Queueing Model of Circular Demand Responsive Transportation System: Theoretical Solution and Heuristic Solution

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Abstract: Sharing mobilities, such as car-sharing and ride-sharing, have been widely spreading recently. In this paper, we consider Car/Ride-Share (CRS) system, which is one of the demand responsive transportations. We consider the scenario where CRS is introduced on the circular bus route. We propose its theoretical and heuristic solutions using queueing theory. We validate the heuristic analysis by comparing with the theoretical result through some numerical examples.

1 INTRODUCTION

Sharing mobilities, such as car-sharing and ride-sharing have been widely spreading recently. Car-sharing is a system such as car-rental for short period time and ride-sharing is a system where people ride cars to their destinations together, e.g., Uber. In this paper, we study Car/Ride-share (CRS) system (Ando et al., 2019), where people carry out car-sharing and ride-sharing simultaneously. We analyzed a queueing model where CRS is introduced between two points (Nakamura et al., 2020a). As an extension, we model and analyze the scenario of the introduction of CRS on a circular bus route. As we will describe later, the theoretical analysis of the multiple points model requires a huge amount of computaiton and memory capacity when conducting numerical experiments. Therefore, we propose a heuristic solution of the model to facilitate the numerical experiments. We compare these two solutions, the theoretical and the heuristic solutions, and discuss the validity of the heuristic solution.

There are various studies about sharing mobilities. Specifically, , various studies using optimization method were conducted, e.g., (Agatz et al., 2012), (Correia and Antunes, 2012). However, almost all of these studies ignore the uncertainty of customer behaviors. In other words, these studies assume that all customer behavior is perfectly understood. In reality, customer behavior has fluctuation. Naturally that sometimes there are many customers and sometimes there are few because of external factors such as other transportations, the road congestion, weather, and so on. A closely related study (Enzi et al., 2021), which deals with the scheduling problem of multimodal car- and ride- sharing problem, also ignores the uncertainty of customers.

As research considering the randomness of customers, Daganzo et al. (Daganzo and Ouyang, 2019) discussed a simple stochastic model of demand-responsive transportations that include ride-sharing. However, they did not consider the coexistence of multiple transportations, e.g., buses and ride-sharing. It is crucial to discuss the coexistence of various mobilities and evaluate the impact on each other from a practical point of view. Research of queueing models for transportation services (e.g., (Tao and Pender, 2020)) also did not consider the coexistence of various types of mobilities.

Based on the above, the novelties of this paper can be summarized as follows:

- Modeling transportation systems considering the uncertainty of customers.
- Modeling the coexistence of multiple transportations, i.e., Car/Ride-Share.
- Presenting both theoretical and heuristic solutions for this new model.

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The rest of the paper is organized as follows. In Section 2, we describe the model based on queueing theory. Sections 3 and 4 present theoretical and heuristic solutions. Section 5 shows some numerical results. The final section of the paper, Section 6, presents a discussion and concluding remarks.

2 MODELING

This section presents the modeling of our scenario for CRS system on a circular bus route.

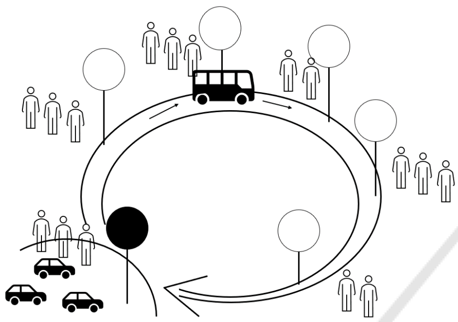

Figure 1: Schematic illustration of the modelling.

We consider a circular bus route as Figure 1. We assume that there is a car parking lot for CRS at one of the bus stops (the bus stop is painted black in Figure 1). In the CRS system, car providers provide their private cars for financial incentives (Nakamura et al., 2020a). This mechanism refers to the concept of peer-to-peer car-sharing, where people lend their private cars. We discussed the beneficial price mechanism for the car providers between two points (Nakamura et al., 2020b). In this paper, we focus on the model extended to capture the travel among multiple points. Therefore, as the first step in the research of the multiple points model, we do not consider detailed financial discussion for system participants such as car providers in this research. Of course, it is vital to discuss its financial perspective to aim for practical use.

On the circular bus route, buses are operated based on a fixed schedule. This model assumes that a bus arrives according to the interval that follows an Erlang distribution with rate r and shape q (this means the sum of q exponentially distributed random variables of parameter r). Erlang distribution can approximate a constant value, i.e., this assumption is suitable for modeling the fixed bus schedule. Erlang distribution can be constructed by a convolution of exponential distributions. Therefore, this approximation makes the model easier to be analyzed as a Markov chain.

Now, we define the concept of "class" of the customers. A class means the combination of the bus stop where the customers first arrive, and the bus stop that is the final destination of the customer. For example, class-AB is for the customers who wait for the bus coming at the bus stop A, and wants to visit bus stop B finally. Besides, we assume that customers of a class arrives at a bus stop according to a Poisson process with a distinct parameter for that class, e.g., the customers of class- AB arrive according to the Poisson Process of rate λ_{AB}. We also assume the customers ride on the buses within the designated number of people for each class, e.g., the customers of class-AB get on a bus up to X_{AB} (which follows an arbitrary distribution) customers.

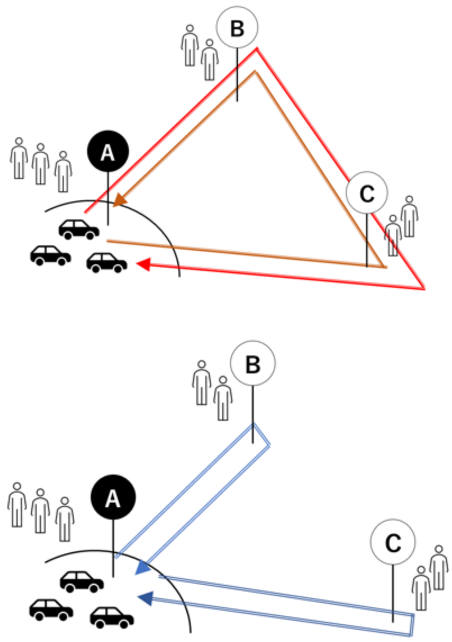

Figure 2: The routes of CRS when $N = 3$.

We consider the scenario of the introduction of CRS on this circular bus route aiming to reduce congestion for the customers. The specific description of the CRS system is as follows:

- Car providers provide their private cars for the parking lot to gain financial incentives, i.e., car-sharing.
- Customers can ride the cars with other customers who have the same direction, i.e., ride-sharing.

From these explanations, we can understand that CRS is a hybrid system of car-sharing and ride-sharing. What is very different from buses is that CRS is demand-responsive type transportation, i.e., CRS is a system where a car does not depart unless more than a certain number of people want to use it. Besides, cars of CRS should be returned to their original position (i.e., the parking lot) after the CRS travel. Therefore, it is necessary to match the trips that enable the car to finally return to the parking lot. This constraint seems to be troublesome at first glance. However, it would reduce the cost of relocating the cars by the operator in traditional car-sharing systems. In our scenario, we assume that the operator of CRS (that can be an operating company dedicated to CRS, and the bus company itself) controls the occurrences of CRS under a particular policy. As a simple policy, we assume the following:

- The minimum and the maximum capacities of the car are m and n customers, respectively.
- We ignore the lack of cars and whether customers have a driver license or not to simplify the model.
- We define the "route" as the course of CRS that starts and ends with the bus stop where there is the parking lot.
- For example, we assume that there are three bus stops (A, B, and C) on circular, and there is the parking lot of CRS at the bus stop-A (see Figure 2). Note that we call a bus stop "node" to the simplified expression and define N as the number of nodes afterward. The possibilities of the routes are as follows:
  - $A \to B \to C \to A$
  - $A \to C \to B \to A$
  - $A \to B \to A$
  - $A \to C \to A$
- Supplementary definitions of the route are as follows: it is impossible to visit the same node more than once except for the bus stop where there is the parking lot, e.g., $A \to B \to C \to B \to A$ is impossible in the setting of Figure 2.
- If there are m or more customers of all classes that composes a route, CRS of its route occurs according to the interval which follows the exponential distribution of parameters given for each route. For example, this is because it takes time for matching of customers by the operator of CRS through the control system, e.g., web application on smartphone. At the moment of the completion of the matching, maximum number of customers not exceeding n (the maximum capacity of the car) ride on the car and start to drive. Note that we do not consider who drives the car in this paper.
- Considering the example in Figure 2, CRS of route $A \to B \to C \to A$ occurs according to the interval which follows the exponential distribution with rate σ_{ABC}, on the condition that the number of all class-AB, class-BC and class-CA is m or more people. In other words, the customers of class-AB start driving from node-A to node-B, and they get off the car at the node-B. The customers of class-BC take over the car at node-B and go to node-C. After that, the customers of class-CA start to drive to node-A (the parking lot). By this series of flow, the car is returned to its original position. The same is true for other routes.
- Supplementary explanation about the car takeover; customers of different classes from same node would not ride the same car, e.g., class-AB and class-AC can not be in same car.

In the next section, we show the theoretical solution of the scenario using queueing theory.

3 THEORETICAL ANALYSIS

In this section, we show the theoretical analysis of the model. For simplicity, we describe the example analysis where the number of nodes (N) is 3, as in the previous section. Then, we summarize the parameters and the variables used in the analysis in Table 1.

Table 1: Parameters used in the analysis.

Parameters	Definitions
N	The number of nodes.
λ_G	The arrival rate for the customers of class-G.
m	The minimum capacity of CRS cars.
n	The maximum capacity of CRS cars.
X_G	The random variable of the number of customers who get into a bus in class-G.
q_G, r_G	Parameters of Erlang distribution of the interval of buses at node-G.
σ_G	The occurrence rate of CRS for route-G.
K_G	The capacity for the customers in node-G.

First of all, we define some sets as follows: $S_{AB} = \{0,1,2,\ldots,K_{AB}-1,K_{AB}\}$, $S_{AC} = \{0,1,2,\ldots,K_{AC}-1,K_{AC}\}$, $S_{BA} = \{0,1,2,\ldots,K_{BA}-1,K_{BA}\}$, $S_{BC} = \{0,1,2,\ldots,K_{BC}-1,K_{BC}\}$, $S_{CA} = \{0,1,2,\ldots,K_{CA}-1,K_{CA}\}$, $S_{CB} = \{0,1,2,\ldots,K_{CB}-1,K_{CB}\}$, $R_A = \{0,1,2,\ldots,r_A-2,r_A-1\}$, $R_B = \{0,1,2,\ldots,r_B-2,r_B-1\}$, $R_C = \{0,1,2,\ldots,r_C-2,r_C-1\}$, and $Z = S_{AB} \times S_{AC} \times S_{BA} \times S_{BC} \times S_{CA} \times S_{CB} \times R_A \times R_B \times R_C$.

Let $S_{AB}(t)$, $S_{AC}(t)$, $S_{BA}(t)$, $S_{BC}(t)$, $S_{CA}(t)$ and $S_{CB}(t)$ denote the number of the waiting customers for each class at time t, respectively, and $R_A(t)$, $R_B(t)$ and $R_C(t)$ denote the progress of the Erlang distributions for buses at each node at time t, respectively. It is easy to find that $\{(S_{AB}(t), S_{AC}(t), S_{BA}(t), S_{BC}(t), S_{CA}(t), S_{CB}(t), R_A(t), R_B(t), R_C(t)); t \geq 0\}$ forms a Markov chain in the state space Z.

Because the Markov chain is finite and irreducible, we define the steady state probabilities as follows:

$$\pi_{(j_{AB}, j_{AC}, j_{BA}, j_{BC}, j_{CA}, j_{CB}, k_A, k_B, k_C)} = \lim_{t \to \infty} \mathrm{P}(S_{AB}(t) = j_{AB}, S_{AC}(t) = j_{AC}, S_{BA}(t) = j_{BA}, S_{BC}(t) = j_{BC}, S_{CA}(t) = j_{CA}, S_{CB}(t) = j_{CB}, R_A(t) = k_A, R_B(t) = k_B, R_C(t) = k_C),$$

where $i_{AB} \in S_{AB}$, $i_{AC} \in S_{AC}$, $i_{BA} \in S_{BA}$, $i_{BC} \in S_{BC}$, $i_{CA} \in S_{CA}$, $i_{CB} \in S_{CB}$, $k_A \in R_A$, $k_B \in R_B$, $k_C \in R_C$.

Sorting the all the states in Z in the lexicographic order, the infinitesimal generator $\boldsymbol{Q} =$ (size: $((K_{AB}+1)(K_{AC}+1)(K_{BA}+1)(K_{BC}+1)(K_{CA}+1)(K_{CB}+1)r_A r_B r_C)$) of our Markov chain is given as follows:

$$\boldsymbol{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ \widehat{Y} \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \dots & \widehat{Y} \end{array} \\ \begin{pmatrix} Q_{\boldsymbol{e}_0,\boldsymbol{e}_0} & Q_{\boldsymbol{e}_0,\boldsymbol{e}_1} & Q_{\boldsymbol{e}_0,\boldsymbol{e}_2} & \ddots & Q_{\boldsymbol{e}_0,\boldsymbol{y}} \\ Q_{\boldsymbol{e}_1,\boldsymbol{e}_0} & Q_{\boldsymbol{e}_1,\boldsymbol{e}_1} & Q_{\boldsymbol{e}_1,\boldsymbol{e}_2} & \ddots & Q_{\boldsymbol{e}_1,\boldsymbol{y}} \\ Q_{\boldsymbol{e}_2,\boldsymbol{e}_0} & Q_{\boldsymbol{e}_2,\boldsymbol{e}_1} & Q_{\boldsymbol{e}_2,\boldsymbol{e}_2} & \ddots & Q_{\boldsymbol{e}_2,\boldsymbol{y}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ Q_{\boldsymbol{y},\boldsymbol{e}_0} & Q_{\boldsymbol{y},\boldsymbol{e}_1} & Q_{\boldsymbol{y},\boldsymbol{e}_2} & \ddots & Q_{\boldsymbol{y},\boldsymbol{y}} \end{pmatrix} \end{array},$$

where $\widehat{Y} = ((K_{AB}+1)(K_{AC}+1)(K_{BA}+1)(K_{BC}+1)(K_{CA}+1)(K_{CB}+1)r_A r_B r_C) - 1$, $\boldsymbol{y} = (K_{AB}, K_{AC}, K_{BA}, K_{BC}, K_{CA}, K_{CB}, r_A - 1, r_B - 1, r_C - 1)$, $\boldsymbol{e}_0 = (0,0,0,0,0,0,0,0,0)$, $\boldsymbol{e}_1 = (0,0,0,0,0,0,0,0,1)$, $\boldsymbol{e}_2 = (0,0,0,0,0,0,0,0,2)$.

Moreover, we also define the elements of $\boldsymbol{Q}$ as follows (for subscripts, we write only where there is a main change in transition):
$Q_{(\dots,k_A,k_B,k_C),(\dots,k_A+1,k_B,k_C)} = q_A, \quad 0 \leqq k_A \leqq r_A - 2,$
$Q_{(\dots,k_A,k_B,k_C),(\dots,k_A,k_B+1,k_C)} = q_B, \quad 0 \leqq k_B \leqq r_B - 2,$
$Q_{(\dots,k_A,k_B,k_C),(\dots,k_A,k_B,k_C+1)} = q_C, \quad 0 \leqq k_C \leqq r_C - 2,$
$Q_{(j_{AB},j_{AC},\dots,r_A-1,\dots),(j_{AB}-\min(X_{AB},j_{AB}),j_{AC}-\min(X_{AC},j_{AC}),\dots,0,\dots)} = q_A$, $Q_{(\dots,r_B-1,k_C),(j_{AB},j_{AC},j_{BA}-\min(\dots,j_{BA}),j_{BC}-\min(X_{BC},j_{BC}),\dots,0,k_C)} = q_B$, $Q_{(\dots,j_{CA},j_{CB},\dots,r_C-1),(\dots,j_{CA}-\min(X_{CA},j_{CA}),j_{CB}-\min(X_{CB},j_{CB}),\dots,0)} = q_C$, $Q_{(j_{AB},\dots),(j_{AB}+1,\dots)} = \lambda_{AB}, \quad j_{AB} \leqq K_{AB} - 1,$
$Q_{(\dots,j_{AC},\dots),(\dots,j_{AC}+1,\dots)} = \lambda_{AC}, \quad j_{AC} \leqq K_{AC} - 1,$
$Q_{(\dots,j_{BA},\dots),(\dots,j_{BA}+1,\dots)} = \lambda_{BA}, \quad j_{BA} \leqq K_{BA} - 1,$
$Q_{(\dots,j_{BC},\dots),(\dots,j_{BC}+1,\dots)} = \lambda_{BC}, \quad j_{BC} \leqq K_{BC} - 1,$
$Q_{(\dots,j_{CA},\dots),(\dots,j_{CA}+1,\dots)} = \lambda_{CA}, \quad j_{CA} \leqq K_{CA} - 1,$
$Q_{(\dots,j_{CB},\dots),(\dots,j_{CB}+1,\dots)} = \lambda_{CB}, \quad j_{CB} \leqq K_{CB} - 1,$
$Q_{(j_{AB},j_{AC},j_{BA},\dots),(j_{AB}-\min(n,j_{AB}),j_{AC},j_{BA}-\min(n,j_{BA}),\dots)} = \sigma_{ABA}, \quad j_{AB} \geqq m, j_{BA} \geqq m,$
$Q_{(\dots),(j_{AB},j_{AC}-\min(n,j_{AC}),j_{BA},j_{BC},j_{CA}-\min(n,j_{CA}),j_{CB},\dots)} = \sigma_{ACA}, \quad j_{AC} \geqq m, j_{CA} \geqq m,$
$Q_{(\dots),(j_{AB}-\min(n,j_{AB}),j_{AC},j_{BA},j_{BC}-\min(n,j_{BC}),j_{CA}-\min(n,j_{CA}),j_{CB},\dots)} = \sigma_{ABCA}, \quad j_{AB} \geqq m, j_{BC} \geqq m, j_{CA} \geqq m,$
$Q_{(\dots),(j_{AB},j_{AC}-\min(n,j_{AC}),j_{BA}-\min(n,j_{BA}),j_{BC},j_{CA},j_{CB}-\min(n,j_{CB}),\dots)} = \sigma_{ACBA}, \quad j_{AC} \geqq m, j_{CB} \geqq m,, j_{BA} \geqq m.$

It should be noted that the diagonal elements are the values such that the sum of a row equals 0.

From the above, we obtain the steady state probabilities $\boldsymbol{\pi} = (\pi_{\boldsymbol{e}_0}, \pi_{\boldsymbol{e}_1}, \pi_{\boldsymbol{e}_2}, \dots, \pi_{\boldsymbol{y}})$ by solving the equilibrium equation and the normalization condition:

$$\boldsymbol{\pi Q} = \boldsymbol{0},$$
$$\boldsymbol{\pi e} = 1,$$

where $\boldsymbol{0}$ is the vector of zeros of an appropriate size and $\boldsymbol{e}$ is the vector of ones of an appropriate size. We can also define the mean number of the customers of class-AB as follows (same for the other classes):

$$\mathrm{E}[L_{AB}] = \sum_{j_{AB}=0}^{K_{AB}} \sum_{j_{AC}=0}^{K_{AC}} \sum_{j_{BA}=0}^{K_{BA}} \sum_{j_{BC}=0}^{K_{BC}} \sum_{j_{CA}=0}^{K_{CA}} \sum_{j_{CB}=0}^{K_{CB}} \sum_{k_A=0}^{r_A-1} \sum_{k_B=0}^{r_B-1} \sum_{k_C=0}^{r_C-1} j_{AB} \pi_{(j_{AB}, j_{AC}, j_{BA}, j_{BC}, j_{CA}, j_{CB}, k_A, k_B, k_C)}.$$

4 HEURISTIC ANALYSIS

In this section, we present the heuristic solution for the model. As we showed in the previous section, the number of the states becomes large even for $N = 3$. The theoretical analysis is not practical due to the limited memory capacity of the computer. Therefore, we would like to propose the heuristic solution concerning Markov chains of a small number of states.

As a premise, the heuristic solution makes a strong assumption that the transitions of each class are independent. Strictly speaking, CRS reduces the number of customers in multiple classes at the same time, that means that they are not independent, but for the sake of simplicity, we make this assumption.

We describe the heuristic solution based on the example where $N = 3$. We consider 6 $(= N(N-1))$ basic models instead of the complicated exact model. In other words, we consider the basic model for each class, and estimate the steady state probabilities numerically.

The detailed procedure is given in Algorithm 1. We define several notations; V is the set of all nodes, P is the set of all classes, and U is the set of all routes. P_A is the set of classes including node-A (same for P_B and P_C). U_{AB} is the set of routes including class- AB (same for other classes). $T_{U_{AB}}$ is the set of classes except for class-AB and consisting of the classes in U_{AB} (same for other classes). $\chi_{ABCA/AB}$ is the probability that

Algorithm 1: $N = 3$.

Input: $V = \{A, B, C\}, \{q_v, r_v; v \in V\}, \{\lambda_p; p \in P\}, \{X_p; p \in P\}, \{\sigma_u; u \in U\}, \varepsilon$.
Output: $\{\boldsymbol{\pi}^*_p; p \in P\}, \{\chi_{u/p}; u \in U, p \in P\}$.
  $P = \{(A \Rightarrow B), (A \Rightarrow C), (B \Rightarrow A), (B \Rightarrow C), (C \Rightarrow A), (C \Rightarrow B)\}$
  $U = \{(A \to B \to A), (A \to C \to A), (A \to B \to C \to A), (A \to C \to B \to A)\}$
  $P_A = \{(A \Rightarrow B), (A \Rightarrow C)\}$, $P_B = \{(B \Rightarrow A), (B \Rightarrow C)\}$, $P_C = \{(C \Rightarrow A), (C \Rightarrow B)\}$
  $U_{AB} = \{(A \to B \to A), (A \to B \to C \to A)\}$
  $U_{AC} = \{(A \to C \to A), (A \to C \to B \to A)\}$
  $U_{BA} = \{(A \to B \to A), (A \to C \to B \to A)\}$
  $U_{BC} = \{(A \to B \to C \to A)\}$
  $U_{CA} = \{(A \to C \to A), (A \to B \to C \to A)\}$
  $U_{CB} = \{(A \to C \to B \to A)\}$
  $T_{U_{AB}} = \{(B \Rightarrow A), (B \Rightarrow C), (C \Rightarrow A)\}$
  $T_{U_{AC}} = \{(B \Rightarrow A), (C \Rightarrow A), (C \Rightarrow B)\}$
  $T_{U_{BA}} = \{(A \Rightarrow B), (A \Rightarrow C), (C \Rightarrow B)\}$
  $T_{U_{BC}} = \{(A \Rightarrow B), (C \Rightarrow A)\}$
  $T_{U_{CA}} = \{(A \Rightarrow B), (A \Rightarrow C), (B \Rightarrow C)\}$
  $T_{U_{CB}} = \{(A \Rightarrow C), (B \Rightarrow A)\}$
  for $p \in P$ **do**
    for $u_p \in U_p$ **do**
      $\chi^{(0)}_{u_p/p} = 0$
      Compute $\boldsymbol{\pi}^{*(0)}_p$ such that $\boldsymbol{\pi}^{*(0)}_p \boldsymbol{Q}^{*(0)}_p = \mathbf{0}$ with $\boldsymbol{\pi}^{*(0)}_p \mathbf{1}^\top = 1$
    end for
  end for
  for $p \in P$ **do**
    for $u_p \in U_p$ **do**
      $\chi^{(1)}_{u_p/p} = \prod_{t \in T_{U_p}} \sum_{j_t=m}^{K_t} \sum_{k_t=0}^{r_{\hat{t}}-1} \pi^{*}_{j_t,k_{\hat{t}}}{}^{(0)}$
    end for
  end for
  $n = 1$
  while $||\boldsymbol{\pi}^{*(n)}_p - \boldsymbol{\pi}^{*(n-1)}_p|| > \varepsilon$ **do**
    for $p \in P$ **do**
      for $u_p \in U_p$ **do**
        Compute $\boldsymbol{\pi}^{*(n)}_p$ such that $\boldsymbol{\pi}^{*(n)}_p \boldsymbol{Q}^{*(n)}_p = \mathbf{0}$ with $\boldsymbol{\pi}^{*(n)}_p \mathbf{1}^\top = 1$
      end for
    end for
    for $p \in P$ **do**
      for $u_p \in U_p$ **do**
        $\chi^{(n+1)}_{u_p/p} = \prod_{t \in T_{U_p}} \sum_{j_t=m}^{K_t} \sum_{k_t=0}^{r_{\hat{t}}-1} \pi^*_{j_t,k_{\hat{t}}}$
      end for
    end for
    $n \leftarrow n + 1$
  end while

class-BC and class-CA (all classes composing route-$ABCA$ except for class-AB) satisfy the conditions for the occurrences of CRS. This is also the same for the other classes and routes. The condition for the occurrences of CRS is the same as the theoretical analysis; there are m or more customers of every class that composes a route.

Let $S^*_{AB}(t)$ and $R^*_A(t)$ denote the number for the waiting customers of class-AB and the progress of the Erlang distribution for the buses on which class-AB rides (i.e., the buses at node-A), at time t, respectively (same for other classes). The basic model becomes two dimensional Markov chain of these two variables, i.e., $\{(S^*_{AB}(t), R^*_A(t)) | t \geqq 0\}$. Our goal is to know the steady state probabilities of these basic models, i.e., $\pi^{AB}_{j_{AB},k_{AB}} = \lim_{t\to\infty} \mathrm{P}(S^*_{AB}(t) = j_{AB}, R^*_A(t) = k_A)$, $\pi^{AC}_{j_{AC},k_{AC}} = \lim_{t\to\infty} \mathrm{P}(S^*_{AC}(t) = j_{AC}, R^*_A(t) = k_A), \ldots$, (same for other classes).

We obtain the steady state probabilities by solving the following formulae (example of class-AB):

$$\boldsymbol{\pi}^*_{AB} \boldsymbol{Q}^*_{AB} = \mathbf{0},$$

$$\boldsymbol{\pi}^*_{AB} \boldsymbol{e} = 1,$$

where,

$$\boldsymbol{\pi}^*_{AB} = \begin{array}{c} \\ 0 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \ldots & K_{AB}(r_A-1) \end{array} \\ \begin{pmatrix} \pi^*_{0,0} & \pi^*_{0,1} & \ddots & \pi^*_{K_{AB},r_A-1} \end{pmatrix} \end{array},$$

$$\boldsymbol{Q}^*_{AB} =$$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ K_{AB}(r_A-1) \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \ldots & K_{AB}(r_A-1) \end{array} \\ \begin{pmatrix} Q^*_{AB_{(0,0),(0,0)}} & Q^*_{AB_{(0,0),(0,1)}} & \ddots & Q^*_{AB_{(0,0),(K_{AB},r_A-1)}} \\ Q^*_{AB_{(0,1),(0,0)}} & Q^*_{AB_{(0,1),(0,1)}} & \ddots & Q^*_{AB_{(0,1),(K_{AB},r_A-1)}} \\ Q^*_{AB_{(0,2),(0,0)}} & Q^*_{AB_{(0,2),(0,1)}} & \ddots & Q^*_{AB_{(0,2),(K_{AB},r_A-1)}} \\ \vdots & \ddots & \ddots & \vdots \\ Q^*_{AB_{(K_{AB},r_A-1),(0,0)}} & Q^*_{AB_{(K_{AB},r_A-1),(0,1)}} & \ddots & Q^*_{AB_{(K_{AB},r_A-1),(K_{AB},r_A-1)}} \end{pmatrix} \end{array},$$

$$Q^*_{AB_{(j_{AB},k_{AB}),(j_{AB}+1,k_{AB})}} = \lambda_{AB}, \qquad j_{AB} \leqq K_{AB} - 1,$$

$$Q^*_{AB_{(j_{AB},k_{AB}),(j_{AB},k_{AB}+1)}} = q_A, \qquad 0 \leqq k_A \leqq r_A - 2,$$

$$Q^*_{AB_{(j_{AB},K_{AB}),(j_{AB}-\min(X_{AB},j_{AB}),0)}} = q_A,$$

$$Q^*_{AB_{(j_{AB},k_{AB}),(j_{AB}-\min(n,j_{AB}),k_{AB})}} = \chi_{ABA/AB}\sigma_{ABA} + \chi_{ABCA/AB}\sigma_{ABCA}, \qquad j_{AB} \geqq m.$$

Note that $\chi_{ABA/AB}$ and $\chi_{ABCA/AB}$ (and also for the other routes and classes; $\chi_{U_p/p}$) are unknown values. Therefore, we propose the following procedure;

- Set all $\chi_{U_p/p} = 0$ at first and solve the basic models.
- Obtain new values of $\chi_{U_p/p}$ and solve all the basic models again.
- Repeat the above until the difference between the steady state probabilities elements of all the basic models is less than an extremely small value ($= \varepsilon$).

See Algorithm 1 for the formal notations. We can also derive the number of waiting customers of each class as (for example class AB):

$$\mathrm{E}[L_{AB}] = \sum_{j_{AB}=0}^{K_{AB}} \sum_{k_A=0}^{r_A-1} j_{AB} \pi^{AB}_{j_{AB},k_A}.$$

Unfortunately, we cannot theoretically guarantee the convergence of the algorithm. However, it was experimentally confirmed that this algorithm works well. We show the numerical results for both theoretical and heuristic solutions in the next section.

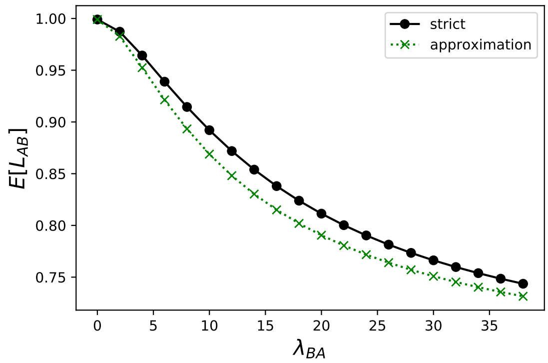

Figure 3: The expected number of the waiting customers.

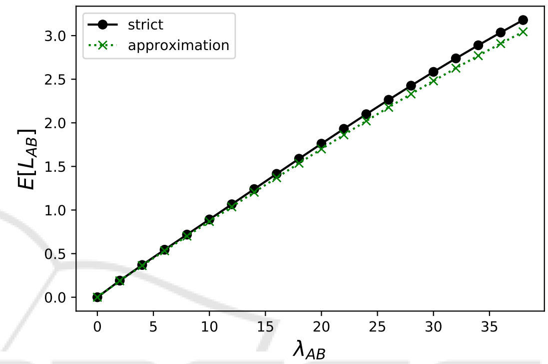

Figure 4: The expected number of the waiting customers.

5 NUMERICAL EXPERIMENTS

This section presents the numerical results for both theoretical (strict) and heuristic (approximation) solutions. We set the parameters as $\varepsilon = 10^{-5}$, $\lambda_{AB} = \lambda_{AC} = \lambda_{BA} = \lambda_{BC} = \lambda_{CA} = \lambda_{CB} = 10$, $m = 2$, $n = 4$, $l = 10$, $K_{AB} = K_{AC} = K_{BA} = K_{BC} = K_{CA} = K_{CB} = 10$, $X_{AB} = X_{AC} = X_{BA} = X_{BC} = X_{CA} = X_{CB} = 10$, $q_A = q_B = q_C = 10$, $r_A = r_B = r_C = 1$ unless otherwise specified.

Figures 3 and 4 show the results for the number of the waiting customers in class-AB (= $\mathrm{E}[L_{AB}]$) according to the values of λ_{BA} and λ_{AB}, respectively. Overall, both solutions show a similar trend. The reason why the heuristic solution is that we assume each class's independence. Therefore, it is easier to satisfy the condition of CRS occurrences compared to the theoretical solution, and thus, customers tend to decrease quickly in the heuristic analysis. However, it should be noted that the heuristic model has a much smaller number of states (i.e., less memory capacity required for the computation). We can consider that the heuristic analysis is practical to obtain the rough trend of the system performance with low computational cost.

6 CONCLUSION

This paper has considered the modeling of the introduction of CRS (one of the demand-responsive trans portations) on a circular bus route. First of all, we have shown the theoretical analysis of the model regarding queueing theory. However, the theoretical model is not practical from the perspective of its computational cost. Therefore, we have proposed the heuristic analysis, which estimates the steady state probabilities algorithmically. We have confirmed the validity of the heuristic analysis compared to the theoretical analysis by conducting some numerical experiments. As future works, we plan to conduct numerical experiments for more various cases, e.g., the case where the number of nodes is more than 3. Moreover, it is also an important issue that we confirm the guarantee of the theoretical convergence of the heuristic model.

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