Application of Graded Fuzzy Preconcept Lattices in Risk Analysis

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Abstract: Fuzzy logic has a wide range of applications in the risk assessment based on expert opinion. Several methods have been used for this purpose with fuzzy implication systems being among the most popular. While these standard tools have proven their usefulness, we propose the application of an alternative approach. In this paper we analyse the possibility of using the graded fuzzy preconcept lattices in introducing the risk assessment model. We have also provided the necessary information on the theoretical basis of the graded fuzzy preconcept lattices introduced by the authors earlier.

1 INTRODUCTION

The risk assessment process is a crucial part of the risk management. It is carried out in different activities from a simple daily behaviours, like participation in the traffic, up to the different industries, science and managing of a large scale projects. The risk assessment process often relies on statistic data, but in many instances these data should be further combined with expert opinions on specific and previously unknown circumstances and their impact. There are also scenarios of unique and previously unknown conditions when historic statistic data are either not available or could be hardly applicable. Therefore the experts should evaluate any possible developments and propose concrete actions based on their experience and judgements. These provisions have provided a solid basis for many studies on application of fuzzy logic in the risk assessment, see, e.g. (Chan and Wang, 2013), (Jones, 2009).

For the purposes of constructing the risk assessment model, first we consider the risk hierarchy with more general risks including further specific risks which, in turn, are characterised by risk factors (those can be considered as subclasses of the specific risks) which are assessed by corresponding risk levels. These risk levels are obtained via evaluations which are usually based on the fuzzy inference systems, with Mamdani and Sugeno systems being the most visible examples, see, e.g. (Lilly, 2010). In our research we propose the risk assessment model based on the fuzzy preconcept lattices which, in our opinion, provides a good illustrative basis for modelling of different scenarios and also allows to study certain mathematical properties embedded in this model.

Formal concept analysis or just concept analysis for short was developed mainly in eighties of the previous century. The principles and fundamental results of concept analysis were exposed in the paper (Wille, 1992) and further expanded in (Ganter and Wille, 1999). The concept analysis starts with the notion of a (formal) context that is a triple (X,Y,R) where X and Y are sets and $R \subseteq X \times Y$ is a relation between the elements of these sets. The elements of X are interpreted as some abstract objects, the elements of Y are interpreted as some abstract properties or attributes, and the entry $(x,y) \in R$ means that an object x has attribute y. The idea of the concept analysis is to reveal all pairs (A,B) of sets $A \subseteq X$ and $B \subseteq Y$ (called concepts) such that every object $x \in A$ has all properties $y \in B$ and every property $y \in B$ holds for all objects $x \in A$. In the first decade of the 21st century different fuzzy counterparts of the formal concept were introduced. The most important work in the first decade of the 21st century in fuzzy concept analysis was carried out by R. Bělohlávek, see, e.g. (Bělohlávek, 1999), (Bělohlávek, 2004), (Bělohlávek and Vychodil, 2005).

Since its inception, crisp concept analysis has found important applications in the study of different "real-world" problems. Starting with illustrative examples of application of crisp lattices given in (Wille,

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1992), there appeared many serious works in which concept lattices were used. Concept analysis has found a very wide range of applications in medical research and teaching of medical students (see, e.g. (Hu et al., 2019), (Keller et al., 2012), (Liu et al., 2010), (Yazdani and Hoseini, 2018)), in particlar in the process of basic studies of nurses (see e.g. (Rodhers et al., 2018)). Methods of concept analysis are used also in the research of problems related to biology (Raza, 2017), (Hashikami et al., 2013), geology (Bělohlávek, 2004, p. 214), social type problems (Missaoui et al., 2017), software engineering (Tonella, 2004) and in other applied sciences. On the other hand, we found only a few works, where fuzzy concept analysis is used in the research of any practical problems. In our opinion, the problem to use the fuzzy concept lattices in case when the scale value L is continuous (like $L = [0,1]$) or lattice having many, probably incomparable elements, is that the request in the concept analysis of the *precise correspondence* between a fuzzy set A of objects and a fuzzy set B of attributes in "real-world" situations is (almost) impracticable. In this case one sooner has to deal with the weaker request asking *that the correspondence between A and B must hold up to a certain degree*. In order to provide a theoretical basis for the research in this situation, in (Šostak et al., 2021), we introduced the notion of a graded preconcept lattice, more flexible than the notion of a concept lattice and initiated the study of the properties of graded preconcept lattices. In this paper we provide a further insight in the practical application of the theoretical models in the assessment of risks related to spread of pandemic crisis.

This paper is organized as follows. In the second section we briefly remind the notions related to lattices, residuated lattices or quantales, fuzzy sets and fuzzy relations. Besides, we remind the concept of a fuzzy inclusion of fuzzy sets that lies in the base of gradation of fuzzy preconcepts. In the third section we define fuzzy preconcepts, introduce partial order relation $\preceq$ on the family of all fuzzy preconcepts of a given fuzzy context (X,Y,L,R) and show that the resulting structure $(\mathbb{P}(X,Y,L,R),\preceq)$ is a lattice. Further in this section we propose a method allowing to determine the grade of the distinction of a fuzzy preconcept from "being a real fuzzy concept" and study the corresponding graded preconcept lattices. In the fourth section we explain the possible application of fuzzy preconcept lattices in the analysis of pandemic scenario and proceed with practical calculations in modelling the spread of Covid-19 pandemics. In the last, conclusion section, we briefly summarize main results obtained and survey some directions for the future work.

2 BACKGROUND

2.1 Lattices, Quantales and Residuated Lattices

We use the standard terminology accepted in theory of lattices, see, e.g. (Birkhoff, 1995), (Gierz et al., 2003), (Morgan and Dilworth, 1939) for details. For the reader convenience, we make clarification of some, possibly less known terms.

A complete lattice $L = (L,\leq,\wedge,\vee)$ is called infinitely distributive if $a \wedge (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\wedge b_i)$ for every $a \in L$ and every $\{b_i \mid i \in I\} \subseteq L$. A complete lattice L is called infinitely co-distributive if $a \vee (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a\vee b_i)$ for every $a \in L$ and every $\{b_i \mid i \in I\} \subseteq L$. A complete lattice is called infinitely bi-distributive if it is infinitely and co-infinitely distributive. It is known, that every completely distributive lattice is infinitely bi-distributive.

Let L be a complete lattice and $* : L\times L \to L$ be a binary associative commutative monotone operation. Then the tuple $(L,\leq,\wedge,\vee,*)$ is called *a (commutative) quantale* (Rosenthal, 1990) if $*$ distributes over arbitrary joins: $a*(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a*b_i)$.

A quantale is called integral if the top element 1 acts as the unit, that is $1*a = a*1 = a$ for every $a \in L$.

In what follows saying *quantale* we mean *a commutative integral quantale*. A typical example of a quantale is the unit interval endowed with a lower semi-continuous t-norm, see, e.g. (Klement et al., 2000).

In a quantale a further binary operation $\mapsto : L\times L \to L$, the residuum, can be introduced as associated with operation $*$ of the quantale $(L,\leq,\wedge,\vee,*)$ via the Galois connection, that is $a*b \leq c \iff a \leq b \mapsto c$ for all $a,b,c \in L$. A quantale $(L,\leq,\wedge,\vee,*)$ provided with the derived operation $\mapsto$, that is the tuple $(L,\leq,\wedge,\vee,*,\mapsto)$, is known also as a (complete) residuated lattice (Morgan and Dilworth, 1939).

2.2 Fuzzy Sets and Fuzzy Relations

Given a set X, its L-fuzzy subset is a mapping $A : X \to L$. The lattice and the quantale structure of L is extended point-wise to the L-exponent of X, that is to the set L^X of all L-fuzzy subsets of X. Specifically, the union and intersection of a family of L-fuzzy sets $\{A_i | i \in I\} \subseteq L$ are defined by their join $\bigvee_{i\in I} A_i$ and meet $\bigwedge_{i\in I} A_i$ respectively. An L-fuzzy relation between two sets X and Y is an L-fuzzy subset of the product $X \times Y$, that is a mapping $R : X \times Y \to L$, see, e.g. (Valverde, 1985), (Zadeh, 1971).

2.3 Measure of Inclusion of L-fuzzy Sets

The gradation of a preconcept lattice presented below is based on the fuzzy inclusion between fuzzy sets. We present here a brief introduction into this field.

In order to fuzzify the inclusion relation $A \subseteq B$ "a fuzzy set A is a subset of a fuzzy set B", we have to interpret it as a certain fuzzy relation $\hookrightarrow$ based on "if - then" rule, that is on some implication $\Rightarrow$ defined on the lattice L. In the result we come to the formula $A \hookrightarrow B = \inf_{x\in X}(A(x) \Rightarrow B(x))$.

In this paper we use implication $\Rightarrow$ defined by means of the residuum: $a \Rightarrow b =_{def} a \mapsto b$.

Definition 2.1. *By setting* $A \hookrightarrow B = \bigwedge_{x\in X}(A(x) \mapsto B(x))$ *for all* $A, B \in L^X$, *we obtain a mapping* $\hookrightarrow$: $L^X \times L^X \to L$. *We call* $A \hookrightarrow B$ *by the (L-valued) measure of inclusion of the L-fuzzy set A into the L-fuzzy set B. We denote* $A \cong B =_{def} (A \hookrightarrow B) \wedge (B \hookrightarrow A)$ *and view it as the degree of equality of L-fuzzy sets A and B.*

Proposition 2.2. (see, e.g. (Han and Šostak, 2016), (Han and Šostak, 2018)) *Mapping* $\hookrightarrow: L^X \times L^X \to L$ *satisfies the following properties:*
(1) $(\bigvee_i A_i) \hookrightarrow B = \bigwedge_i (A_i \hookrightarrow B)$ *for all* $\{A_i \mid i \in I\} \subseteq L^X$ *and for all* $B \in L^X$;
(2) $A \hookrightarrow (\bigwedge_i B_i) = \bigwedge_i (A \hookrightarrow B_i)$ *for all* $A \in L^X$ *and for all* $\{B_i \mid i \in I\} \subseteq L^X$;
(3) $A \hookrightarrow B = 1_L$ *whenever* $A \leq B$;
(4) $1_X \hookrightarrow A = \bigwedge_x A(x)$ *for all* $A \in L^X$ *where* $1_X : X \to L$ *is a constant function with the value* $1_L \in L$;
(5) $(A \hookrightarrow B) \leq (A * C \hookrightarrow B * C)$ *for all* $A, B, C \in L^X$;
(6) $(A \hookrightarrow B) * (B \hookrightarrow C) \leq (A \hookrightarrow C)$ *for all* $A, B, C \in L^X$;
(7) $(\bigwedge_i A_i) \hookrightarrow (\bigwedge_i B_i) \geq \bigwedge_i (A_i \hookrightarrow B_i)$ *for all* $\{A_i : i \in I\}$, $\{B_i : i \in I\} \subseteq L^X$;
(8) $(\bigvee_i A_i) \hookrightarrow (\bigvee_i B_i) \geq \bigwedge_i (A_i \hookrightarrow B_i)$ *for all* $\{A_i : i \in I\}$, $\{B_i : i \in I\} \subseteq L^X$.

3 GRADED PRECONCEPT LATTICES

3.1 Preconcepts and Preconcept Lattices

Let L be a complete lattice. Further, let X, Y be sets and $R : X \times Y \to L$ be a fuzzy relation. Following terminology accepted in the theory of (fuzzy) concept lattices, see, e.g. (Wille, 1992), (Bělohlávek, 1999), (Bělohlávek, 2004), (Bělohlávek and Vychodil, 2005), we refer to the tuple (X, Y, L, R) as a fuzzy context.

Definition 3.1. *Given a fuzzy context* (X, Y, L, R), *a pair* $P = (A, B) \in L^X \times L^Y$ *is called a fuzzy preconcept.*

On the set $L^X \times L^Y$ of all fuzzy preconcepts we introduce a partial order $\preceq$ as follows. Given $P_1 = (A_1, B_1)$ and $P_2 = (A_2, B_2)$, we set $P_1 \preceq P_2$ if and only if $A_1 \leq A_2$ and $B_1 \geq B_2$. Let $(\mathbb{P}, \preceq)$ be the set $L^X \times L^Y$ endowed with this partial order. Further, given a family of fuzzy concepts $\{P_i = (A_i, B_i) : i \in I\} \subseteq L^X \times L^Y$, we define its join (supremum) by $\underline{\vee}_{i\in I} P_i = (\bigvee_{i\in I} A_i, \bigwedge_{i\in I} B_i)$ and its meet (infimum) as $\overline{\wedge}_{i\in I} P_i = (\bigwedge_{i\in I} A_i, \bigvee_{i\in I} B_i)$.

Theorem 3.2. $\mathbb{P}$ *is a complete lattice. Besides, if L is a infinitely bi-distributive lattice, then* $(\mathbb{P}, \preceq, \overline{\wedge}, \underline{\vee})$ *is also a infinitely bi-distributive lattice.*

In the sequel we write just $\mathbb{P}$ or $(\mathbb{P}, \preceq)$ instead of $(\mathbb{P}, \preceq, \overline{\wedge}, \underline{\vee})$ if no misunderstanding is possible, or $(\mathbb{P}, X, Y, L, R)$ in case when we need to specify the fuzzy context we are working in.

3.2 Operators $R^{\uparrow}$ and $R^{\downarrow}$ on L-powersets and Fuzzy Concept Lattices

Let X and Y be sets and let $R : X \times Y \to L$ be a fuzzy relation, where L is a fixed *quantale*. Given a fuzzy context (X, Y, L, R), we define operators $R^{\uparrow} : L^X \to L^Y$ and $R^{\downarrow} : L^Y \to L^X$ as follows:

Definition 3.3. (see, e.g. (Bělohlávek and Vychodil, 2005)) *Given* $A \in L^X$ *and* $B \in L^Y$, *we define* $A^{\uparrow} \in L^Y$ *and* $B^{\downarrow} \in L^X$ *by setting*
$A^{\uparrow}(y) = \bigwedge_{x\in X}(A(x) \mapsto R(x,y))\ \forall y \in Y$,
$B^{\downarrow}(x) = \bigwedge_{y\in Y}(B(y) \mapsto R(x,y))\ \forall x \in X$.
By changing A over L^X *and B over* L^Y, *we get operators* $R^{\uparrow} : L^X \to L^Y$ *and* $R^{\downarrow} : L^Y \to L^X$.

In the crisp case, that is when $A \subseteq X, B \subseteq Y$ and $R : X \times Y \to \{0, 1\}$, this definition is obviously equivalent to the original definition of operators $A \longrightarrow A'$ and $B \longrightarrow B'$ in (Wille, 1992). Informally these sets can be described as follows. Let a set X of objects with a subset $A \subseteq X$ and a set Y of properties with a subset $B \subseteq Y$ be given. Further, let $R \subseteq X \times Y$ be a relation where $(x, y) \in R$ means "object x has property y". Now $A^{\uparrow}$ is the set of such properties $y \in Y$ which have all objects $x \in A$ while $B^{\downarrow}$ is the set of such objects $x \in X$ which have all properties $y \in B$. From the properties of the residuum one can easily justify the following:

Proposition 3.4. *Operators $R^{\uparrow} : L^X \to L^Y$ and $R^{\downarrow} : L^Y \to L^X$ are decreasing: $A_1 \le A_2 \Rightarrow A_1^{\uparrow} \ge A_2^{\uparrow}$, $B_1 \le B_2 \Rightarrow B_1^{\downarrow} \ge B_2^{\downarrow}$.*

In the sequel we write $A^{\uparrow\downarrow}$ instead of $(A^{\uparrow})^{\downarrow}$ and $B^{\downarrow\uparrow}$ instead of $(B^{\downarrow})^{\uparrow}$. We write also $R^{\uparrow\downarrow}$ for the composition $R^{\downarrow} \circ R^{\uparrow} : L^X \to L^X$ and $R^{\downarrow\uparrow}$ for the composition $R^{\uparrow} \circ R^{\downarrow} : L^Y \to L^Y$.

Proposition 3.5. (cf, e.g. (Wille, 1992) in crisp case, (Bělohlávek, 2004)) *$A^{\uparrow\downarrow} \ge A$ for every $A \in L^X$ and $B^{\downarrow\uparrow} \ge B$ for every $B \in L^Y$.*

Proposition 3.6. (cf, e.g. (Wille, 1992) in crisp case, (Bělohlávek, 2004)) *$A^{\uparrow} = A^{\uparrow\downarrow\uparrow}$ for every $A \in L^X$ and $B^{\downarrow} = B^{\downarrow\uparrow\downarrow}$ for every $B \in L^Y$.*

Example 3.7. Let a fuzzy context (X,Y,L,R) be given and let $A \subseteq X$.[1] Then for every $y \in Y$ $A^{\uparrow}(y) = \bigwedge_{x\in X} A(x) \mapsto R(x,y) = \bigwedge_{x\in A} R(x,y)$. In the same way we prove that if $B \subseteq Y$, then $B^{\downarrow}(x) = \bigwedge_{y\in B} R(x,y)$. Hence, even in case when $A \subseteq X$, $B \subseteq Y$ a pair (A,B) can be a concept (either crisp or fuzzy) only in case when R is also crisp, that is $R : X \times Y \to \{0,1\}$. This shows the limitation for the use of concept lattices in the case of a fuzzy context and gives an additional evidence in favour of the graded approach to fuzzy preconcept lattices.

Continuing the previous example we calculate $A^{\uparrow\downarrow}$ and $B^{\downarrow\uparrow}$ in case of crisp sets A and B:
$A^{\uparrow\downarrow}(x) = \bigwedge_{y\in Y} \left(\bigwedge_{x'\in A}(R(x',y) \mapsto R(x,y))\right)$,
$B^{\downarrow\uparrow}(y) = \bigwedge_{x\in X} \left(\bigwedge_{y'\in B}(R(x,y') \mapsto R(x,y))\right)$. □

Proposition 3.8. (cf, e.g. (Wille, 1992) for the crisp case, (Bělohlávek, 2004)) *Given a family $\{A_i \mid i \in I\} \subseteq L^X$, we have $(\bigvee_{i\in I} A_i)^{\uparrow} = \bigwedge_{i\in I} A_i^{\uparrow}$. Given a family $\{B_i \mid i \in I\} \subseteq L^Y$, we have $(\bigvee_{i\in I} B_i)^{\downarrow} = \bigwedge_{i\in I} B_i^{\downarrow}$.*

3.3 Concepts and Concept Lattices

Referring to the definition of a (fuzzy) concept given in (Wille, 1992), (Bělohlávek, 2004) we reformulate it as follows:

Definition 3.9. *A fuzzy preconcept (A,B) is called a (formal) fuzzy concept if $A^{\uparrow} = B$ and $B^{\downarrow} = A$.*

Let $\mathbb{C} = \mathbb{C}(X,Y,L,R)$ be the subset of $\mathbb{P} = \mathbb{P}(X,Y,L,R)$ consisting of fuzzy concepts (A,B) and let $\preceq$ be the partial order on $\mathbb{C}$ induced by the partial order $\preceq$ from the lattice $(\mathbb{P},\preceq)$. Then $(\mathbb{C},\preceq)$ is a partially ordered subset of the lattice $(\mathbb{P},\preceq)$. However generally $(\mathbb{C},\preceq)$ is not a sublattice of the lattice $(\mathbb{P},\preceq,\barwedge,\veebar)$ and we need to define joins and meets in $(\mathbb{C},\preceq)$ differently from $\barwedge$ and $\veebar$. To do this first we show the following simple lemma:

Lemma 3.10. *Let (A_1,B_1), (A_2,B_2) be fuzzy concepts. If $A_1 \le A_2$, then $B_1 \ge B_2$ and if $B_1 \ge B_2$ then $A_1 \le A_2$.*

Corollary 3.11. *Let $(A_1,B_1),(A_2,B_2) \in \mathbb{C}$. Then $(A_1,B_1) \preceq (A_2,B_2)$ if and only if $A_1 \le A_2$ if and only if $B_1 \ge B_2$.*

Proposition 3.12. *If $A \in L^X$, then $(A^{\uparrow\downarrow},A^{\uparrow})$ is the smallest concept containing A as the fuzzy set of objects. If $B \in L^Y$, then $(B^{\downarrow},B^{\downarrow\uparrow})$ is the smallest concept containing B as the fuzzy set of attributes.*

Theorem 3.13. *Let (X,Y,L,R) be a fuzzy context and let $\preceq$ be the partial order induced from the lattice $\mathbb{P}(X,Y,L,R,\preceq)$. Then $\mathbb{C}(X,Y,L,R \preceq)$ is a complete lattice.*

Taking into account that in a fuzzy concept (A_i,B_i) it holds $A_i^{\uparrow} = B_i$ and $B_i^{\downarrow} = A_i$, we get the following corollary:

Corollary 3.14. *Let $\mathcal{C} = \{C_i = (A_i,B_i) \mid i \in I\} \subseteq \mathbb{C}$ be a family of fuzzy concepts. Then*
$\curlywedge_{i\in I} C_i = \left(\bigwedge_{i\in I} A_i, (\bigwedge_{i\in I} A_i)^{\uparrow}\right)$ is its infimum in the lattice $(\mathbb{C},\preceq)$,
$\curlyvee_{i\in I} C_i = ((\bigwedge_{i\in I} B_i)^{\downarrow}, \bigwedge_{i\in I} B_i)$ is its supremum in the lattice $(\mathbb{C},\preceq)$.

3.4 Conceptuality Degree of a Fuzzy Preconcept

Let (X,Y,L,R) be a fuzzy context and $(A,B) \in \mathbb{P}(X,Y,L,R)$.

Definition 3.15. *The degree of contentment of the fuzzy set A of objects by the fuzzy set B of attributes or the degree object based contentment of the preconcept (A,B) for short is defined by $\mathcal{D}^{\uparrow}(A,B) =_{def} A^{\uparrow} \cong B$.*

Definition 3.16. *The degree of contentment of the fuzzy set B of of attributes by the fuzzy set A of objects or the attribute based contentment of the preconcept (A,B) is defined by $\mathcal{D}^{\downarrow}(A,B) =_{def} A \cong B^{\downarrow}$.*

Definition 3.17. *The degree of conceptuality of the preconcept (A,B) in the preconcept lattice $\mathbb{P}$ is defined by $\mathcal{D}(A,B) = \mathcal{D}^{\uparrow}(A,B) \wedge \mathcal{D}^{\downarrow}(A,B)$.*

Changing pairs $(A,B) \in \mathbb{P}$, we obtain mappings $\mathcal{D}^{\uparrow} : \mathbb{P} \to L$, $\mathcal{D}^{\downarrow} : \mathbb{P} \to L$ and $\mathcal{D} : \mathbb{P} \to L$.

Referring to our previous paper *"Graded Fuzzy Preconcept Lattices: Theoretical Basis"*, we illustrate the evaluation of conceptuality degree in the fuzzy context (X,Y,L,R) in case when A and B are crisp sets. Besides, to make exposition as simple as

[1] Here and in the sequel we do not distinguish between a crisp set $A \subseteq X$ and its characteristic function $\chi_A : X \to \{0,1\}$.

possible, we assume that the relation $R : X \times Y \to L$ satisfies the following two conditions:
$(\dagger_{R_{BA}})$ $\bigvee_{y \in B^c} (\bigwedge_{x \in A} (R(x,y)) = 0$;
$(\dagger_{R_{AB}})$ $\bigvee_{y \in A^c} (\bigwedge_{x \in B} (R(x,y)) = 0$.
Note that if $B = Y$ then $(\dagger_{R_{BA}})$ is satisfied and if $A = X$ then $(\dagger_{R_{AB}})$ is satisfied.

Example 3.18. Let $A \subseteq X, B \subseteq Y$, let $(L, \leq, \wedge, \vee, *)$ be an arbitrary quantale, $\mapsto: L \times L \to L$ its residuum, and $R : X \times Y \to L$ a fuzzy relation. Then
$A^{\uparrow} \hookrightarrow B = 1$;
$B \hookrightarrow A^{\uparrow} = \bigwedge_{y \in B} \bigwedge_{x \in A} R(x,y)$;
$\mathcal{D}^{\uparrow}(A,B) = \bigwedge_{x \in A, y \in B} R(x,y)$.
$A \hookrightarrow B^{\downarrow} = \bigwedge_{x \in A, y \in B} R(x,y)$;
$B^{\downarrow} \hookrightarrow A = 1$;
$\mathcal{D}^{\downarrow}(A,B) = \bigwedge_{x \in A, y \in B} R(x,y)$
and hence $\mathcal{D}(A,B) = \bigwedge_{x \in A, y \in B} R(x,y)$.

Example 3.19. Let now $B \subseteq Y$, $a \in (0,1)$ and fuzzy set $A : X \to L = [0,1]$ be defined by $A(x) = a \; \forall x \in X$. Then $B \hookrightarrow A^{\uparrow} = \bigwedge_{y \in B, x \in X} (a \mapsto R(x,y))$; $A^{\uparrow} \hookrightarrow B = 1$; hence $\mathcal{D}^{\uparrow}(A,B) = \bigwedge_{y \in B, x \in X} (a \mapsto R(x,y))$.
$A \hookrightarrow B^{\downarrow} = \bigwedge_{x \in X} \left(\bigwedge_{y \in B} R(x,y) \mapsto a \right)$; $B^{\downarrow} \hookrightarrow A = 1$
and hence $D^{\downarrow}(A,B) = \bigwedge_{x \in X} \left(\bigwedge_{y \in B} R(x,y) \mapsto a \right)$ and
$\mathcal{D}(A,B) = \left(\bigwedge_{y \in B, x \in X} (a \mapsto R(x,y)) \right) \wedge \left(\bigwedge_{x \in X} \left(\bigwedge_{y \in B} R(x, y \mapsto a) \right) \right)$.

We demonstrate the formulas obtained in the previous example for calculating $\mathcal{D}(A,B)$ for the three basic t-norms $*$ on $[0,1]$: $*_{\wedge} = \wedge$ - the minimum t-norm, $*_L$ - the Łukasiewicz t-norm and $*_P$ - the product t-norm, see, e.g. (Klement et al., 2000). Besides, we restrict to the case when $X_a = X$ and $B = Y$.
(1) In the case of Łukasiewicz t-norm $\mathcal{D}^{\uparrow}(A,B) = \left(\bigwedge_{x \in X} (1 - a + \bigwedge_{y \in Y} R(x,y)) \right) \wedge 1$;
$\mathcal{D}^{\downarrow}(A,B) = \bigwedge_{x \in X} \left(1 - \bigwedge_{y \in Y} R(x,y) + a \right) \wedge 1$ and hence

$$\mathcal{D}(A,B) = \bigwedge_{x \in X, y \in Y} (1 - |a - R(x,y)|).$$

(2) In the case of product t-norm for describing $\mathcal{D}(A,B)$ we denote
$X_1 = \{x \in X \mid a \leq \bigwedge_{y \in B} R(x,y)\}$,
$X_2 = \{x \in X \mid a \geq \bigwedge_{y \in B} R(x,y)\}$. Then $\mathcal{D}(A,B) = \left(\bigwedge_{x \in X_1} \frac{a}{\bigwedge_{y \in B} R(x,y)} \right) \wedge \left(\bigwedge_{x \in X_2} \frac{\bigwedge_{y \in B} R(x,y)}{a} \right)$ if $X_1 \cup X_2 \neq \emptyset$ and $\mathcal{D}(A,B) = 1$ otherwise.

(3) In the case of minimum t-norm applying the notations from the previous example we have
$\mathcal{D}(A,B) = \begin{cases} \bigwedge_{x \in X_2, y \in Y} R(x,y) & \text{if } X_2 \neq \emptyset; \\ a & \text{otherwise .} \end{cases}$ □

3.5 $\mathcal{D}$-graded Preconcept Lattices

Given a fuzzy context (X,Y,L,R), let $(\mathbb{P}(X,Y,L,R), \preceq)$ be the corresponding fuzzy preconcept lattice, and let $\mathcal{D}^{\uparrow}$, $\mathcal{D}^{\downarrow}$ be the operators defined in the previous subsection. Then the tuple $(\mathbb{P}, \preceq, \mathcal{D}^{\uparrow}, \mathcal{D}^{\downarrow})$ will be referred to as a $\mathcal{D}$-graded fuzzy preconcept lattice of the fuzzy preconcept (X,Y,L,R). In the next theorems we provide the most important properties of the operators of $\mathcal{D}$-gradation and $\mathcal{D}$-graded fuzzy preconcept lattices.

Theorem 3.20. *Let $\mathbb{P} = (\mathbb{P}, \preceq \curlyvee, \barwedge)$ be a fuzzy preconcept lattice. Given a family of fuzzy preconcepts $\mathcal{P} = \{P_i = (A_i, B_i) \mid i \in I\} \subseteq \mathbb{P}$ it holds*

$$\mathcal{D}^{\uparrow}(\curlyvee_{i \in I} P_i) \geq \bigwedge_{i \in I} \mathcal{D}^{\uparrow}(P_i)$$

Theorem 3.21. *Given a family of fuzzy preconcepts $\mathcal{P} = \{P_i = (A_i, B_i) \mid i \in I\} \subseteq \mathbb{P}$ it holds*

$$\mathcal{D}^{\downarrow}(\barwedge_{i \in I} P_i) \geq \bigwedge_{i \in I} \mathcal{D}^{\downarrow}(P_i).$$

4 RISK ASSESSMENT AND FUZZY PRECONCEPT LATTICES

4.1 Risk Assessment Model

The risk assessment process often allows a combination of statistical data and qualitative evaluations based on expert opinions. When dealing with any risks of rare occurrence the whole society usually faces unique circumstances with no historic patterns. Such circumstances are often called the tail events in the terminology of the risk management. These events are characterised by low possibility, but huge impact. Pandemic scenario is among the most obvious examples. Therefore the current spread of Covid-19 pandemic provides a unique opportunity in studying possible pandemic developments from different angles. While a lot of statistical data have been available for almost a year, we should still combine the statistics with expert based opinions on possible further developments related to sudden virus outbreaks and mutations in order to propose the actions for, e.g. national governments or other decision making institutions. In its essence, the whole Covid-19 crisis management is dependent on fuzzy logic based decisions and provides a favourable preconditions for enhancing the fuzzy logic methods. Several medical studies based on fuzzy logic applications have already been carried out, see, e.g. (Orozco-del-Castillo et al., 2020), (Shaban et al., 2021). From a different perspective, it is always important to introduce the models for taking different decisions under these circumstances, such as modelling of availability of medical services, limiting or restricting public activities and services etc. We propose the analysis of data containing estimated total maximum number of infected inhabitants vs estimated total maximum number of hospitalised inhabitants at any fixed time moment as explained in the Table 1. It should be admitted that the estimated total number of infected inhabitants can be replaced with any other parameter characterizing the spread of pandemics, e.g. estimated cumulative cases, while the estimated total maximum number of hospitalised inhabitants at any fixed time moment is the most important parameter in the model characterising the crisis severity. In many instances such model could be expanded further to the analysis of mortality rate, but this stage illustrates the efficiency of the healthcare system and its ability to save the lives of inhabitants. In the example we also consider the values of total number of inhabitants of particular country and the maximum admissible number of hospitalised inhabitants, but these values are used just for the illustrative reference purposes as the benchmark values for assessment of severity of the pandemics. We introduce the model containing the following objects and attributes: set X contains estimated quantities of infected inhabitants and set Y contains estimated quantities of hospitalised inhabitants. In this context we treat X as a set of objects expressed in a vector format $x = (x_1, x_2, \dots, x_m)$ and Y as a set of attributes expressed in a vector format $y = (y_1, y_2, \dots, y_n)$. Furthermore, we introduce a fuzzy relation $R(x_i, y_j) = \alpha_{ij}$ which is a matrix containing corresponding expert opinion:

$$R(x_i, y_j) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}$$

This model can be applied for various purposes, including illustration of expert assessments, comparing of these assessments against real statistics, adjusting these assessments depending on new circumstances and taking decisions on implementation of public restrictions. For the sake of clarity, it should be noted that the proposed setup can be used for developing the model for any risk assessment, but we have chosen the tail events and Covid-19 crisis in particular for highlighting the importance of expert opinion. In the following section we show how the fuzzy preconcept lattice can be applied for estimation of the possible impact of Covid-19 crisis on the healthcare system in Latvia.

Table 1: The data for modelling of pandemic scenario. The following data are provided in the respective columns: 1 - Total Number of Country Inhabitants w, 2 - Estimated Total Maximum Number of Infected Inhabitants x_i, 3 - Estimated Total Maximum Number of Hospitalised Inhabitants y_j, 4 - Maximal Admissible Number of Hospitalised Inhabitants z.

1	2	3	4
w	x_1 x_2 $\vdots$ x_m	y_1 y_2 $\vdots$ y_n	z

Table 2: The data for modelling of pandemic scenario in Latvia. The following data are provided in the respective columns: 1 - Total Number of Inhabitants, 2 - Estimated Total Maximum Number of Infected Inhabitants, 3 - Estimated Total Maximum Number of Hospitalised Inhabitants, 4 - Maximal Admissible Number of Hospitalised Inhabitants.

1	2	3	4
1,920,000	50,000 80,000 110,000 140,000 170,000	500 1,000 1,500	3,000

4.2 Assessment of Possible Covid-19 Impact on the Healthcare System in Latvia

The management of Covid-19 crisis has created almost similar challenges in all countries around the world. Different public restrictions have been implemented and lifted based on actual rates of infected inhabitants, risks of new Covid-19 mutations and vaccination rates. In our opinion, the management of the public restrictions can be based on the proposed risk assessment model. Therefore the corresponding rates for Latvia provided in the Table 2 can be replaced with corresponding numbers for any country.

In the example we link the estimated total maximum number of infected inhabitants x with estimated total maximum number of hospitalised inhabitants y by applying the following fuzzy relation reflecting hypothetical expert opinion[2]:

$$R(x, y) = \begin{pmatrix} 0.9 & 0.9 & 0.4 \\ 0.9 & 0.8 & 0.4 \\ 0.6 & 0.5 & 0.3 \\ 0.5 & 0.4 & 0.3 \\ 0.4 & 0.3 & 0.2 \end{pmatrix}$$

[2]The values closer to 0 indicate lower possibility while the values closer to 1 indicate higher possibility.

This matrix is further analysed in such way that we consider the values of $R(x,y)$ for each particular row. First of all we analyse scenario when the values of x are considered as precise and calculate the values of $\mathcal{D}(A,B)$ as provided in the Example 2.21. In such case $a = 1$ and we obtain the following values for corresponding rows for the Łukasiewicz t-norm, the product t-norm and the minimum t-norm (all results are equal for the corresponding t-norms in this case, indices of $\mathcal{D}(A,B)$ denote the corresponding rows for which the corresponding values have been calculated): $\mathcal{D}_1(A,B) = 0.4$, $\mathcal{D}_2(A,B) = 0.4$, $\mathcal{D}_3(A,B) = 0.3$, $\mathcal{D}_4(A,B) = 0.3$, $\mathcal{D}_5(A,B) = 0.2$.

Furthermore we can assume that the values of x are not precise. In such case we fuzzify these values by applying different values of a. The following example shows the values of $\mathcal{D}(A,B)$ calculated for $a = 0.8$. For the Łukasiewicz t-norm we obtain the following values: $\mathcal{D}_1(A,B) = 0.6$, $\mathcal{D}_2(A,B) = 0.6$, $\mathcal{D}_3(A,B) = 0.5$, $\mathcal{D}_4(A,B) = 0.5$, $\mathcal{D}_5(A,B) = 0.4$. For the product t-norm we obtain the following values: $\mathcal{D}_1(A,B) = 0.5$, $\mathcal{D}_2(A,B) = 0.5$, $\mathcal{D}_3(A,B) = 0.375$, $\mathcal{D}_4(A,B) = 0.375$, $\mathcal{D}_5(A,B) = 0.25$. For the minimum t-norm we obtain the following values: $\mathcal{D}_1(A,B) = 0.4$, $\mathcal{D}_2(A,B) = 0.4$, $\mathcal{D}_3(A,B) = 0.3$, $\mathcal{D}_4(A,B) = 0.3$, $\mathcal{D}_5(A,B) = 0.2$. Another example shows the values of $\mathcal{D}(A,B)$ calculated for $a = 0.4$. Consequently, for the Łukasiewicz t-norm we obtain the following values: $\mathcal{D}_1(A,B) = 0.5$, $\mathcal{D}_2(A,B) = 0.5$, $\mathcal{D}_3(A,B) = 0.8$, $\mathcal{D}_4(A,B) = 0.9$, $\mathcal{D}_5(A,B) = 0.8$. For the product t-norm we obtain the following values: $\mathcal{D}_1(A,B) = 0.44$, $\mathcal{D}_2(A,B) = 0.44$, $\mathcal{D}_3(A,B) = 0.67$, $\mathcal{D}_4(A,B) = 0.75$, $\mathcal{D}_5(A,B) = 0.5$. For the minimum t-norm we obtain the following values: $\mathcal{D}_1(A,B) = 0.4$, $\mathcal{D}_2(A,B) = 0.4$, $\mathcal{D}_3(A,B) = 0.3$, $\mathcal{D}_4(A,B) = 0.3$, $\mathcal{D}_5(A,B) = 0.2$.

These results reflect differences in the impact of the t-norms used in the above calculations. On the practical side we can foresee that application of Łukasiewicz t-norm returns higher values indicating possibility of more severe pandemic impact. At the same time we should admit that these results should not be interpreted as correct or incorrect, but just as illustration of different expert opinions and application of different t-norms in their analysis.

5 CONCLUSIONS

The research has been focused on the application of an alternative method in risk assessment that is based on application of graded fuzzy preconcept lattices. Graded fuzzy preconcept lattices were introduced and studied in the paper (Šostak et al., 2021), their basic properties are expounded also in our earlier papers. The use of graded preconcept lattices could be more appropriate for practical problems, in particular, those ones that are related to risk assessment, than fuzzy concept lattices due to the possibility to establish a more flexible relationship between the fuzzy set of objects and corresponding fuzzy set of attributes. In this paper, graded fuzzy preconcept lattices are applied for the analysis of pandemic spread by introducing the model of estimated total number of infected inhabitants vs estimated levels of hospitalised inhabitants which have been assessed based on expert opinion. The application of different t-norms in the calculation of the degree of conceptuality revealed differences in the impact of the selected t-norms on the assessment of the crisis severity. The obtained results can be further applied in the implementation of public restrictions aimed at containment of Covid-19 pandemics. The results of our research can be extended to comparing opinions from different experts. The aggregation of multiple opinions analysed by means of fuzzy preconcept lattices is subject to our further research.

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