Nonlinear Complementarity Problems for n-Player Strategic Chance-constrained Games

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Abstract: In this paper, we focus on n-player strategic chance-constrained games where the payoff of each player follows either Cauchy or normal distribution. We transform the Nash equilibrium problem into its equivalent nonlinear complementarity problem (NCP) through the Karush-Kuhn-Tucker (KKT) conditions. Then, we prove the existence of the Nash equilibrium by the mean of Brouwer's fixed-point theorem. In order to show the efficiency of our approach, we perform numerical experiments on a set of randomly generated instances.

1 INTRODUCTION

Nash equilibrium is a crucial concept widely studied in game theory literature. John Von Neumann proved the existence of mixed strategy saddle point equilibrium for two-player finite zero-sum games (Neumann, 1928). John Nash extended this result to finite games with n players and deterministic payoffs (Nash et al., 1950).

In real-life problems, games input data might be affected by different uncertainty sources leading to numerous studies on games under uncertainty, namely stochastic games. The oligopoly market is a typical example where the payoff of each player is a random variable. Generally, the players in an oligopoly market are risk neutral. Therefore, they consider the expectation of random payoffs and constraints (De Wolf and Smeers, 1997; DeMiguel and Xu, 2009; Jadamba and Raciti, 2015; Ravat and Shanbhag, 2011; Valenzuela and Mazumdar, 2007; Xu and Zhang, 2013).

When the players are risk averse, chance-constrained games can be used efficiently (Charnes and Cooper, 1963; Cheng and Lisser, 2012; Prékopa, 2013). In (Singh et al., 2016a), the authors prove the existence of Nash equilibrium for an n-player finite strategic chance-constrained game under elliptical distributions. Furthermore, (Singh et al., 2016b) show that a Nash equilibrium problem for a two-player random bi-matrix game is equivalent to a linear complementarity problem (LCP, for short) when each player's payoff follows independent Cauchy distributions. When the player's payoffs are normally distributed, Nash equilibrium is equivalent to a nonlinear complementarity problem (NCP, for short). In (Singh and Lisser, 2018), the authors characterized the set of Nash equilibria of a chance-constrained game using the solution set of a variational inequality problem. In the case where the probability distributions are not known in advance, (Singh et al., 2017) studied distributionally robust chance-constrained games. Various approaches were considered in the literature for chance-constrained two-player stochastic zero-sum games (Blau, 1974; Cassidy et al., 1972; Charnes et al., 1968; Cheng et al., 2016; Song, 1992).

In addition, stochastic games with deterministic payoffs and chance-constrained strategies were also studied in the literature. For the case of two-player zero sum games, (Singh and Lisser, 2019) show that the saddle point is equivalent to a primal-dual pair of second order cone programs. As for the n-player general sum games with joint chance constraints, (Peng et al., 2018) show the existence of Nash equilibrium when the random linear constraints are independently normally distributed.

In this paper, we extend the two-player results in (Singh et al., 2016b) to the case of n-player stochastic games. We show that the Nash equilibrium problem

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can be reformulated as an NCP when the player's payoff follows either Logistic or Normal distributions. We also prove the existence of Nash equilibrium under different conditions using Brouwer's fixed-point theorem. As for the numerical experiments, we solve several randomly generated game instances to show the performance of our approaches. Unlike (Singh et al., 2016b), we solve instances where the size ranges from (2×2) to $(6 \times 6 \times 6 \times 6 \times 6 \times 6)$.

The chance-constrained game model can be applied to solve real-life problems, e.g., competition in electricity markets (Lee and Baldick, 2003) and decision-making for autonomous vehicles (Blackmore et al., 2006). When it comes to the electricity market, companies seek to maximize their profits by controlling prices or production quantities. As the reward function is random, we can model this situation by our chance-constrained game model to determine each company's Nash equilibrium strategy. In the decision-making process, autonomous vehicles seek to avoid potential collisions with obstacles while taking into account perceptional errors and environmental disturbances (Blackmore et al., 2006). In the case of multiple vehicles, the chance-constrained game theoretical framework can be used to model the vehicles decisions under uncertainty.

The remainder of this paper is organized as follows: Section 2 introduces our chance-constrained modelling framework. In Section 3, we prove the existence of Nash equilibrium by Brouwer's fixed-point theorem, and reformulate our stochastic chance-constrained games as an NCP. Section 4 is dedicated to numerical simulations. Finally, Section 5 concludes the paper.

2 CHANCE-CONSTRAINED GAMES

We consider an n-player chance-constrained finite strategic game with random payoffs. Let $I = \{1,2,3,\dots n\}$ be the set of players. $A_i, i \in I$ is the action set of player i with components a_i. The set of mixed strategies of player i includes all probability distributions over its action set, defined by the following k-simplex:

$$X_i = \{\tau_i \in \mathbb{R}^{k+1} | \sum_{j=1}^{k+1} \tau_{ij} = 1, \tau_{ij} \geq 0\}, \tag{1}$$

where τ_{ij} is the jth component of vector τ_i, $k = |A_i| - 1$ with $|A_i|$ the cardinality of the set A_i. Specifically, τ_{ij} is the probability for the player i to choose the jth action in A_i. Let $X = \prod_{i=1}^{n} X_i$ be the set of strategy profiles for all players with components $\tau \in X$. The pure strategy set of player i is defined by

$$Y_i = \{y_i \in X_i \mid \exists j \in \{1,2,..|A_i|\},\ s.t.\ y_{ij} = 1\}, \tag{2}$$

which is a subset of X_i. The set of pure strategy profiles for all players is defined by $Y = \prod_{i=1}^{n} Y_i$, with $y \in Y$ its element. In order to describe the strategy of one specific player in response to other players, we denote $X_{-i} = \prod_{j=1,j\neq i}^{n} X_j$ the strategy set of all players except player i, with components $\tau_{-i} \in X_{-i}$. Similarly, we denote $Y_{-i} = \prod_{j=1,j\neq i}^{n} Y_j$ where $y_{-i} \in Y_{-i}$ is the related generic element.

We assume that the pure strategy y based payoff of player i denoted by $r_i^{\omega}(y)$ is a random variable.

Given the payoff corresponding to each pure strategy, the payoff of player i for a mixed strategy $\tau \in X$ is a linear combination of pure-strategy payoffs, i.e.,

$$r_i^{\omega}(\tau) = \sum_{y \in Y} \prod_{k=1}^{n} \tau_{kj_{y_k}} r_i^{\omega}(y), \tag{3}$$

where j_{y_k} is the index of y_i's component.

In a chance-constrained game, the objective of each player is to maximize the expected payoff under a given level of confidence, i.e.,

$$u_i^{\alpha_i}(\tau) = \sup\{u | P(r_i^{w}(\tau) \geq u) \geq \alpha_i\}. \tag{4}$$

In the next section, we show the existence of Nash equilibrium for the chance-constrained games, and derive the NCP reformulations.

3 NCP FOR n-PLAYER CHANCE-CONSTRAINED GAME

In this section, we assume that the random payoffs of each player follow two probability distributions, namely Cauchy and Normal distributions. For each distribution, we derive a deterministic equivalent NCP.

3.1 Independent Cauchy Distributed Payoffs

We assume that the pure strategy payoffs for all players follow independent Cauchy distribution, i.e. $r_i^{\omega}(y) \sim C(\mu_i(y), \sigma_i(y))$ for all $y \in Y$. Then, for a mixed strategy $\tau \in X$, the payoff $r_i^{\omega}(\tau) = \sum_{y \in Y} \prod_{k=1}^{n} \tau_{kj_{y_k}} r_i^{\omega}(y)$ of player i is Cauchy distributed with $\mu_i(\tau) = \sum_{y \in Y} \prod_{k=1}^{n} \tau_{kj_{y_k}} \mu_i(y)$ and $\sigma_i(\tau) = \sum_{y \in Y} \prod_{k=1}^{n} \tau_{kj_{y_k}} \sigma_i(y)$. Therefore, $Z_i^C =$

$\frac{r_i^{\omega}-\mu_i(\tau)}{\sigma_i(\tau)}$ follows a standard Cauchy distribution $C(0,1)$. Let $F^{-1}_{Z_i^C}$ be the quantile function of the standard Cauchy distribution.

For each player i, the chance-constrained payoff with confidence level α_i is:

$$\begin{aligned}
u_i^{\alpha_i}(\tau) &= \sup\{u|P(r_i^w(\tau)\geq u)\geq \alpha_i\} \\
&= \sup\{u|P(\frac{r_i^w(\tau)-\mu_i(\tau)}{\sigma_i(\tau)} \geq \\
&\quad \frac{u-\sum_{y\in Y}\prod_{k=1}^n \tau_{kj_{y_k}}\mu_i(y)}{\sum_{y\in Y}\prod_{k=1}^n \tau_{kj_{y_k}}\sigma_i(y)})\geq \alpha_i\} \\
&= \sup\{u|F_{Z_i^C}(\frac{u-\sum_{y\in Y}\prod_{k=1}^n \tau_{kj_{y_k}}\mu_i(y)}{\sum_{y\in Y}\prod_{k=1}^n \tau_{kj_{y_k}}\sigma_i(y)}) \\
&\quad \leq 1-\alpha_i\} \\
&= \sum_{y\in Y}\prod_{k=1}^n \tau_{kj_{y_k}}\mu_i(y) \\
&\quad + F^{-1}_{Z_i^C}(1-\alpha_i)\sum_{y\in Y}\prod_{k=1}^n \tau_{kj_{y_k}}\sigma_i(y) \\
&= \sum_{y\in Y}\prod_{k=1}^n \tau_{kj_{y_k}}(\mu_i(y)+F^{-1}_{Z_i^C}(1-\alpha_i)\sigma_i(y)) \\
&= \sum_{y\in Y}\prod_{k=1}^n \tau_{kj_{y_k}}A_i(y) \\
&= V_i^T(\tau_{-i})\tau_i,
\end{aligned} \tag{5}$$

where $V_i(\tau_{-i})\in\mathbb{R}^{|A_i|}$.

$$V_i(\tau_{-i}) = \begin{pmatrix} \sum_{y_{-i}\in Y_{-i}} A_i(y_i^1,y_{-i})\prod_{k=1,\,k\neq i}^n \tau_{kj_{y_k}}) \\ \vdots \\ \sum_{y_{-i}\in Y_{-i}} A_i(y_i^m,y_{-i})\prod_{k=1,\,k\neq i}^n \tau_{kj_{y_k}}) \\ \vdots \\ \sum_{y_{-i}\in Y_{-i}} A_i(y_i^{|A_i|},y_{-i})\prod_{k=1,\,k\neq i}^n \tau_{kj_{y_k}}) \end{pmatrix}, \tag{6}$$

where $y_i^j\in\mathbb{R}^{|A_i|}$ is a unit vector with jth element equal to 1.

3.1.1 Existence of Nash Equilibrium

In the following, we prove the existence of Nash equilibrium for stochastic games with Cauchy distribution.

Theorem 1. *There always exists a Nash equilibrium for every n-player strategic chance-constrained game, where the payoff of each player is independently Cauchy distributed.*

The proof of this theorem is similar to the proof given in (Nash, 1951).

3.1.2 NCP Formulation

Given a strategy profile τ of all players, the chance-constrained payoff of player i is $u_i^{\alpha_i}(\tau)=V_i^T(\tau_{-i})\tau_i$. The best response of player i, given the strategy profile τ_{-i} for all other players, can be obtained by solving the following optimization problem:

$$\begin{aligned}
\max_{\tau_i}\ & V_i^T(\tau_{-i})\tau_i \\
s.t.\ & \sum_{j=1}^{|A_i|}\tau_{ij}=1, \\
& \tau_{ij}\geq 0, \quad \forall j\in\{1,2,...,|A_i|\}.
\end{aligned} \tag{7}$$

The objective function in (7) is concave subject to linear constraints. Hence, Slater's condition is satisfied and the KKT conditions are necessary and sufficient for optimality.

By KKT conditions, the best response of player i can be reformulated as follows:

$$\begin{aligned}
&0\leq \tau_i \ \perp\ -V_i-\lambda_1^i\mathbb{1}_{|e_i|}+\lambda_2^i\mathbb{1}_{|e_i|}\geq 0, \\
&0\leq \lambda_1^i \ \perp\ \sum_{j=1}^{|A_i|}\tau_{ij}-1\geq 0, \\
&0\leq \lambda_2^i \ \perp\ 1-\sum_{j=1}^{|A_i|}\tau_{ij}\geq 0,
\end{aligned} \tag{8}$$

where $\mathbb{1}_n$ denotes all-ones vector with size n.

Putting together the KKT conditions for all players, we obtain the Nash equilibrium of the chance-constrained game by solving the following NCP:

$$0\leq \zeta \ \perp\ G(\zeta)\geq 0, \tag{9}$$

where

$$\zeta=(\tau_1,\tau_2,...,\tau_n,\lambda_1^1,\lambda_2^1,...,\lambda_1^n,\lambda_2^n)\in\mathbb{R}^{\sum_{i=1}^n |A_i|+2n}, \tag{10}$$

and

$$G(\zeta)=\begin{pmatrix} -V_1-\lambda_1^1\mathbb{1}_{|A_1|}+\lambda_2^1\mathbb{1}_{|A_1|} \\ -V_2-\lambda_1^2\mathbb{1}_{|A_2|}+\lambda_2^2\mathbb{1}_{|A_2|} \\ \vdots \\ -V_n-\lambda_1^n\mathbb{1}_{|A_n|}+\lambda_2^n\mathbb{1}_{|A_n|} \\ \sum_{j=1}^{|A_1|}\tau_{1j}-1 \\ 1-\sum_{j=1}^{|A_1|}\tau_{1j} \\ \sum_{j=1}^{|A_2|}\tau_{2j}-1 \\ 1-\sum_{j=1}^{|A_2|}\tau_{2j} \\ \vdots \\ \sum_{j=1}^{|A_n|}\tau_{nj}-1 \\ 1-\sum_{j=1}^{|A_n|}\tau_{nj} \end{pmatrix}. \tag{11}$$

3.2 Independent Normally Distributed Payoffs

In the following, we consider normally distributed pure strategy payoffs for all the players. Thus, for a mixed strategy $\tau \in X$, the payoff $r_i^{\omega}(\tau) = \sum_{y\in Y}\prod_{k=1}^{n}\tau_{kj_{y_k}} r_i^{\omega}(y)$ of player i follows a normal distribution $N(\mu_i(\tau), \sigma_i^2(\tau))$ with $\mu_i(\tau) = \sum_{y\in Y}\prod_{k=1}^{n}\tau_{kj_{y_k}}\mu_i(y)$ and $\sigma_i^2(\tau) = \sum_{y\in Y}\prod_{k=1}^{n}\tau_{kj_{y_k}}^2\sigma_i^2(y)$.

Therefore, $Z_i^N = \frac{r_i^{\omega}-\mu_i(\tau)}{\sigma_i(\tau)}$ follows a standard normal distribution $N(0,1)$. Let $F_{Z_i^N}^{-1}$ be the quantile function of the standard normal distribution.

For each player i, the chance-constrained payoff with confidence level α_i is:

$$\begin{aligned}
u_i^{\alpha_i}(\tau) &= \sup\{u|P(r_i^w(\tau)\geq u)\geq \alpha_i\}\\
&= \sup\{u|P(\frac{r_i^w(\tau)-\mu_i(\tau)}{\sigma_i(\tau)} \geq \\
&\quad \frac{u-\sum_{y\in Y}\prod_{k=1}^{n}\tau_{kj_{y_k}}\mu_i(y)}{\sqrt{\sum_{y\in Y}\prod_{k=1}^{n}\tau_{kj_{y_k}}^2\sigma_i^2(y)}})\geq\alpha_i\}\\
&= \sup\{u|F_{Z_i^N}(\frac{u-\sum_{y\in Y}\prod_{k=1}^{n}\tau_{kj_{y_k}}\mu_i(y)}{\sqrt{\sum_{y\in Y}\prod_{k=1}^{n}\tau_{kj_{y_k}}^2\sigma_i^2(y)}})\\
&\quad \leq 1-\alpha_i\}\\
&= \sum_{y\in Y}\prod_{k=1}^{n}\tau_{kj_{y_k}}\mu_i(y)+\\
&\quad F_{Z_i^C}^{-1}(1-\alpha_i)\sqrt{\sum_{y\in Y}\prod_{k=1}^{n}\tau_{kj_{y_k}}^2\sigma_i^2(y)}\\
&= P_i^T(\tau_{-i})\tau_i + C_i\|Q_i^{\frac{1}{2}}(\tau_{-i})\tau_i\|,
\end{aligned} \tag{12}$$

where $C_i = F_{Z_i^C}^{-1}(1-\alpha_i)$ and $P_i(\tau_{-i}) \in \mathbb{R}^{|A_i|}$,

$$P_i(\tau_{-i}) = \begin{pmatrix} \sum_{y_{-i}\in Y_{-i}}(\mu_i(y_i^1,y_{-i})\prod_{k=1,\,k\neq i}^{n}\tau_{kj_{y_k}}) \\ \vdots \\ \sum_{y_{-i}\in Y_{-i}}(\mu_i(y_i^m,y_{-i})\prod_{k=1,\,k\neq i}^{n}\tau_{kj_{y_k}}) \\ \vdots \\ \sum_{y_{-i}\in Y_{-i}}(\mu_i(y_i^{|A_i|},y_{-i})\prod_{k=1,\,k\neq i}^{n}\tau_{kj_{y_k}}) \end{pmatrix}, \tag{13}$$

and $Q_i^{\frac{1}{2}}(\tau_{-i}) \in \mathcal{M}_{|A_i|\times|A_i|}$ is a diagonal matrix

$$Q_i^{\frac{1}{2}}(\tau_{-i}) = \begin{pmatrix} q_1^{\frac{1}{2}}(\tau_{-i}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & q_{|A_i|}^{\frac{1}{2}}(\tau_{-i}) \end{pmatrix}, \tag{14}$$

where $q_m^{\frac{1}{2}}(\tau_{-i}) = \sqrt{\sum_{y_{-i}\in Y_{-i}}(\sigma_i^2(y_i^m,y_{-i})\prod_{k=1,\,k\neq i}^{n}\tau_{kj_{y_i}}^2)}$.

3.2.1 Existence of Nash Equilibrium

Lemma 1. *If a function $f(x)$ is strictly concave and continuous on a compact convex set, then* $\arg\max_x f(x)$ *is a single-valued correspondence.*

Proof. A continuous function can always reach its maximum on a compact set. A strictly concave function on a convex set has no more than one maximum. Thus, the function f has one maximum, which implies that $\arg\max_x f(x)$ is a single-valued correspondence. □

Theorem 2. *Consider an n-player chance-constrained strategic game. If the pure strategy payoff of each player follows an independent normal distribution, then the Nash equilibrium exists for confidence level $\alpha \in [0.5,1)$.*

Proof. Firstly we construct a function

$$br_i(\tau) = \arg\max_{\tau_i^*}(u_i(\tau_i^*,\tau_{-i}) - ||\tau_i^*-\tau_i||). \tag{15}$$

For $\alpha\in[0.5,1)$, br_i is well-defined since $f_i(\tau_{-i}) = u_i(\tau_i^*,\tau_{-i}) - ||\tau_i^*-\tau_i||$ is a strictly concave function. Therefore $\arg\max_{\tau_{-i}} f_i(\tau_{-i})$ is a singleton by lemma 1. As $f_i(\tau_{-i})$ is continuous, we can prove that $br_i(\tau)$ is a continuous function by the Maximum theorem.

By concatenating br_i, we have

$$br(\tau) = \prod_{i=1}^{n} br_i(\tau_{-i}). \tag{16}$$

Since br is a continuous function from a convex compact subset to itself, according to Brouwer's fixed-point theorem, there exists a point τ^* where $\tau^* = br(\tau^*)$. Based on the definition of br, we can conclude that τ^* is a Nash equilibrium for this game. □

3.2.2 NCP Formulation

For a given strategy profile τ_{-i} for all other players and $\alpha\in[0.5,1)$, a best response strategy of player i can be obtained by solving the following optimization problem:

$$\begin{aligned}
\max_{\tau_i}\ & P_i^T(\tau_{-i})\tau_i + C_i\|Q_i^{\frac{1}{2}}(\tau_{-i})\tau_i\|\\
s.t.\ & \sum_{j=1}^{|A_i|}\tau_{ij} = 1,\\
& \tau_{ij}\geq 0,\quad \forall j\in\{1,2,...,|A_i|\}.
\end{aligned} \tag{17}$$

Here the objective function is concave and the constraints are linear, thus Slater's condition holds and the KKT conditions are both necessary and sufficient for optimality.

By KKT conditions, the best response of player i can be reformulated as follows

$$
\begin{aligned}
&0 \le \tau_i \;\perp\; -P_i(\tau_{-i}) - C_i \frac{Q_i(\tau_{-i})\tau_i}{\|Q_i^{\frac{1}{2}}(\tau_{-i})\tau_i\|} - \lambda_1^i \mathbb{1}_{|e_i|} \\
&\qquad\qquad + \lambda_2^i \mathbb{1}_{|e_i|} \ge 0, \\
&0 \le \lambda_1^i \;\perp\; \sum_{j=1}^{|A_i|} \tau_{ij} - 1 \ge 0, \\
&0 \le \lambda_2^i \;\perp\; 1 - \sum_{j=1}^{|A_i|} \tau_{ij} \ge 0.
\end{aligned}
\tag{18}
$$

Putting together the KKT conditions for all players, we can obtain the Nash equilibrium of the stochastic game by solving the following NCP:

$$0 \le \zeta \;\perp\; G(\zeta) \ge 0, \tag{19}$$

where

$$\zeta = (\tau_1, \tau_2, ..., \tau_n, \lambda_1^1, \lambda_2^1, ..., \lambda_1^n, \lambda_2^n) \in \mathbb{R}^{\sum_{i=1}{}^{n}|A_i| + 2n}, \tag{20}$$

and

$$
G(\zeta) = \begin{pmatrix}
-P_1(\tau_{-1}) - C_1 \frac{Q_1(\tau_{-1})\tau_1}{\|Q_1^{\frac{1}{2}}(\tau_{-1})\tau_1\|} - \lambda_1^1 \mathbb{1}_{|e_1|} + \lambda_2^1 \mathbb{1}_{|e_1|} \\
-P_2(\tau_{-2}) - C_2 \frac{Q_2(\tau_{-2})\tau_2}{\|Q_2^{\frac{1}{2}}(\tau_{-2})\tau_2\|} - \lambda_1^2 \mathbb{1}_{|e_2|} + \lambda_2^2 \mathbb{1}_{|e_2|} \\
\vdots \\
-P_n(\tau_{-n}) - C_n \frac{Q_n(\tau_{-n})\tau_n}{\|Q_n^{\frac{1}{2}}(\tau_{-n})\tau_n\|} - \lambda_1^n \mathbb{1}_{|e_n|} + \lambda_2^n \mathbb{1}_{|e_n|} \\
\sum_{j=1}^{|A_1|} \tau_{1j} - 1 \\
1 - \sum_{j=1}^{|A_1|} \tau_{1j} \\
\sum_{j=1}^{|A_2|} \tau_{2j} - 1 \\
1 - \sum_{j=1}^{|A_2|} \tau_{2j} \\
\vdots \\
\sum_{j=1}^{|A_n|} \tau_{nj} - 1 \\
1 - \sum_{j=1}^{|A_n|} \tau_{nj}
\end{pmatrix}. \tag{21}
$$

4 NUMERICAL EXPERIMENTS

In this section, we generate random instances in Matlab and we use PATH Solver to come up with Nash equilibrium.

PATH Solver is an implementation of a stabilized Newton method for the solution of the Mixed Complementarity Problem (MCP) (Dirkse and Ferris, 1995). For our concern, once the analytic form of $G(x)$ and its Jacobian is known and coded, we can directly use PATH to solve our NCP.

As a matter of illustration, we give two examples of $(3 \times 3 \times 3)$ random games with different distributions and then analyze the corresponding results.

4.1 $(3 \times 3 \times 3)$ Random Games with Cauchy Distribution

Three examples of randomly generated $(3 \times 3 \times 3)$ games, following independent Cauchy distribution $C_i(a) \sim (\mu(a), \sigma(a))$, are given below. The mean μ and deviation σ are uniformly generated between 1 and 3 as follows:

1.

$$\mu_1(:,:,1) = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix}, \mu_1(:,:,2) = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix},$$

$$\mu_1(:,:,3) = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix}$$

$$\sigma_1(:,:,1) = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{pmatrix}, \sigma_1(:,:,2) = \begin{pmatrix} 3 & 2 & 2 \\ 3 & 3 & 3 \\ 2 & 2 & 3 \end{pmatrix},$$

$$\sigma_1(:,:,3) = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 3 \\ 2 & 2 & 1 \end{pmatrix}$$

$$\mu_2(:,:,1) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & 3 \end{pmatrix}, \mu_2(:,:,2) = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix},$$

$$\mu_2(:,:,3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\sigma_2(:,:,1) = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \sigma_2(:,:,2) = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 2 & 1 \end{pmatrix},$$

$$\sigma_2(:,:,3) = \begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 1 \\ 3 & 2 & 3 \end{pmatrix}$$

$$\mu_3(:,:,1) = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \end{pmatrix}, \mu_3(:,:,2) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix},$$

$$\mu_3(:,:,3) = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

$$\sigma_3(:,:,1)=\begin{pmatrix}3&1&1\\2&3&2\\2&1&1\end{pmatrix},\ \sigma_3(:,:,2)=\begin{pmatrix}3&1&3\\3&2&2\\3&2&3\end{pmatrix},$$
$$\sigma_3(:,:,3)=\begin{pmatrix}2&1&2\\2&1&3\\3&1&3\end{pmatrix}$$

2.

$$\mu_1(:,:,1)=\begin{pmatrix}1&1&1\\1&1&2\\1&2&3\end{pmatrix},\ \mu_1(:,:,2)=\begin{pmatrix}2&1&3\\2&2&1\\2&3&1\end{pmatrix},$$
$$\mu_1(:,:,3)=\begin{pmatrix}2&1&1\\2&3&1\\2&1&1\end{pmatrix}$$

$$\sigma_1(:,:,1)=\begin{pmatrix}2&3&1\\2&3&3\\1&3&3\end{pmatrix},\ \sigma_1(:,:,2)=\begin{pmatrix}3&2&1\\3&1&1\\3&2&3\end{pmatrix},$$
$$\sigma_1(:,:,3)=\begin{pmatrix}1&2&3\\1&2&2\\1&1&3\end{pmatrix}$$

$$\mu_2(:,:,1)=\begin{pmatrix}2&1&3\\2&2&1\\1&1&3\end{pmatrix},\ \mu_2(:,:,2)=\begin{pmatrix}2&1&3\\3&2&3\\3&2&1\end{pmatrix},$$
$$\mu_2(:,:,3)=\begin{pmatrix}1&2&1\\2&2&3\\1&1&3\end{pmatrix}$$

$$\sigma_2(:,:,1)=\begin{pmatrix}2&3&1\\3&3&2\\1&2&1\end{pmatrix},\ \sigma_2(:,:,2)=\begin{pmatrix}1&1&2\\1&1&1\\3&1&3\end{pmatrix},$$
$$\sigma_2(:,:,3)=\begin{pmatrix}1&2&1\\1&1&1\\1&3&1\end{pmatrix}$$

$$\mu_3(:,:,1)=\begin{pmatrix}1&3&1\\1&2&3\\2&1&3\end{pmatrix},\ \mu_3(:,:,2)=\begin{pmatrix}2&2&1\\1&1&1\\3&3&3\end{pmatrix},$$
$$\mu_3(:,:,3)=\begin{pmatrix}2&3&3\\2&3&2\\3&1&2\end{pmatrix}$$

$$\sigma_3(:,:,1)=\begin{pmatrix}1&2&1\\3&3&2\\2&1&2\end{pmatrix},\ \sigma_3(:,:,2)=\begin{pmatrix}3&1&3\\3&1&3\\3&1&3\end{pmatrix},$$
$$\sigma_3(:,:,3)=\begin{pmatrix}1&1&2\\3&2&3\\1&1&2\end{pmatrix}$$

3.

$$\mu_1(:,:,1)=\begin{pmatrix}1&2&3\\1&2&2\\2&2&2\end{pmatrix},\ \mu_1(:,:,2)=\begin{pmatrix}2&1&2\\3&2&3\\3&2&1\end{pmatrix},$$
$$\mu_1(:,:,3)=\begin{pmatrix}1&2&2\\1&2&2\\3&2&2\end{pmatrix}$$

$$\sigma_1(:,:,1)=\begin{pmatrix}3&1&1\\2&1&2\\3&1&2\end{pmatrix},\ \sigma_1(:,:,2)=\begin{pmatrix}2&3&1\\3&3&1\\2&3&3\end{pmatrix},$$
$$\sigma_1(:,:,3)=\begin{pmatrix}1&3&3\\2&2&3\\2&2&2\end{pmatrix}$$

$$\mu_2(:,:,1)=\begin{pmatrix}2&1&1\\1&1&2\\2&3&3\end{pmatrix},\ \mu_2(:,:,2)=\begin{pmatrix}2&3&1\\3&3&1\\2&2&2\end{pmatrix},$$
$$\mu_2(:,:,3)=\begin{pmatrix}1&2&3\\1&3&2\\3&2&3\end{pmatrix}$$

$$\sigma_2(:,:,1)=\begin{pmatrix}3&3&3\\2&3&2\\1&1&3\end{pmatrix},\ \sigma_2(:,:,2)=\begin{pmatrix}3&3&2\\1&2&1\\3&3&1\end{pmatrix},$$
$$\sigma_2(:,:,3)=\begin{pmatrix}2&3&1\\3&1&1\\3&2&1\end{pmatrix}$$

$$\mu_3(:,:,1)=\begin{pmatrix}3&2&3\\2&1&2\\1&1&3\end{pmatrix},\ \mu_3(:,:,2)=\begin{pmatrix}3&3&3\\2&3&2\\3&1&3\end{pmatrix},$$
$$\mu_3(:,:,3)=\begin{pmatrix}2&1&2\\2&2&3\\3&3&3\end{pmatrix}$$

$$\sigma_3(:,:,1)=\begin{pmatrix}2&1&3\\1&2&3\\2&2&3\end{pmatrix},\ \sigma_3(:,:,2)=\begin{pmatrix}3&2&3\\2&1&2\\3&1&3\end{pmatrix},$$
$$\sigma_3(:,:,3)=\begin{pmatrix}1&1&1\\2&2&2\\2&2&1\end{pmatrix}$$

For the above randomly generated examples, the mean and the deviation of each player's payoff use $(3\times3\times3)$ tensors since there are 3 players in the game and each player has 3 actions to choose. For instance, if each player chooses the first action as their strategy, then the payoff for player 1 follows a Cauchy distribution with mean parameter $\mu = 1$ and scale parameter $\sigma = 1$. Table 1 summarizes the Nash equilibrium of the three examples for different confidence levels α. Column 1 presents the index of examples. Columns 2-4 contain the different confidence levels α for the chance-constrained game. The Nash equilibrium of the game is given in Columns 5-7.

4.2 $(3 \times 3 \times 3)$ Random Games with Normal Distribution

Similarly, three instances of randomly generated $(3 \times 3 \times 3)$ games, following independent normal distribution $N_i(a) \sim (\mu(a), \sigma^2(a))$, are given below. The mean μ and deviation σ are uniformly generated between 1 and 3 as follows:

1.

$$\mu_1(:,:,1) = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 2 \\ 3 & 2 & 3 \end{pmatrix}, \mu_1(:,:,2) = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 2 \\ 3 & 2 & 3 \end{pmatrix},$$
$$\mu_1(:,:,3) = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{pmatrix}$$

$$\sigma_1^2(:,:,1) = \begin{pmatrix} 9 & 1 & 1 \\ 4 & 9 & 1 \\ 4 & 1 & 4 \end{pmatrix}, \sigma_1^2(:,:,2) = \begin{pmatrix} 4 & 1 & 1 \\ 9 & 9 & 1 \\ 9 & 1 & 1 \end{pmatrix},$$
$$\sigma_1^2(:,:,3) = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 1 & 9 & 1 \end{pmatrix}$$

$$\mu_2(:,:,1) = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \mu_2(:,:,2) = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix},$$
$$\mu_2(:,:,3) = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$

$$\sigma_2^2(:,:,1) = \begin{pmatrix} 4 & 4 & 1 \\ 1 & 9 & 4 \\ 1 & 4 & 9 \end{pmatrix}, \sigma_2^2(:,:,2) = \begin{pmatrix} 9 & 4 & 4 \\ 1 & 4 & 9 \\ 4 & 1 & 1 \end{pmatrix},$$
$$\sigma_2^2(:,:,3) = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 1 \\ 9 & 1 & 9 \end{pmatrix}$$

$$\mu_3(:,:,1) = \begin{pmatrix} 3 & 3 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 2 \end{pmatrix}, \mu_3(:,:,2) = \begin{pmatrix} 3 & 3 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix},$$
$$\mu_3(:,:,3) = \begin{pmatrix} 3 & 3 & 2 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\sigma_3^2(:,:,1) = \begin{pmatrix} 1 & 9 & 9 \\ 9 & 9 & 1 \\ 9 & 4 & 1 \end{pmatrix}, \sigma_3^2(:,:,2) = \begin{pmatrix} 4 & 9 & 1 \\ 1 & 4 & 1 \\ 1 & 9 & 9 \end{pmatrix},$$
$$\sigma_3^2(:,:,3) = \begin{pmatrix} 9 & 1 & 9 \\ 1 & 4 & 9 \\ 4 & 4 & 9 \end{pmatrix}$$

2.

$$\mu_1(:,:,1) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 3 & 2 & 3 \end{pmatrix}, \mu_1(:,:,2) = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{pmatrix},$$
$$\mu_1(:,:,3) = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\sigma_1^2(:,:,1) = \begin{pmatrix} 4 & 9 & 4 \\ 1 & 1 & 4 \\ 9 & 9 & 9 \end{pmatrix}, \sigma_1^2(:,:,2) = \begin{pmatrix} 9 & 1 & 9 \\ 4 & 9 & 9 \\ 1 & 9 & 9 \end{pmatrix},$$
$$\sigma_1^2(:,:,3) = \begin{pmatrix} 4 & 4 & 9 \\ 4 & 1 & 4 \\ 1 & 4 & 4 \end{pmatrix}$$

$$\mu_2(:,:,1) = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix}, \mu_2(:,:,2) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix},$$
$$\mu_2(:,:,3) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}$$

$$\sigma_2^2(:,:,1) = \begin{pmatrix} 4 & 9 & 4 \\ 9 & 4 & 4 \\ 4 & 4 & 9 \end{pmatrix}, \sigma_2^2(:,:,2) = \begin{pmatrix} 4 & 4 & 9 \\ 1 & 4 & 9 \\ 9 & 9 & 1 \end{pmatrix},$$
$$\sigma_2^2(:,:,3) = \begin{pmatrix} 4 & 9 & 4 \\ 4 & 1 & 1 \\ 9 & 9 & 4 \end{pmatrix}$$

$$\mu_3(:,:,1) = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 3 & 3 & 2 \end{pmatrix}, \mu_3(:,:,2) = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$
$$\mu_3(:,:,3) = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

Table 1: Nash equilibrium for various values of α for Cauchy distribution.

No.	α			Nash Equilibrium		
	α_1	α_2	α_3	x^*	y^*	z^*
1	0.4	0.4	0.4	(0, 1, 0)	(1, 0, 0)	(0.667, 0, 0.333)
	0.5	0.5	0.5	(0, 1, 0)	(0, 1, 0)	(1, 0, 0)
	0.7	0.7	0.7	(0, 1, 0)	(0, 1, 0)	(0, 0, 1)
2	0.4	0.4	0.4	(0.442, 0, 0.558)	(0, 0, 1)	(0, 0, 1)
	0.5	0.5	0.5	(0, 0, 1)	(1, 0, 0)	(0, 1, 0)
	0.7	0.7	0.7	(0, 0, 1)	(0, 0, 1)	(1, 0, 0)
3	0.4	0.4	0.4	(0, 0.775, 0.225)	(0, 1, 0)	(0, 1, 0)
	0.5	0.5	0.5	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)
	0.7	0.7	0.7	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)

$$\sigma_3^2(:,:,1)=\begin{pmatrix}1&9&4\\4&1&1\\1&1&1\end{pmatrix},\ \sigma_3^2(:,:,2)=\begin{pmatrix}9&9&9\\9&4&9\\9&1&4\end{pmatrix},$$
$$\sigma_3^2(:,:,3)=\begin{pmatrix}1&4&1\\1&9&9\\9&4&1\end{pmatrix}$$

3.

$$\mu_1(:,:,1)=\begin{pmatrix}2&3&1\\2&3&3\\2&1&2\end{pmatrix},\ \mu_1(:,:,2)=\begin{pmatrix}2&2&2\\2&3&1\\2&2&2\end{pmatrix},$$
$$\mu_1(:,:,3)=\begin{pmatrix}2&1&1\\1&1&2\\3&3&1\end{pmatrix}$$

$$\sigma_1^2(:,:,1)=\begin{pmatrix}1&4&1\\9&1&1\\1&1&4\end{pmatrix},\ \sigma_1^2(:,:,2)=\begin{pmatrix}4&9&4\\4&4&1\\9&4&4\end{pmatrix},$$
$$\sigma_1^2(:,:,3)=\begin{pmatrix}9&9&9\\9&1&4\\1&1&1\end{pmatrix}$$

$$\mu_2(:,:,1)=\begin{pmatrix}3&2&2\\3&3&1\\3&1&2\end{pmatrix},\ \mu_2(:,:,2)=\begin{pmatrix}1&1&1\\3&1&1\\1&1&1\end{pmatrix},$$
$$\mu_2(:,:,3)=\begin{pmatrix}1&2&1\\1&1&3\\1&2&1\end{pmatrix}$$

$$\sigma_1^2(:,:,1)=\begin{pmatrix}1&4&1\\9&1&1\\1&1&4\end{pmatrix},\ \sigma_1^2(:,:,2)=\begin{pmatrix}4&9&4\\4&4&1\\9&4&4\end{pmatrix},$$
$$\sigma_1^2(:,:,3)=\begin{pmatrix}9&9&9\\9&1&4\\1&1&1\end{pmatrix}$$

$$\mu_3(:,:,1)=\begin{pmatrix}3&3&2\\1&3&3\\1&1&2\end{pmatrix},\ \mu_3(:,:,2)=\begin{pmatrix}2&3&3\\3&3&1\\2&1&3\end{pmatrix},$$
$$\mu_3(:,:,3)=\begin{pmatrix}2&3&3\\2&2&3\\3&3&2\end{pmatrix}$$

$$\sigma_3^2(:,:,1)=\begin{pmatrix}1&1&9\\4&4&9\\9&1&1\end{pmatrix},\ \sigma_3^2(:,:,2)=\begin{pmatrix}9&9&1\\1&4&1\\1&9&1\end{pmatrix},$$
$$\sigma_3^2(:,:,3)=\begin{pmatrix}4&4&9\\9&4&4\\4&1&1\end{pmatrix}$$

For the above randomly generated examples, the mean and the deviation of each player's payoff use $(3\times3\times3)$ tensors. Considering the first example, if each player chooses the first action as their strategy, the payoff for player 1 follows a normal distribution with mean parameter $\mu=3$ and variance parameter $\sigma^2=9$. Table 2 summarizes the Nash equilibrium in the same way as Table 1. Column 1 presents the index of examples. Columns 2-4 show the different confidence levels α for the chance-constrained game. The Nash equilibrium of the game is given in Columns 5-7.

Table 2: Nash equilibrium for various values of α for normal distribution.

No.	α			Nash Equilibrium		
	α_1	α_2	α_3	x^*	y^*	z^*
1	0.6	0.6	0.6	(0, 0, 1)	(0, 0.187, 0.813)	(0.092, 0.909, 0)
	0.7	0.7	0.7	(0, 0, 1)	(0, 0, 1)	(0.673, 0.327, 0)
	0.8	0.8	0.8	(0, 0, 1)	(0.194, 0.229, 0.577)	(0.764, 0.236, 0)
2	0.6	0.6	0.6	(0.623, 0, 0.377)	(1, 0, 0)	(0.424, 0.576, 0)
	0.7	0.7	0.7	(0.851, 0.149, 0)	(0.395, 0.386, 0.219)	(0, 1, 0)
	0.8	0.8	0.8	(0.067, 0.861, 0.069)	(0.322, 0.678, 0)	(0.771, 0.229, 0)
3	0.6	0.6	0.6	(0, 0.457, 0.543)	(0, 0.291, 0.71)	(0.058, 0 , 0.942)
	0.7	0.7	0.7	(0, 0.626, 0.374)	(0.072, 0.206, 0.722)	(0.264, 0 , 0.736)
	0.8	0.8	0.8	(0, 0, 1)	(0, 1, 0)	(0, 0, 1)

Table 3: Comparison of success rate and running time.

Game type	Cauchy distribution			Normal distribution		
	α	success rate	average time(s)	α	success rate	average time(s)
2×2	0.2	100%	0.0128	0.6	99%	0.0389
	0.4	100%	0.0122	0.7	95%	0.0449
	0.6	100%	0.0107	0.8	90%	0.0487
	0.8	100%	0.0098	0.9	86%	0.0509
$3 \times 3 \times 3$	0.2	100%	0.0463	0.6	98%	0.2198
	0.4	100%	0.0378	0.7	92%	0.2419
	0.6	100%	0.0420	0.8	86%	0.3398
	0.8	100%	0.0337	0.9	83%	0.3264
$4 \times 4 \times 4 \times 4$	0.2	100%	1.1165	0.6	96%	4.5894
	0.4	100%	1.0237	0.7	90%	5.4259
	0.6	100%	0.8908	0.8	87%	7.4005
	0.8	100%	0.7670	0.9	86%	6.5441
$5 \times 5 \times 5 \times 5 \times 5$	0.2	81%	39.8803	0.6	64%	144.8211
	0.4	80%	48.8849	0.7	52%	196.6912
	0.6	89%	44.9324	0.8	67%	201.4756
	0.8	94%	36.1606	0.9	86%	126.2685

4.3 Numerical Results for Large Size Game Instances

Here we solve large size instances with up to $(5 \times 5 \times 5 \times 5 \times 5)$ games.

The NCPs are implemented in Matlab and solved by PATH on Intel Core i72,6 GHz with 32GB RAM. We randomly generated 100 tests of several groups of different game sizes and confidence levels for both distributions, then we computed the average running time and the success rate in relation of solved instances by PATH solver. Table 3 summarizes the numerical results for different sizes of the chance-constrained games. Column 1 presents the size of the game instances. Columns 2-4 show the confidence level α, success rate and average CPU time for problems under Cauchy distribution, respectively. Columns 5-7 provide the same information as Columns 2-4 for problems under normal distribution.

As shown in Table 3, the average CPU time for the instances up to $(4\times4\times4\times4)$ is within 1 second, whilst $(5\times5\times5\times5)$ instances are solved within 49 seconds. Games with Cauchy distributions have 100% success rates for all the instances except $(5\times5\times5\times5)$ instances where the success rates range from 81% up to 94%. As for normal distribution games, the success rate ranges from 52% for the large instance to 99% for the smallest instances. In addition, we also solve game instances with size $(6\times6\times6\times6\times6\times6)$ within 30 minutes. PATH failed to solve game instances with more than $(6\times6\times6\times6\times6\times6)$.

5 CONCLUSION

In this paper, we solved the Nash equilibrium problem with n-player chance-constrained games. We proved the existence of Nash Equilibrium for stochastic games with Cauchy and normal distributions. We derive a deterministic equivalent NCP for these games.

In order to show the performances of our approaches, we generated random instances and used the PATH solver to solve the related NCPs.

For future work, we will consider different distributions for the addressed stochastic games and apply our approach to real-life applications, e.g., autonomous vehicles.

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