particular fuzzy rules. In fact, it works as some kind of shift (and rotate) operator because for each fuzzy rule in DNF_G, it holds

$$(p_j \approx_1 x) \,\&\, (q_j \approx_2 y) \,\&\, G(p_j,q_j) \le G(p_j,q_j), \quad (13)$$

for all $j \in J$, $x \in X_1$, $y \in X_2$, and analogously, for each fuzzy rule in CNF_G, it holds

$$G(p_j,q_j) \le (p_j \approx_1 x) \,\&\, (q_j \approx_2 y) \to G(p_j,q_j), \quad (14)$$

for all $j \in J$, $x \in X_1$, $y \in X_2$. Roughly speaking, applying the weights to fuzzy rules as done in normal forms gradually switches on or off the particular fuzzy rules.

Figures 9–14 demonstrate the behavior described above. Figure 11 shows the aggregation of fuzzy sets from Figures 9 and 10 in the Łukasiewicz algebra. The weight 0.5 lowers the values of DNF below 0.5 as shown in Figure 12. Analogously, the same weight rotates and shifts the values of DNF above 0.5, see Figure 14. And Figure 13 demonstrates the duality of DNF and CNF.

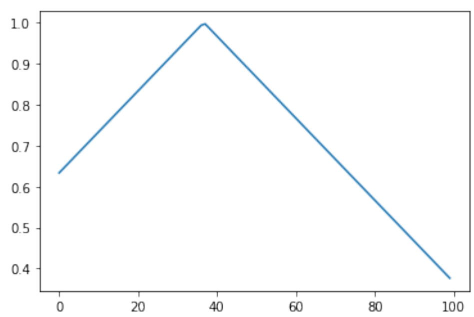

Figure 9: $x \approx_1 c$.

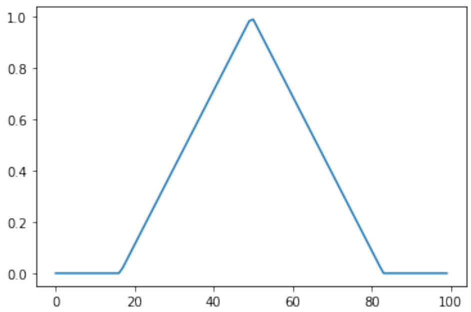

Figure 10: $y \approx_2 d$.

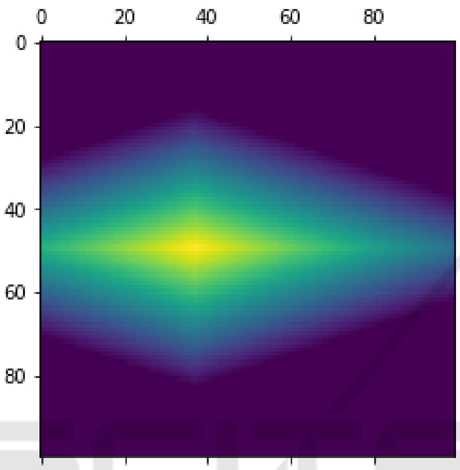

Figure 11: $(X \approx_1 c) \,\&\, (y \approx_2 d)$ in Łukasiewicz standard algebra.

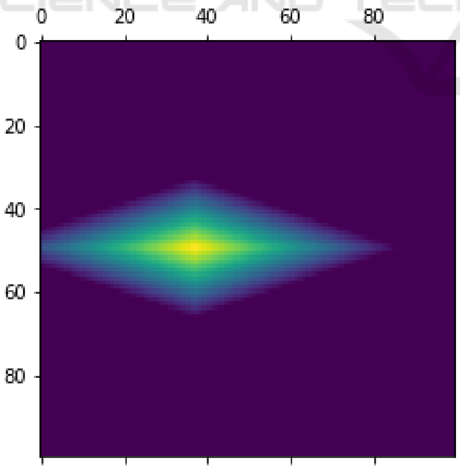

Figure 12: $(X \approx_1 c) \,\&\, (y \approx_2 d) \,\&\, 0.5$ in Łukasiewicz standard algebra.

If G is an extensional binary fuzzy relation, then the following inequalities hold; see (Perfiljeva, 2004):

$$\mathrm{DNF}_G(x,y) \leq G(x,y), \tag{15}$$

$$G(x,y) \leq \mathrm{CNF}_G(x,y), \tag{16}$$

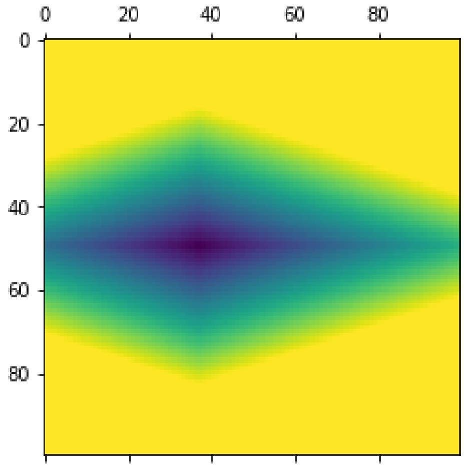

Figure 13: $(X \approx_1 c) \,\&\, (y \approx_2 d) \to 0$ in Łukasiewicz standard algebra.

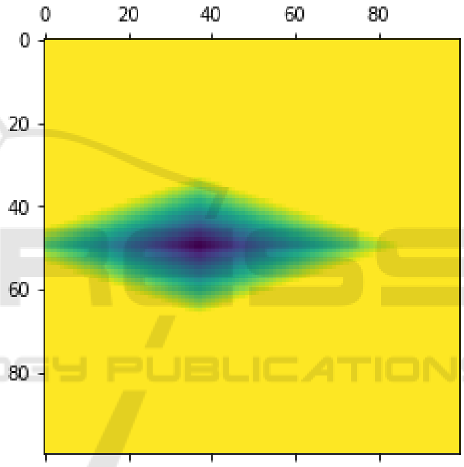

Figure 14: $(X \approx_1 c) \,\&\, (y \approx_2 d) \to 0.5$ in Łukasiewicz standard algebra.

$$\bigvee_{i \in I} [(c_i \approx_1 x)^2 \,\&\, (y \approx_1 d_i)^2] \leq \mathrm{DNF}_G(x,y) \leftrightarrow G(x,y), \tag{17}$$

$$\bigvee_{i \in I} [(c_i \approx_1 x)^2 \,\&\, (y \approx_1 d_i)^2] \leq \mathrm{CNF}_G(x,y) \leftrightarrow G(x,y), \tag{18}$$

$$\bigvee_{i \in I} [(c_i \approx_1 x)^2 \,\&\, (y \approx_1 d_i)^2] \leq \mathrm{DNF}_G(x,y) \leftrightarrow \mathrm{CNF}_G(x,y), \tag{19}$$

for all $x \in X_1, y \in X_2$

The above-listed inequalities seem to be analogous to the results of Hájek's but they are more general. Since the fuzzy relation G is not functional, we have to lower estimate the above equivalences by the respective degrees of similarity in domain X_2. However, in the case of Hájek's approach, it is sufficient to take into consideration only the respective similarities in the domain X_1. More properties on approximate reasoning with normal forms can be found in (Daňková, 2007).

4 WEIGHTED FUZZY RULES FROM THE RELATIONAL DATA MODELS

Obviously, normal forms cannot be applied directly for given input data. Naturally, the question arises of how to obtain weights inside normal forms based on observational positive data? In this section, we propose a particular solution to this question.

First, let us summarize the requirements:

- $\approx_j$ be a similarity relation on X_j, $j = 1,2$.
- $(c_i, d_i) \in X_1 \times X_2$, for $i \in I$.
- $(p_j, q_j) \in X_1 \times X_2$, for $j \in j$.
- F be a binary fuzzy relation on $X_1 \times X_2$.
- $D = \{(c_i, d_i)\}_{i \in I}$ be a set of positive data, that is, $F(c_i, d_i) = 1$ for all $i \in I$.

Notice that (c_i, d_i), for $i \in I$ are input data for Mamd and Rules, while (p_j, q_j), $j \in J$ are nodes in which we construct normal forms.

As shown in Section 2, the relational data models Mamd and Rules given by (4) and (5), respectively, are extensional; consequently, all the results valid for the normal forms pass to the following normal forms:

$$\mathrm{DNF}_{\mathrm{Mamd}_F}, \quad (20)$$
$$\mathrm{DNF}_{\mathrm{Rules}_F}, \quad (21)$$
$$\mathrm{CNF}_{\mathrm{Mamd}_F}, \quad (22)$$
$$\mathrm{CNF}_{\mathrm{Rules}_F}, \quad (23)$$

that will be called *weighted fuzzy rules for positive samples*.

In the following, we list the valid properties for the introduced weighted fuzzy rules, i.e., the particular normal forms, derived from (15)–(19).

$$\mathrm{DNF}_{\mathrm{Mamd}_F}(x,y) \leq \mathrm{Mamd}_F(x,y), \quad (24)$$
$$\mathrm{Mamd}_F(x,y) \leq \mathrm{CNF}_{\mathrm{Mamd}_F}(x,y), \quad (25)$$
$$\mathrm{DNF}_{\mathrm{Rules}_F}(x,y) \leq \mathrm{Rules}_F(x,y), \quad (26)$$
$$\mathrm{Rules}_F(x,y) \leq \mathrm{CNF}_{\mathrm{Rules}_F}(x,y), \quad (27)$$

$$\bigvee_{j\in J} (x \approx_1 p_j)^2 \,\&\, (y \approx_2 q_j)^2 \leq \mathrm{DNF}_{\mathrm{Mamd}_F}(x,y) \leftrightarrow \mathrm{CNF}_{\mathrm{Mamd}_F}(x,y). \quad (28)$$

$$\bigvee_{j\in J} (x \approx_1 p_j)^2 \,\&\, (y \approx_2 q_j)^2 \leq \mathrm{DNF}_{\mathrm{Rules}_F}(x,y) \leftrightarrow \mathrm{CNF}_{\mathrm{Rules}_F}(x,y). \quad (29)$$

for all $x \in X_1, y \in X_2$.

Moreover, if F is extensional and functional, then we can use Hájek's results (6)–(10), and we obtain:

$$\mathrm{DNF}_{\mathrm{Mamd}_F}(x,y) \leq F(x,y), \quad (30)$$
$$F(x,y) \leq \mathrm{CNF}_{\mathrm{Rules}_F}(x,y), \quad (31)$$

$$\bigvee_{j\in J} (x \approx_1 p_j)^2 \,\&\, (y \approx_2 q_j)^2 \,\&\, \bigvee_{i\in I} (x \approx_1 c_i)^2 \leq \mathrm{DNF}_{\mathrm{Mamd}_F}(x,y) \leftrightarrow F(x,y), \quad (32)$$

$$\bigvee_{j\in J} (x \approx_1 p_j)^2 \,\&\, (y \approx_2 q_j)^2 \,\&\, \bigvee_{i\in I} (x \approx_1 c_i)^2 \leq F(x,y) \leftrightarrow \mathrm{CNF}_{\mathrm{Rules}}(x,y). \quad (33)$$

for all $x \in X_1, y \in X_2$.

Many more interesting results on approximate inference with $\mathrm{DNF}_{\mathrm{Mamd}_F}$ and $\mathrm{CNF}_{\mathrm{Mamd}_F}$ are obtained directly from the results on normal forms published in (Daňková, 2007).

Let us provide an illustrative example for the data given in Figure 15 and the weighted fuzzy rules for positive samples of the form $\mathrm{DNF}_{\mathrm{Mamd}_F}$. We consider 11 nodes

$$\{(p_i, q_i)\}_{i=0}^{10} = \{(5.4 \cdot i, i^2)\}_{i=0}^{10}.$$

The resulting $\mathrm{DNF}_{\mathrm{Mamd}_F}$ for the three basic algebras are shown in Figures 16–18. The respective weights are as follows:

$\mathcal{L}_L$	$\mathcal{L}_P$	$\mathcal{L}_G$
0.81	0.81	0.81
0.91	0.91	0.93
0.95	0.96	0.96
0.94	0.94	0.96
0.86	0.87	0.93
0.99	0.99	0.99
0.84	0.85	0.92
0.92	0.92	0.95
0.9	0.9	0.94
0.8	0.81	0.87
0.99	0.99	0.99

where $\mathcal{L}_L$ denotes Łukasiewicz algebra, $\mathcal{L}_P$ the product algebra, and $\mathcal{L}_G$ the Gödel algebra.

5 CONCLUSION

In this contribution, we presented a novel approach to the construction of weighted fuzzy rules based

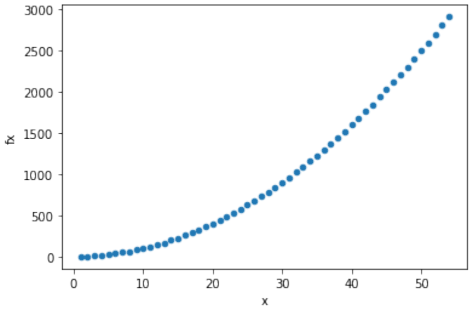

Figure 15: Input data.

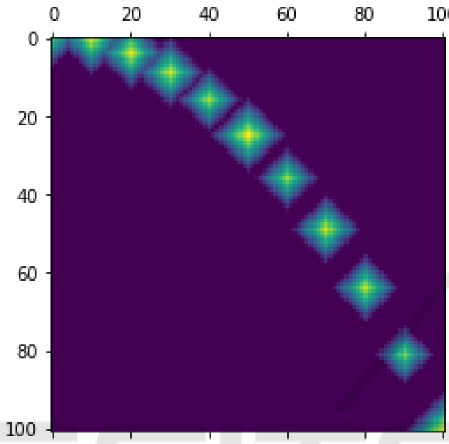

Figure 16: $\mathrm{DNF}_{\mathrm{Mamd}_F}$ in Łukasiewicz algebra.

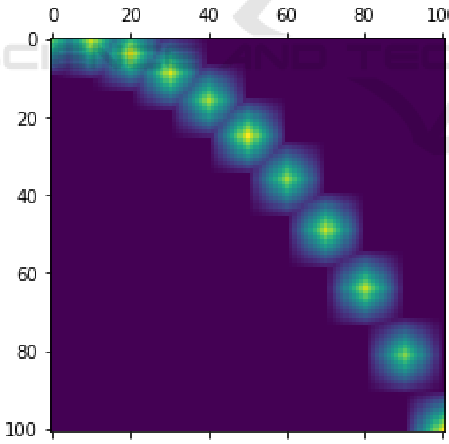

Figure 17: $\mathrm{DNF}_{\mathrm{Mamd}_F}$ in Product algebra.

on positive samples. We combine Perfilieva's normal forms and the relational data models proposed by Hájek. It allows us to fix a number of weighted fuzzy rules and overcome the problem of a huge unclear fuzzy rule basis. Additionally, all theoretical results known in both approaches are applicable for the special normal forms, too, as was presented some selected results in the preceding section.

In this phase of research, only theoretical results

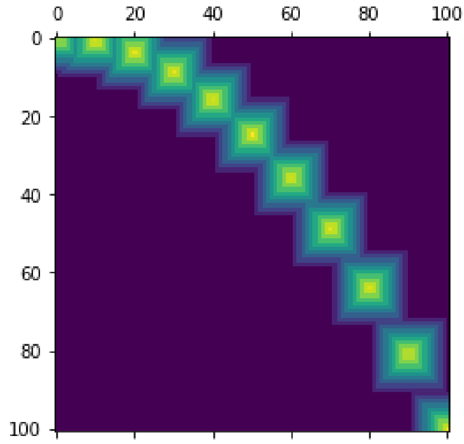

Figure 18: $\mathrm{DNF}_{\mathrm{Mamd}_F}$ in Gödel algebra.

are at the disposal. Therefore, it is now the next step to test the proposed method. Clearly, the number of weighted fuzzy rules can be optimized as well as the position of nodes in which the special normal forms will be constructed. Intuitively, in the case of $\mathrm{DNF}_{\mathrm{Mamd}_F}, \mathrm{DNF}_{\mathrm{Rules}_F}$, the nodes (p_j, q_j), $j \in J$ closer to the data points (c_i, d_i), $i \in I$ will propagate a better approximation in the sense of the degree of equivalence. In contrast to the previous fact, the more distant the nodes are from the given data in $\mathrm{CNF}_{\mathrm{Mamd}_F}$ and $\mathrm{CNF}_{\mathrm{Rules}_F}$, the better the approximation (higher degree of equivalence).

ACKNOWLEDGEMENTS

This research was supported by the Czech Science Foundation project No. 20–07851S.

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