A Modified Polynomial Preserving Recovery Technique

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Abstract: In this work, the polynomial preserving recovery method is enhanced by increasing the order of the fitting polynomial within the same patch. This is achieved by adding more sample points inside the elements of the patch then substitute them in the discretized form of the differential equation. These sample points are the set of superconvergent points of the patch under consideration. Numerical results show that the recovered gradient at the nodes with linear elements is superconvergent. The proposed method improves the accuracy of the recovered gradient over the domain of the solution with the same rate of convergence of the polynomial preserving recovery technique.

1 INTRODUCTION

Gradient recovery techniques are post-processing methods in which the finite element solution is utilized to build a recovered gradient. Several techniques were developed as in (Levine, 1985) and (Bramble and Schatz, 1977), but they suffered from long execution time, complexity, requiring special mesh structure, or the necessity of using low order finite elements. The super-convergent patch recovery (SPR) technique or the ZZ recovery technique (Zienkiewicz and Zhu, 1992) was the first method to overcome these obstacles. The SPR has been introduced to recover the accuracy and continuity in the gradient field. The method least-squares fits the stresses at super-convergent points or points whose accuracy is high on element patches (Barlow, 1976) to obtain the gradient at the nodes. The super-convergence properties of the SPR are proved in (Li and Zhang, 1999) and (Zhang, 2000), and the efficiency of the ZZ error estimator is presented in (Ainsworth et al., 1989). Several treatments to improve the recovered gradient of the SPR have been proposed. The authors in (Blacker and Belytschko, 1994) improved the accuracy of the gradients by including the squares of the residuals of the equilibrium equations and the boundary conditions. They introduced a new conjoint polynomial for interpolating the stresses of the local patch to improve the projection scheme. Li and Wiberg (Li and Wiberg, 1994) fitted a higher order polynomial expansion to the finite element solution at super-convergent points in a patch of elements that include the targeted element and its neighbors. Wiberg et al. (Wiberg et al., 1994) proposed an enhancement of the SPR for linear elasticity problems and achieved an improvement in the gradient recovery near the boundaries. Gu et al. (Gu et al., 2004) modified the SPR for the nonlinear problems by using integration points as sampling points, applying weighted average procedure, and introducing additional nodes.

Zhang and Naga (Zhang and Naga, 2005) and (Naga and Zhang, 2005) introduced the polynomial preserving recovery (PPR) method. The idea of the method starts by constructing a patch of elements around a targeted node. Then the numerical solution at the nodes of the patch is fitted to a polynomial one degree higher than the solution. The fitting polynomial is differentiated to obtain the gradient at the node. The PPR was also employed to improve eigenvalue approximation (Naga et al., 2006) and (Shen and Zhou, 2006). In (Guo et al., 2016), the authors provided two strategies to enhance the gradi-

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ent at the boundaries. The study of the PPR was generalized to high frequency wave problems (Guo and Yang, 2017). The recovered gradient of the PPR is also used in adaptive refinement for Fredholm integral equation (Adel et al., 2016). The PPR is used to recover the heat flux for skin tissues in the presence of a tumor (Essam et al., 2019).

In this paper, we propose a modification to the PPR technique. Our modified technique builds a solution of a higher order polynomial within the same patch of the classical PPR. We substitute sample points within the elements of the patch in the discretized form of the differential equation. The sample points chosen in this work are the superconvergent points used in the SPR method. Then the resultant system of equations together with the system resulting from the PPR are solved together in the least-squares sense to evaluate the coefficients of our solution. Finally, the higher order solution is differentiated to get the gradient at the desired node.

The rest of the paper is organized as follows. In section 2, the basic concepts of the PPR are introduced. The proposed technique is presented in section 3. Section 4 contains some numerical examples to validate the accuracy and the robustness of the proposed technique. Finally, the findings of this work are summarized in section 5.

2 THE PPR METHOD

We firstly give some notations and review the basic concepts of the PPR method introduced in (Naga and Zhang, 2005) and (Zhang and Naga, 2005).

Let v denote a vertex in a finite element mesh, n be a positive integer, and $\ell(v,n)$ denote the union of the elements in the first n layers around v, i.e.,

$$\ell(v,n) := \bigcup\{\tau : \tau \in \tau_h, \tau \cap \ell(v,n-1) \neq \emptyset\}, \quad (1)$$

where τ is an element in a finite element triangulation τ_h and $\ell(v,0) := \{v\}$.

Let N_h denote all the mesh nodes and V_h be a finite element space of degree m over the triangulation τ_h. The standard Lagrange basis of V_h is denoted by $\{\phi_v : v \in N_h\}$. The PPR gradient recovery operator is denoted by G_h such that $G_h : V_h \longrightarrow V_h \times V_h$. For a mesh node v, a patch of elements around v is denoted by α_v. To recover the gradient at v, a polynomial $p_v \in P_{m+1}(\alpha_v)$ is to be fitted in the least-square sense using the set of nodes $N_h \cap \alpha_v$, i.e.

$$p_v = \arg \min_{p \in P_{m+1}(\alpha_v)} \sum_{\overline{v} \in N_h \cap \alpha_v} (u_h - p)^2(\overline{v}), \quad (2)$$

where $P_{m+1}(\alpha_v)$ is the space of all polynomials defined on the patch α_v with degree less than or equal $m+1$. Then the gradient at the node v is given by

$$(G_h u_h)(v) = \nabla p_v(v). \quad (3)$$

After getting the gradient at all vertices, we utilize interpolation with the original basis function of V_h to evaluate the gradient representation over the whole domain as

$$(G_h u_h) := \sum_{v \in N_h} (G_h u_h)(v)\phi_v. \quad (4)$$

For an interior node v we define α_v as the smallest $\ell(v,n)$ that ensures the uniqueness of the fitting polynomial (Naga and Zhang, 2005) and (Zhang and Naga, 2005). In the case that v is a boundary node, consider r to be the smallest positive integer to ensure that $\ell(v,r)$ has at least one interior mesh node. Then, we define

$$\alpha_v = \ell(v,r) \cup \{\alpha_{\tilde{v}} : \tilde{v} \in \ell(v,r) : \textit{ and } \tilde{v} \textit{ an interior vertex}\}. \quad (5)$$

The superconvergence analysis of G_h in (Zhang and Naga, 2005) showed that if $u \in W_\infty^{m+2}(\alpha_v)$, then

$$\|\nabla u - G_h u_h\|_{L_\infty(\alpha_v)} \leqslant Ch^{m+1}|u|_{W_\infty^{m+2}(\alpha_v)}. \quad (6)$$

where the PPR operator G_h preserves polynomials of degrees up to $m+1$ in the domain.

3 MODIFIED PPR METHOD

We refer to the proposed recovery technique as (MPPR). The objective of this method is to construct an enhanced recovered gradient $(\overline{G}_h u_h)$ at each node then use them to get a recovered gradient over the domain by interpolation. This method is only applicable when the source term f can be evaluated at any point in the domain. To recover the gradient at a node v, we assume that the solution over the patch of this node is a high order polynomial $p_m(x)$. We use two sets of sampling points to determine the coefficients of this polynomial. The first set contains inner-element points inside the patch. The points of this set are substituted in the discretized form of the differential equation. We choose these points to be the superconvergent points or the points whose finite element stress is of high accuracy. Then we follow the same steps of the classical PPR to get the other set which consists of the nodes of the patch. Finally, we get a system of algebraic equations that we solve in the least-squares sense to get the desired coefficients. If the nodes $N_h \cap \alpha_v$ from the first patch α_v together with inner-element nodes are not enough for the uniqueness condition of the desired polynomial, we go further to include the nodes from the following patch. We keep this process until the uniqueness condition is met.

In this work, for the one-dimensional case, we consider the following differential equation

$$L(u(x)) = f(x);\ \ x \in \Omega. \tag{7}$$

We construct a polynomial of an order m (higher than the order of the finite element solution) $p_m(x)$ that approximates the solution $u(x)$ of the considered problem.

$$p_m(x) = Pa, \tag{8}$$

where

$$P = (1,\, x,\, x^2, ...,\, x^m),\ \ a^T = (a_0,\, a_1,\, a_2, ...,\, a_m). \tag{9}$$

We consider an interior node v and its patch α_v. Then, for all the nodes ϑ in $N_h \cap \alpha_v$, we have

$$p_m(x_\vartheta) = u_h(x_\vartheta), \tag{10}$$

where x_ϑ are the coordinates of the nodes ϑ of the patch

Then we substitute the polynomial p_m in the differential equation to evaluate

$$L(p_m(x_j)) = f(x_j). \tag{11}$$

where x_j are the coordinates of the superconvergent points inside the elements of the patch

We solve this system of equations to obtain the coefficients of the polynomial p_m . Then the recovered gradient at the node v is given by

$$(\overline{G}_h u_h)(v) = \frac{d}{dx}(p_m(v)), \tag{12}$$

The recovered gradient over the entire domain is obtained using the original shape function as

$$\overline{G}_h u_h = \sum_{v \in N_h} (\overline{G}_h u_h)(v)\phi_v. \tag{13}$$

3.1 Adaptive Refinement

Adaptive refinement based on an *a posteriori* error estimator is widely used in FEMs. It is proved to be a useful tool for reducing the computational cost of solving differential equations as the refinement is directed toward regions where the solution is of a low quality. The new operator $\overline{G}_h u_h$ is tested as an error estimator in an adaptive FEM algorithm. The main steps of the algorithm are solving, estimating then refining. The considered differential equation is solved using FEM with linear elements. The recovered gradient obtained by the MPPR is calculated at each node. Then the refinement process is controlled by the following estimator:

$$e = \|\overline{G}_h u_h - \nabla u_h\|.$$

This estimator is evaluated at each element. All elements where the estimator exceeds a prescribed value are refined. We keep this process till the stoppage criterion is satisfied.

4 NUMERICAL RESULTS

In this section, we solve various problems using linear finite element method to illustrate the efficiency of the proposed approach. The fitting polynomial is chosen to be of a fourth order. The first two examples compare the new technique to the PPR over uniform refinement. We define the L_∞ norm error as the maximum error at the inner nodes. The other two examples test the gradient operator of the proposed technique as an *a posteriori* error estimator for adaptive refinement.

Example 1 (Zienkiewicz and Zhu, 1992): Consider the 1D linear differential equation

$$-\frac{d}{dx}\left(\frac{du}{dx}\right) + u = f;\ \ x \in (0,1),$$

with boundary conditions

$$u(0) = 0,\ \ u(1) = 0,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$u(x) = x^2 - \frac{\sinh 4x}{\sinh 4}.$$

Figures (1) and (2) show a comparison of the errors between the exact gradient and the recovered gradient obtained using PPR and MPPR methods in L_∞ and L_2 norms, respectively. The MPPR technique yields better results than those obtained by the standard PPR technique. Using the same original basis function to construct the new gradient affects the error in L_2 norm. The new method keeps the same order of convergence of the PPR

The values of $G_h u_h$ and $\overline{G}_h u_h$ together with the exact gradient over the entire domain are plotted in figure (3) for four linear elements. The nodal values of the recovered gradient obtained by the MPPR are better than those obtained by the PPR. It is apparent that the proposed approach performs better at the boundaries of the domain.

Example 2 (Mitchell, 2013): Consider the 1D linear differential equation

$$-\frac{d}{dx}\left(\frac{du}{dx}\right) = f;\ \ x \in (0,1),$$

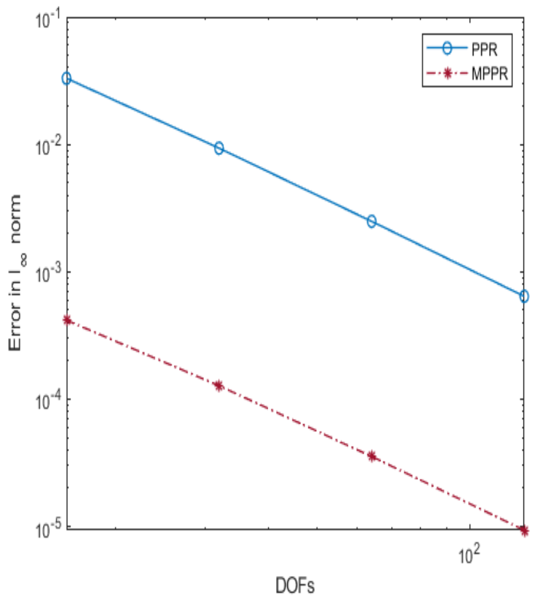

Figure 1: The L_∞ norm error between the exact gradient and the recovered gradients obtained by by the PPR and the MPRR for example (1).

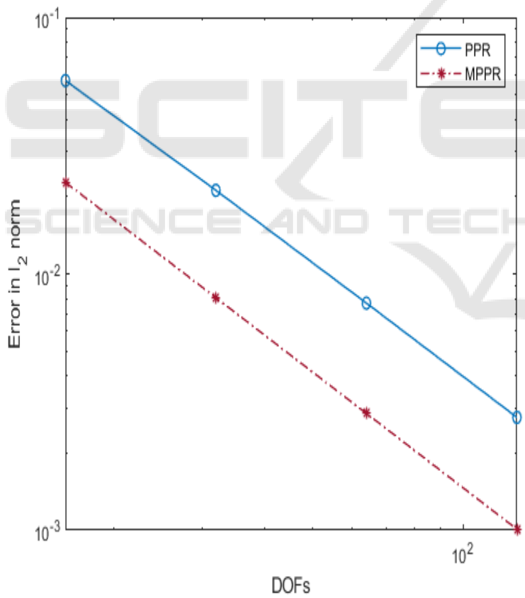

Figure 2: The L_2 norm error between the exact gradient and the recovered gradient obtained by the PPR and the MPRR for example (1).

with boundary conditions

$$u(0) = 0, \;\; u(1) = 0,$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$u(x) = 2^{2a} x^a (1-x)^a.$$

The given problem is well behaved with no singularities where PPR can perform well. For $a = 10$, figure (4) and figure (5) clarify that the MPPR method presents leading results to the PPR in both norms. Again, the same order of convergence of the PPR is preserved in the proposed technique.

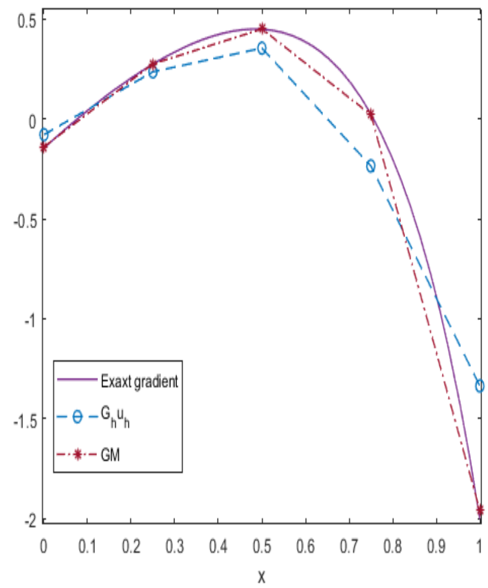

Figure 3: Distribution of the exact gradient and the recovered gradients $G_h u_h$ and $(\overline{G}_h u_h)$ of example (1).

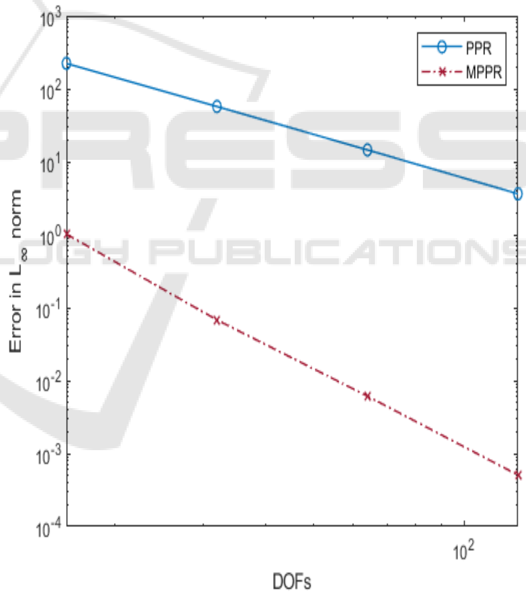

Figure 4: The L_∞ norm error between the exact gradient and the recovered gradients obtained by the PPR and the MPRR for example (2).

Figure (6) shows a comparison between the recovered gradients and the exact one over 16 linear elements. The accuracy of the gradient of the MPRR is higher than that of the PPR especially at the nodes.

Table (1) shows a comparison between the average computing time of both methods for examples (1) and (2). The MPPR takes slightly longer time to compute the gradient.

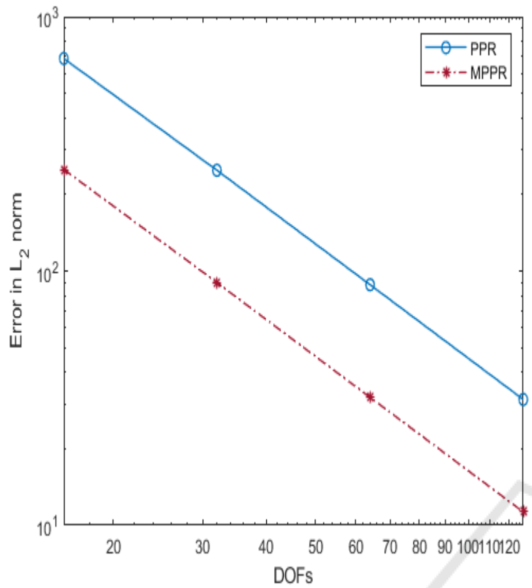

Figure 5: The L_2 norm error between the exact gradient and the recovered gradient obtained by the PPR and the MPRR for example (2).

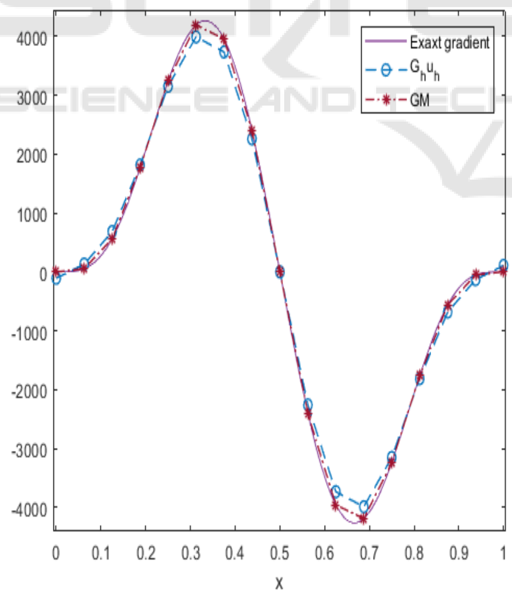

Figure 6: Distribution of the exact gradient and the recovered gradients $G_h u_h$ and $(\overline{G}_h u_h)$ of example (2).

The author in (Mitchell, 2013) made a collection of problems for testing adaptive refinement techniques. We utilize two of these problems to show the efficiency of the new operator as an error estimator in adaptive refinement algorithm.

Table 1: Comparison between average computing time of the PPR and the MPPR.

	DOFs	average time (sec.) PPR	MPPR
Example (1)	16	0.19	0.23
	32	0.38	0.45
	64	0.74	0.87
	128	1.43	1.69
Example (2)	16	0.19	0.22
	32	0.36	0.43
	64	0.73	0.88
	128	1.44	1.68

Example 3: Consider the 1D linear differential equation

$$-\frac{d}{dx}\left(\frac{du}{dx}\right) = f; \quad x \in (0,1),$$

with boundary conditions

$$u(0) = \exp(-\mu(x_c)^2), \quad u(1) = \exp(-\mu(1-x_c)^2),$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$u(x) = \exp(-\mu(x-x_c)^2),$$

where $0 < x_c < 1$.

This problem has an exponential peak inside the domain of its solution. The location of the peak is at $x = x_c$, and the strength of the peak depends on the parameter μ.

We use the L_2 norm error between the finite element gradient and the recovered gradient from the two methods as an error estimator. In each case, we evaluate the L_2 norm error between this recovered gradient and the exact gradient. Results for different values of μ and $x_c = 0.5$ are presented in Figure (7) and figure (8). It is apparent that the grids resulting when applying the two methods in adaptive refinement are very close. Results emphasize the superiority of the proposed technique over the PPR regarding the accuracy of the gradient.

Example 4: Consider the Helmholtz equation

$$-\frac{d}{dx}\left(\frac{du}{dx}\right) - \frac{1}{(\lambda+x)^4}u = f; \quad x \in (0,1),$$

with boundary conditions

$$u(0) = 0, \quad u(1) = \sin\left(\frac{1}{\lambda+1}\right),$$

where the function $f(x)$ is chosen so that the exact solution is given by

$$u(x) = sin\left(\frac{1}{\lambda+x}\right).$$

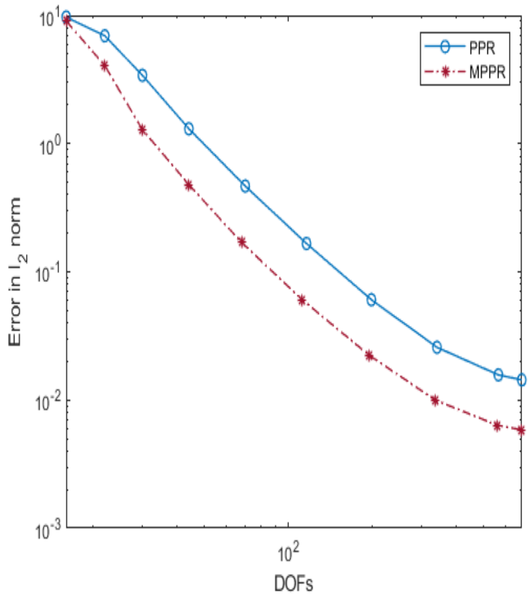

Figure 7: The L_2 norm error between the exact gradient and the recovered gradient obtained by the PPR and the MPRR for $\mu = 10^3$ for example (3).

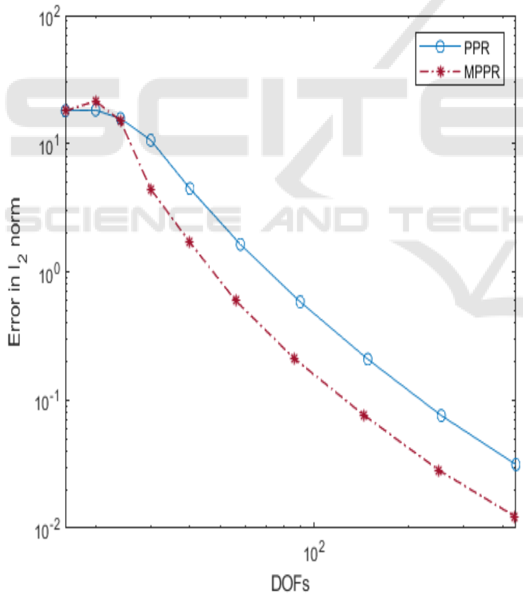

Figure 8: The L_2 norm error between the exact gradient and the recovered gradient obtained by the PPR and the MPRR for $\mu = 10^4$ for example (3).

This problem exhibits an oscillatory behaviour near the origin where the wavelength decreases. The number of the oscillations N depends on the parameter λ, where $\lambda = \frac{1}{N\pi}$.

The error between the exact gradient and the recovered gradients in the L_2 norm form is presented in figures (9) and (10) for $\lambda = \frac{1}{2\pi}, \frac{1}{4\pi}$ respectively. Again, we can see that the results obtained by the MPPR approach are preferable to those obtained by the classical PPR method.

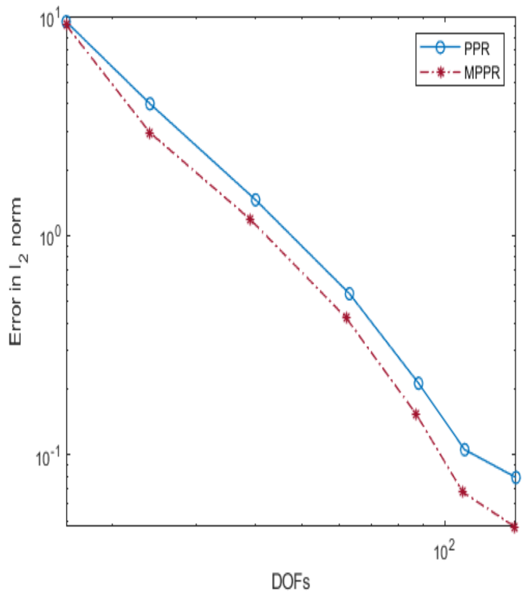

Figure 9: The L_2 norm error between the exact gradient and the recovered gradients obtained by the PPR and the MPRR for $\lambda = \frac{1}{2\pi}$ for example (4).

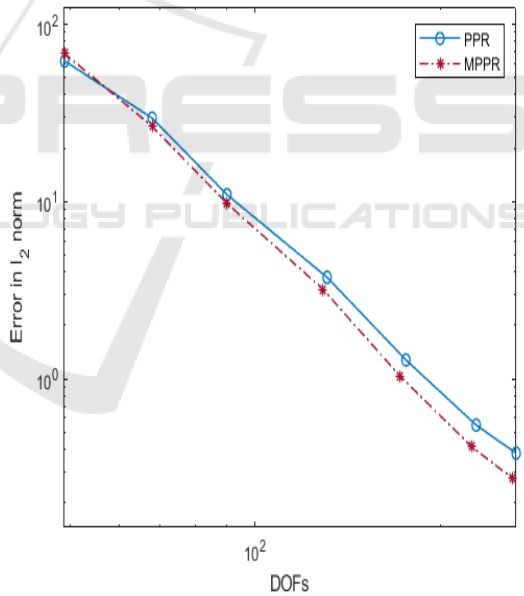

Figure 10: The L_2 norm error between the exact gradient and the recovered gradients obtained by the PPR and the MPRR for $\lambda = \frac{1}{4\pi}$ for example (4).

5 CONCLUSION

An modified PPR technique is presented. The main idea is to increase the order of the fitting polynomial without including other patches. To achieve that, sample points are substituted in the discretized form of the differential equation. The sample points are chosen to be the superconvergent points in the considered patch. The proposed technique benefits from the high order of the polynomial capturing oscillations and rapid changes in the solution. It also keeps the local behavior of the solution around the targeted node. The new operator of the proposed method is also used as an *a posteriori* error estimator in adaptive refinement.

Numerical results show that the accuracy of the new recovered gradient is higher than that obtained with the PPR and that the proposed method keeps the same order of convergence as the PPR.

Our future goals are to extend our method to the higher-dimensional cases and perform a convergence analysis of the new operator. We also aim to develop an approach for choosing the inner-elements points to achieve the best accuracy.

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