Finite-time Stability Analysis for Nonlinear Descriptor Systems

N. Shopa[a], D. Konovalov[b], A. Kremlev[c] and K. Zimenko[d]
Faculty of Control Systems and Robotics, ITMO University, Russia Federation

Keywords: Descriptor Systems, Finite-time Stability, Nonlinear Systems, Stability Analysis.

Abstract: Sufficient conditions of finite-time stability are presented for the class of nonlinear descriptor systems. Both, explicit and implicit Lyapunov function methods, are extended for finite-time stability analysis of descriptor systems and the corresponding settling time estimates are obtained. The theoretical results are supported by numerical examples.

1 INTRODUCTION

Frequently, in control practice there are nonlinear systems for which it is hard to derive useful representation in the form of Ordinary Differential Equations (ODEs). Compared to ODEs, descriptor (singular) systems in addition to the dynamic part also include a static (uncausal) one (e.g., algebraic constraints). In this way, descriptor models are more flexible for system description. Additionally, descriptor models allow preserving physical meaning of variables. Therefore, descriptor systems have often been a subject of research (see, for example, (Sun et al., 2014; Zheng and Cao, 2013; Mo et al., 2017; Ikeda et al., 2004; Yang et al., 2012; Wu and Mizukami, 1994)).

Stability (stabilizability) analysis based on the Lyapunov function method for nonlinear descriptor systems was considered in (Ikeda et al., 2004; Hill and Mareels, 1990; Wu and Mizukami, 1994; Yang et al., 2012; Chen and Yang, 2016). *Finite-time* stability analysis (stabilization) is important if all transitions of the system has to be terminated in a finite (specified in advance) time (see, for example, (Bhat and Bernstein, 2000; Roxin, 1966; Polyakov et al., 2015; Bacciotti and Rosier, 2005; Moulay and Perruquetti, 2006; Orlov, 2004), etc.). The papers (Sun et al., 2014; Zheng and Cao, 2013; Konovalov et al., 2021) are devoted to the finite-time control design problem for descriptor systems. In these papers an analysis of finite-time stability is based on preliminary transformation to the canonical semi-explicit form (i.e., ODEs with constraints) and subsequent finite-time stability analysis for ODEs. A finite-time stability analysis based on the Lyapunov function method was proposed in the paper (Chen and Yang, 2016) for nonlinear descriptor systems. However, stability conditions proposed in (Chen and Yang, 2016) are too conservative and can be applied only for rather specific examples.

In this paper, sufficient Lyapunov-based conditions of finite-time stability are presented for the class of nonlinear descriptor systems. The corresponding estimates of settling time functions are also derived. Compared to the paper (Chen and Yang, 2016), stability conditions are significantly relaxed. The proposed conditions are applicable for descriptor systems in not necessarily semi-explicit form.

The paper is organized in the following way. Section II introduces notation used in the paper. Section III recalls some basics on descriptor systems and finite-time stability. Section IV presents the main result on finite-time stability conditions for nonlinear descriptor systems. Some numerical examples are also presented there. Finally, concluding remarks are given in Section V.

2 NOTATION

Through the paper the following notation will be used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, where $\mathbb{R}$ is the field of real numbers;
- $\mathbb{R}^n$ denotes the n dimensional Euclidean space

[a] https://orcid.org/0000-0001-7518-6346
[b] https://orcid.org/0000-0002-9973-8202
[c] https://orcid.org/0000-0002-7024-3126
[d] https://orcid.org/0000-0001-6220-7494

Shopa, N., Konovalov, D., Kremlev, A. and Zimenko, K.
Finite-time Stability Analysis for Nonlinear Descriptor Systems.
DOI: 10.5220/0011347500003271
In *Proceedings of the 19th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2022)*, pages 711-716
ISBN: 978-989-758-585-2; ISSN: 2184-2809

with vector norm $\|\cdot\|$;

- $\mathrm{O}_{m\times n}$ denotes zero matrix with dimension $m\times n$;
- $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix;
- the order relation $P>0$ (<0; ≥ 0; ≤ 0) for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive (negative) definite (semidefinite);
- A continuous function $\sigma:\mathbb{R}_+\cup\{0\}\to\mathbb{R}_+\cup\{0\}$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0)=0$. It belongs to class $\mathcal{K}_\infty$ if it is also unbounded.

3 PRELIMINARIES

Let us consider a nonlinear descriptor system

$$E\dot{x}(t)=f(x(t)),\quad x(0)=x_0,\quad t\geq 0, \tag{1}$$

where $x\in\mathbb{R}^n$ is the state vector, $f:\mathbb{R}^n\to\mathbb{R}^n$, $f(0)=0$ and $E\in\mathbb{R}^{n\times n}$ is a constant matrix, which is singular in general (rank$E=r<n$).

Definition 1 (Newcomb, 1981). *Initial conditions x_0 are consistent at $t=0$ if there exists a solution $\Phi(t,x_0)$ to the system (1), such that $x_0=\lim_{t\to 0^+}\Phi(t,x_0)$.*

Assumption 1. The system (1) has a unique solution $\Phi(t,x_0)$, $t\geq 0$, and initial value $x_0\in\mathbb{R}^n$ is consistent.

Definition 2 (Wu and Mizukami, 1994). *The system (1) is said to be*

- *globally Lyapunov stable if for any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $\|x_0\|<\delta$, then $\|\Phi(t,x_0)\|<\varepsilon$ for all $t\geq 0$.*
- *globally asymptotically stable if it is Lyapunov stable and there exists $\delta(\varepsilon)>0$ such that $\|x_0\|<\delta$ implies $\lim_{t\to+\infty}\Phi(t,x_0)=0$.*

Similar to systems presented by ODEs (see (Bhat and Bernstein, 2000), (Polyakov, 2011)) let us give finite-time stability definitions for the descriptor system (1).

Definition 3. *The origin of (1) is said to be globally finite-time stable if it is globally asymptotically stable and any solution $\Phi(t,x_0)$ of the system (1) reaches the equilibrium point at some finite time moment, i.e., $\Phi(t,x_0)=0$, $\forall t\geq T(x_0)$ and $\Phi(t,x_0)\neq 0$, $\forall t\in[0,T(x_0))$, $x_0\neq 0$, where $T:\mathbb{R}^n\to\mathbb{R}_+\cup\{0\}$, $T(0)=0$ is a settling-time function.*

Definition 4. *The set M is said to be finite-time attractive for (1) if any solution $\Phi(t,x_0)$ of (1) reaches M in a finite instant of time $t=T_M(x_0)$ and remains there $\forall t\geq T_M(x_0)$. As before, $T_M:\mathbb{R}^n\to\mathbb{R}_+\cup\{0\}$ is a settling-time function.*

Define a full column rank matrix $U\in\mathbb{R}^{n\times(n-r)}$ whose column vectors consist of the bases of NullE^T. Define the set $\mathcal{M}=\{x\in\mathbb{R}^n : Ex=0\}$.

The following theorem gives a sufficient condition for asymptotic stability analysis of nonlinear descriptor systems.

Theorem 1 (Ikeda et al., 2004). *Let there exist a continuously differentiable function $V:\mathbb{R}^n\to\mathbb{R}_+$, a continuous function $W:\mathbb{R}^n\times\mathbb{R}^{n-r}\to\mathbb{R}$, functions a, $b\in\mathcal{K}_\infty$ and $c\in\mathcal{K}$ satisfying the following conditions:*

1) $a(\|Ex\|)\leq V(Ex)\leq b(\|Ex\|)$ for $\forall x\in\mathbb{R}^n$;

2) $\dot{V}(Ex)+W(x,U^Tf(x))\leq -c(\|x\|)$ for $\forall x\in\mathbb{R}^n$, where $\dot{V}(Ex)=\mathrm{grad}V(z)\Big|_{z=Ex}\cdot f(x)$;

3) $W(x,0)\equiv 0$ for $\forall x\in\mathbb{R}^n$

Then the zero solution $x\equiv 0$ of the descriptor system (1) is globally asymptotically stable.

In (Chen and Yang, 2016) is proposed a sufficient condition for finite-time stability analysis of descriptor systems.

Theorem 2 (Chen and Yang, 2016). *Let there exist a continuous function $V:\mathbb{R}^n\to\mathbb{R}_+$ and two functions $a,b\in\mathcal{K}_\infty$ such that the following conditions hold:*

1) $a(\|E\|\|x\|)\leq V(x)\leq b(\|E\|\|x\|)$;

2) $\dot{V}(x)\leq -\beta V(x)^\sigma$, where $\sigma\in(0,1)$, $\beta\in\mathbb{R}_+$;

Then the system (1) is finite-time stable with a settling time satisfying the inequality

$$T(x_0)\leq\frac{V^{1-\sigma}(x_0)}{\beta(1-\sigma)}.$$

Remark 1. The dynamical part of the system (1) is represented by state-space equations whose state variable corresponds to the variable Ex, i.e., Ex represents the dynamical behavior of the systems by itself (Ikeda et al., 2004). Thus, it is 'natural' to consider a Lyapunov function as a positive definite function of the variable Ex. In this sense, the condition 1) in Theorem 2 is too restrictive and it significantly narrows the class of descriptor systems, where Theorem 2 can be useful.

4 MAIN RESULT

The next theorem extends the result of Theorem 1 for finite-time stability analysis.

Theorem 3. *Suppose there exist continuously differentiable functions $V_1,V_2:\mathbb{R}^n\to\mathbb{R}_+$, continuous functions $W_1,W_2:\mathbb{R}^n\times\mathbb{R}^{n-r}\to\mathbb{R}$, functions $a,b\in\mathcal{K}_\infty$, $c\in\mathcal{K}$ and real numbers $\beta\in\mathbb{R}_+$ and $\sigma\in[0,1)$, such that:*

1) $a(\|Ex\|)\leq V_i(Ex)\leq b(\|Ex\|)$ for $\forall x\in\mathbb{R}^n$ and $i=1,2$;

2) $\dot{V}_1(Ex)+W_1(x,U^Tf(x))\leq -c(\|x\|)$ for $\forall x\in\mathbb{R}^n$;

3) $W_i(x,0) \equiv 0$ for $i = 1,2$ and $\forall x \in \mathbb{R}^n$;

4) $\dot{V}_2(Ex) + W_2(x, U^T f(x)) \le -\beta V_2^{\sigma}(Ex)$ for $\forall x \in \mathbb{R}^n \setminus \mathcal{M}$;

where $\dot{V}_i(Ex) = \mathrm{grad}V_i(Ex) \cdot f(x)$ for $i = 1,2$. Then the system (1) is globally finite-time stable and

$$T(x_0) \le \frac{V_2^{1-\sigma}(Ex_0)}{\beta(1-\sigma)}$$

Proof. Conditions $1) - 3)$ provide asymptotic stability of the system (1) according to Theorem 1.

Since U satisfies $U^T E = O_{(n-r)\times n}$, the term W_2 (as well as W_1 in the condition 2)) becomes zero along the solutions of the system (1), and 4) provides

$$\dot{V}_2(Ex) \le -\beta V_2^{\sigma}(Ex), \quad \forall x \in \mathbb{R}^n \setminus \mathcal{M}. \tag{2}$$

Then, 1) and (2) provide the set $\mathcal{M}$ is finite-time attractive with the settling-time estimate

$$T_{\mathcal{M}}(x_0) \le \frac{V_2^{1-\sigma}(Ex_0)}{\beta(1-\sigma)}$$

that can be checked by integration of (2) (see (Lopez-Ramirez et al., 2018), (Zimenko et al., 2021) for more details):

$$\begin{aligned} T_{\mathcal{M}}(x) &= \int_0^{V_2(Ex)} \frac{ds}{-\dot{V}_2(E\Phi(\theta_x(s),x))} \\ &\le \int_0^{V_2(Ex)} \frac{ds}{\beta V_2(E\Phi(\theta_x(s),x))^{\sigma}} \\ &= \int_0^{V_2(Ex)} \frac{ds}{\beta s^{\sigma}} \\ &= \frac{1}{\beta(1-\sigma)} V_2(Ex)^{1-\sigma} < +\infty, \end{aligned} \tag{3}$$

where θ_x is the inverse of $t \to V_2(E\Phi(t,x))$. Moreover, $\dot{V}_2(0) = 0$. On the other hand, due to $Ex(t) = 0$ for $\forall t \ge T_{\mathcal{M}}(x_0)$, then from 1) and 2) we have that $\|x\| = 0$ for $\forall t \ge T_{\mathcal{M}}(x_0)$, i.e. the system is finite-time stable with $T(x_0) \equiv T_{\mathcal{M}}(x_0)$. ■

Note that Theorem 2 is a particular case of Theorem 3.

Example 1. Consider the following nonlinear descriptor system:

$$\begin{aligned} \dot{x}_1 &= -x_1^{1/3} + 2x_2, \\ 0 &= x_1 + x_2 + \tfrac{1}{4}\sin(x_1 x_2), \end{aligned} \tag{4}$$

where $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Let us choose

$$V(Ex) = V_1(Ex) = V_2(Ex) = \frac{1}{2}x_1^2, \tag{5}$$

and

$$\begin{aligned} W(x, U^T f(x)) &= W_1(x, U^T f(x)) = W_2(x, U^T f(x)) \\ &= -(x_1 + x_2 + \tfrac{1}{4}\sin(x_1 x_2))^2, \end{aligned} \tag{6}$$

for which the conditions 1) and 3) of Theorem 3 are satisfied. Then, we obtain

$$\begin{aligned} &\dot{V}(Ex) + W(x, U^T f(x)) \\ &= -x_1^{4/3} + 2x_1x_2 - (x_1 + x_2 + \tfrac{1}{4}\sin(x_1x_2))^2 \\ &= -x_1^{4/3} - x_1^2 - x_2^2 - \tfrac{1}{2}(x_1 + x_2)\sin(x_1x_2) \\ &\qquad - \tfrac{1}{16}\sin^2(x_1x_2) \\ &\le -x_1^{4/3} - \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2 - \tfrac{1}{16}\sin^2(x_1x_2). \end{aligned} \tag{7}$$

From (7) we have

$$\dot{V}(Ex) + W(x, U^T f(x)) \le -\tfrac{1}{2}\|x\|^2$$

and, on the other hand,

$$\begin{aligned} \dot{V}(Ex) + W(x, U^T f(x)) &\le -x_1^{4/3} \\ &= -2V^{2/3}(Ex), \end{aligned}$$

i.e., the conditions 2) and 4) of Theorem 3 are also satisfied. Thus, all conditions of Theorem 3 are satisfied, and the system is finite-time stable with the following settling time estimate $T(x_0) \le \frac{3}{2^{4/3}} x_{10}^{2/3}$, where $x_{10} = x_1(0)$. The results of simulation with using the logarithmic scale are shown in Fig. 1 in order to demonstrate finite-time convergence rate of the Euclidean norm $\|x\|$.

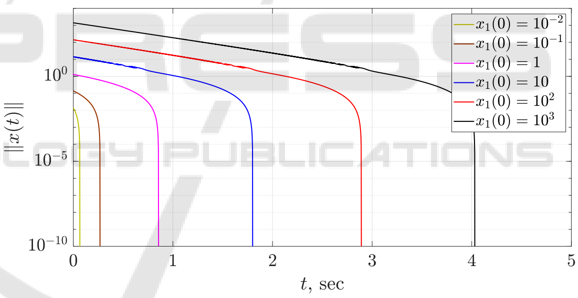

Figure 1: The results of simulation for different initial conditions.

The advantages of the proposed result are based on the following observations:

- under the condition 2) the finite-time attractiveness of the set $\mathcal{M}$ is equivalent to finite-time stability of the origin;
- the condition 1) allows to choose a Lyapunov function depending only on the variable Ex (e.g., $V_i = x^T E^T P E x$, $P > 0$ and its nonlinear variations);
- the terms W_1 and W_2 in the conditions 2) and 4) become zero along the solutions of the system (1). In spite of this, these terms are crucial for the satisfaction of the conditions 2) and 4) in practice (see, for example, the results of (Ikeda et al., 2004), (Uezato and Ikeda, 1999) on asymptopic stability analysis).

If $f(x) = 0$ only at the origin for $x \in \mathcal{M}$ then the following result one can obtain:

Corollary 1. *Suppose there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$, a continuous function $W : \mathbb{R}^n \times \mathbb{R}^{n-r} \to \mathbb{R}$, functions $a, b \in \mathcal{K}_\infty$ and real numbers $\beta \in \mathbb{R}_+$ and $\sigma \in [0,1)$, such that:*

1) $a(\|Ex\|) \le V(Ex) \le b(\|Ex\|)$ for $\forall x \in \mathbb{R}^n$;

2) $\dot{V}(Ex) + W(x, U^T f(x)) \le -\beta V^\sigma(Ex)$ for $\forall x \in \mathbb{R}^n \setminus \mathcal{M}$;

3) $W(x,0) \equiv 0$ for $\forall x \in \mathbb{R}^n$;

4) $\{x \in \mathcal{M} : f(x) = 0\} \equiv \{0\}$.

Then the system (1) is globally finite-time stable and

$$T(x_0) \le \frac{V^{1-\sigma}(Ex_0)}{\beta(1-\sigma)}.$$

Proof. The proof is straightforward. According to the proof of Theorem 3 the conditions 1) – 3) provide finite-time attractiveness of the set $\mathcal{M}$. Due to $Ex(t) = 0$ for $\forall t \ge T_{\mathcal{M}}(x_0) = \frac{V^{1-\sigma}(Ex_0)}{\beta(1-\sigma)}$, then $f(x(t)) = 0$ for $\forall t \ge T_{\mathcal{M}}(x_0)$ and by the condition 4) we have that $\|x\| = 0$ for $\forall t \ge T_{\mathcal{M}}(x_0)$, i.e. the system is finite-time stable with $T(x_0) \equiv T_{\mathcal{M}}(x_0)$. ■

Example 2. Consider the three-tank hydraulic system (Fig. 2) from (Duro et al., 2008). The system consists of three cylinders T_1, T_2, and T_3 with the same cross-section A. These cylinders are connected serially to each other by cross-section S_n pipes.

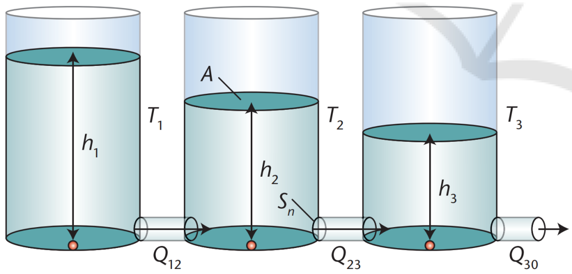

Figure 2: Schematic diagram of the three-tank system.

The mathematical model of the plant can be represented by the following descriptor system:

$$\begin{aligned} A\dot{h}_1 &= -Q_{12}, \\ A\dot{h}_2 &= Q_{12} - Q_{23}, \\ A\dot{h}_3 &= Q_{23} - Q_{30}, \\ Q_{12} &= S_n\sqrt{2g}\lfloor h_1 - h_2 \rceil^{0.5}, \\ Q_{23} &= S_n\sqrt{2g}\lfloor h_2 - h_3 \rceil^{0.5}, \\ Q_{30} &= S_n\sqrt{2g}\lfloor h_3 \rceil^{0.5}, \end{aligned} \tag{8}$$

where h_1, h_2 and h_3 are the liquid levels in each tank, their derivatives represent the balance equations; Q_{12} and Q_{23} are the flow rates between tanks, Q_{30} is the rate of flow exiting the system; g is the acceleration due to gravity; $\lfloor \cdot \rceil^\alpha = |\cdot|^\alpha \mathrm{sign}(\cdot)$. The system is in the form (1) with $E = \begin{bmatrix} AI_3 & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}$, $x^T = \begin{bmatrix} h_1 & h_2 & h_3 & Q_{12} & Q_{23} & Q_{30} \end{bmatrix}^T$. It is easy to check that the condition 4) of Corollary 1 is satisfied.

Now let us choose the following candidate Lyapunov function

$$V(Ex) = |x_1 - x_2|^{1.5} + |x_2 - x_3|^{1.5} + |x_3|^{1.5}. \tag{9}$$

Recall that by Minkowski inequality for any $z_1, z_2 \in \mathbb{R}$ and $p \ge 1$, the inequality

$$|z_1 + z_2|^p \le 2^{p-1}(|z_1|^p + |z_2|^p)$$

hold. Applying this inequality one can obtain

$$\begin{aligned} \tfrac{1}{6}(|x_1|^{1.5} + |x_2|^{1.5} + |x_3|^{1.5}) &\le V(Ex) \\ &\le 2^{3/2}(|x_1|^{1.5} + |x_2|^{1.5} + |x_3|^{1.5}), \end{aligned}$$

i.e., the condition 1) of Corollary 1 is satisfied.

Let

$$\begin{aligned} W(x, U^T f(x)) = {} & \tfrac{1.5}{A}\lfloor x_1 - x_2 \rceil^{0.5}(2x_4 \\ & -2f_1(x) - x_5 + f_2(x)) \\ & + \tfrac{1.5}{A}\lfloor x_2 - x_3 \rceil^{0.5}(f_1(x) \\ & -x_4 + 2x_5 - 2f_2(x) - x_6 + f_3(x)) \\ & + \tfrac{1.5}{A}\lfloor x_3 \rceil^{0.5}(-x_5 + f_2(x) + x_6 - f_3(x)), \end{aligned} \tag{10}$$

where $f_1(x) = S_n\sqrt{2g}\lfloor x_1 - x_2 \rceil^{0.5}$, $f_2(x) = S_n\sqrt{2g}\lfloor x_2 - x_3 \rceil^{0.5}$ and $f_3(x) = S_n\sqrt{2g}\lfloor x_3 \rceil^{0.5}$. According to (8) the function W become zero along the solutions of the system (the condition 3) of Corollary 1 hold).

Since

$$\begin{aligned} \dot{V} = {} & \tfrac{1.5}{A}\lfloor x_1 - x_2 \rceil^{0.5}(-2x_4 + x_5) \\ & + \tfrac{1.5}{A}\lfloor x_2 - x_3 \rceil^{0.5}(x_4 - 2x_5 + x_6) \\ & + \tfrac{1.5}{A}\lfloor x_3 \rceil^{0.5}(x_5 - x_6), \end{aligned} \tag{11}$$

then, taking into account the equivalence of norms $\|\cdot\|_{1.5} \le \|\cdot\|_1$, we obtain

$$\begin{aligned} & \dot{V}(Ex) + W(x, U^T f(x)) \\ & = \tfrac{S_n\sqrt{g}}{\sqrt{2}A} \begin{bmatrix} \lfloor x_1 - x_2 \rceil^{0.5} \\ \lfloor x_2 - x_3 \rceil^{0.5} \\ \lfloor x_3 \rceil^{0.5} \end{bmatrix}^T (H + H^T) \begin{bmatrix} \lfloor x_1 - x_2 \rceil^{0.5} \\ \lfloor x_2 - x_3 \rceil^{0.5} \\ \lfloor x_3 \rceil^{0.5} \end{bmatrix} \\ & \le -\tfrac{1.5S_n\sqrt{g}}{\sqrt{2}A}(|x_1 - x_2| + |x_2 - x_3| + |x_3|) \\ & \le -\tfrac{1.5S_n\sqrt{g}}{\sqrt{2}A} V^{2/3}(Ex), \end{aligned} \tag{12}$$

where $H = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -1.5 \end{bmatrix}$.

Thus, by Corollary 1 the system (8) is finite-time stable.

Remark 2. Consider the system (1) in the canonical semi-explicit form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ 0 &= f_2(x_1, x_2) \end{aligned}, \tag{13}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $E = \begin{bmatrix} I_r & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{bmatrix}$, $f(x) = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix}$. Then under the assumption the function $f_2(x)$ satisfies the condition that there exists a continuous function h such that $x_2 = h(x_1)$ and $h(0) = 0$, the system (13) can be reduced to an ODE

$$\dot{x}_1 = f_1(x_1, h(x_1)),$$

and finite-time stability analysis methods corresponding to ODEs can be applied (for example, (Bhat and Bernstein, 2000), (Roxin, 1966), (Bacciotti and Rosier, 2005), (Moulay and Perruquetti, 2006), etc.). Note, the representation in the canonical form (13) and the subsequent transition to ODEs in some cases can be accompanied by computational complexity and errors. It is also worth noting that for the system in the form (13) the condition 4) of Corollary 1 is satisfied and the principal conditions are $1) - 3)$.

The next theorem presents the Implicit Lyapunov Function method (see (Korobov, 1979), (Adamy and Flemming, 2004)) for finite-time stability analysis of descriptor systems (1).

Theorem 4. *Suppose that there exists a continuous function*

$$\begin{aligned} Q\colon \mathbb{R}_+ \times \mathbb{R}^n &\to \mathbb{R} \\ (V,z) &\mapsto Q(V,z) \end{aligned}$$

such that

1) $Q(V,z)$ *is continuously differentiable* $\forall z \in \mathbb{R}^n\backslash\{0\}$ *and* $\forall V \in \mathbb{R}_+$;

2) *for any* $z \in \mathbb{R}^n\backslash\{0\}$ *there exist* $V^- \in \mathbb{R}_+$ *and* $V^+ \in \mathbb{R}_+$*:*

$$Q(V^-,z) < 0 < Q(V^+,z); \tag{14}$$

3) *for* $\Omega = \{(V,z) \in \mathbb{R}^{n+1}\colon Q(V,z) = 0\}$

$$\lim_{\substack{z\to 0\\ (V,z)\in\Omega}} V = 0,\ \lim_{\substack{V\to 0^+\\ (V,z)\in\Omega}} \|z\| = 0,\ \lim_{\substack{\|z\|\to\infty\\ (V,z)\in\Omega}} V = +\infty;$$

4) *the inequality*

$$-\infty < \frac{\partial Q(V,z)}{\partial V} < 0$$

holds $\forall V \in \mathbb{R}_+$ *and* $\forall z \in \mathbb{R}^n\backslash\{0\}$;

5) *the inequality*

$$\frac{\partial Q(V,z)}{\partial z} f(x) + W(V,x,U^T f(x)) \leq \beta V^\sigma \frac{\partial Q(V,z)}{\partial V}$$

hold $\forall x \in \mathbb{R}^n$; $\forall (V,x) \in \mathbb{R}^{n+1}\colon Q(V,Ex) = 0$, *where* $z = Ex$, $W : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n-r} \to \mathbb{R}$ *is such that* $W(V,x,0) \equiv 0$, *and* $\sigma \in [0,1)$, $\beta \in \mathbb{R}_+$ *are some constants;*

6) $\{x \in \mathcal{M} : f(x) = 0\} \equiv \{0\}$.

Then the origin of the system (1) is globally finite-time stable and

$$T(x_0) \leq \frac{V_0^{1-\sigma}}{\beta(1-\sigma)},$$

where $V_0 \in \mathbb{R}_+ : Q(V_0, Ex_0) = 0$.

Proof. The proof follows the same arguments as one of Corollary 1 and the Implicit Lyapunov Function method for finite-time stability analysis of ODE systems (Polyakov et al., 2015, Theorem 4). The conditions 1), 2), 4) and the implicit function theorem (Courant and John, 2000) imply that the equation $Q(V,z) = 0$ implicitly defines a unique function $V : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_+$ such that $Q(V(z),z) = 0$ for all $z \in \mathbb{R}^n \setminus \{0\}$. Due to the condition 3) the function V can be continuously prolonged at the origin by setting $V(0) = 0$. In addition, V is radially unbounded and positive definite. Then there exist functions $a, b \in \mathcal{K}_\infty$ such that $a(\|z\|) \leq V(z) \leq b(\|z\|)$ (Khalil, 1992), i.e., the condition 1) of Corollary 1 is satisfied with $z = Ex$.

Since by means of Implicit Function Theorem (Courant and John, 2000) $\dot{V} = -\left[\frac{\partial Q}{\partial V}\right]^{-1} \frac{\partial Q}{\partial z}\dot{z}$, then with $z = Ex$ the condition 5) repeats the conditions 2) and 3) of Corollary 1. Finally, the condition 6) repeats the condition 4) of Corollary 1. Thus, all conditions of Corollary 1 are satisfied and the system (1) is finite-time stable. ∎

Remark 3. In (Konovalov et al., 2021) a finite-time homogeneous control is proposed for linear descriptor systems, and the following implicitly defined Lyapunov function is considered

$$Q(V,x) = x^T e^{-G_{\mathbf{d}}^T \ln V} X^T E e^{-G_{\mathbf{d}} \ln V} x - 1,$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix (i.e., $-G_{\mathbf{d}}$ is Hurwitz); $X \in \mathbb{R}^{n\times n}$ is such that $X^T E = E^T X \geq 0$ and $x^T X^T E x = 0$ iff $Ex = 0$. In view of Theorem 4, the result of (Konovalov et al., 2021) can be revisited in order to consider an implicitly defined Lyapunov function in the form

$$\begin{aligned} &Q(V,x) = z^T e^{-L^T \ln V} P e^{-L \ln V} z - 1, \\ &P > 0,\ L \text{ is anti-Hurwitz},\ z = Ex, \end{aligned} \tag{15}$$

which can be considered as homogeneous generalization (see (Konovalov et al., 2021) for more details) of the quadratic Lyapunov function $V(Ex) = x^T E^T P E x$. Based on the stability analysis of linear descriptor systems proposed in (Xu and Lam, 2006), it is expected that basing on Theorem 4 the choice of a Lyapunov function in the form (15) may allow one to obtain more reliable LMI-based stability conditions and investigate the robustness properties of the control scheme given in (Konovalov et al., 2021). This is one of the main directions for future research.

5 CONCLUSION

In this paper, the sufficient conditions of finite-time stability are proposed for the class of nonlinear descriptor systems. The settling time estimates are obtained. Both, explicit and implicit Lyapunov function methods are considered. The conditions are sufficiently less restrictive than those proposed in (Chen and Yang, 2016). The presented finite-time stability analysis opens a lot of topics for future research. For example, control and observer design for descriptor systems based on the proposed stability conditions.

ACKNOWLEDGEMENTS

This work is supported by RSF under grant 22-29-00344 in ITMO University.

REFERENCES

Adamy, J. and Flemming, A. (2004). Soft variable-structure controls: a survey. *Automatica*, 40(11):1821–1844.

Bacciotti, A. and Rosier, L. (2005). *Liapunov functions and stability in control theory*. Springer Science & Business Media.

Bhat, S. P. and Bernstein, D. S. (2000). Finite-time stability of continuous autonomous systems. *SIAM Journal on Control and optimization*, 38(3):751–766.

Chen, G. and Yang, Y. (2016). Finite time stabilization of nonlinear singular systems. In *2016 35th Chinese Control Conference (CCC)*, pages 516–520. IEEE.

Courant, R. and John, F. (2000). Introduction to calculus and analysis volume ii/1.

Duro, N., Dormido, R., Vargas, H., Dormido-Canto, S., Sánchez, J., Farias, G., and Esquembre, F. (2008). An integrated virtual and remote control lab: The three-tank system as a case study. *Computing in Science & Engineering*, 10(4):50–59.

Hill, D. J. and Mareels, I. M. (1990). Stability theory for differential/algebraic systems with application to power systems. *IEEE transactions on circuits and systems*, 37(11):1416–1423.

Ikeda, M., Wada, T., and Uezato, E. (2004). Stability analysis of nonlinear systems via descriptor equations. *IFAC Proceedings Volumes*, 37(11):557–562.

Khalil, H. (1992). Nonlinear system. macmillan publishing company. *New York: Wiley*, pages 461–483.

Konovalov, D., Zimenko, K., Belov, A., and Wang, H. (2021). Homogeneity-based finite-time stabilization of linear descriptor systems. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 3901–3905. IEEE.

Korobov, V. I. (1979). A solution of the problem of synthesis using a controllability function. In *Doklady Akademii Nauk*, volume 248, pages 1051–1055. Russian Academy of Sciences.

Lopez-Ramirez, F., Efimov, D., Polyakov, A., and Perruquetti, W. (2018). On necessary and sufficient conditions for fixed-time stability of continuous autonomous systems. In *2018 European control conference (ECC)*, pages 197–200. IEEE.

Mo, X., Niu, H., and Lan, Q. (2017). Finite-time stabilization for a class of nonlinear differential-algebraic systems subject to disturbance. *Discrete Dynamics in Nature and Society*, 2017.

Moulay, E. and Perruquetti, W. (2006). Finite time stability and stabilization of a class of continuous systems. *Journal of Mathematical analysis and applications*, 323(2):1430–1443.

Newcomb, R. (1981). The semistate description of nonlinear time-variable circuits. *IEEE Transactions on Circuits and Systems*, 28(1):62–71.

Orlov, Y. (2004). Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal on Control and Optimization*, 43(4):1253–1271.

Polyakov, A. (2011). Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, 57(8):2106–2110.

Polyakov, A., Efimov, D., and Perruquetti, W. (2015). Finite-time and fixed-time stabilization: Implicit lyapunov function approach. *Automatica*, 51:332–340.

Roxin, E. (1966). On finite stability in control systems. *Rendiconti del Circolo Matematico di Palermo*, 15(3):273–282.

Sun, L., Feng, G., and Wang, Y. (2014). Finite-time stabilization and h∞ control for a class of nonlinear hamiltonian descriptor systems with application to affine nonlinear descriptor systems. *Automatica*, 50(8):2090–2097.

Uezato, E. and Ikeda, M. (1999). Strict lmi conditions for stability, robust stabilization, and h/sub/spl infin//control of descriptor systems. In *Proceedings of the 38th IEEE Conference on Decision and Control (Cat. No. 99CH36304)*, volume 4, pages 4092–4097. IEEE.

Wu, H. and Mizukami, K. (1994). Stability and robust stabilization of nonlinear descriptor systems with uncertainties. In *Proceedings of 1994 33rd IEEE Conference on Decision and Control*, volume 3, pages 2772–2777. IEEE.

Xu, S. and Lam, J. (2006). *Robust control and filtering of singular systems*, volume 332. Springer.

Yang, C., Sun, J., Zhang, Q., and Ma, X. (2012). Lyapunov stability and strong passivity analysis for nonlinear descriptor systems. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 60(4):1003–1012.

Zheng, C. and Cao, J. (2013). Finite-time synchronization of singular hybrid coupled networks. *Journal of Applied Mathematics*, 2013.

Zimenko, K., Efimov, D., Polyakov, A., and Kremlev, A. (2021). On necessary and sufficient conditions for output finite-time stability. *Automatica*, 125:109427.