Multiobjective Bimatrix Game with Fuzzy Payoffs and Its Solution Method using Necessity Measure and Weighted Tchebycheff Norm

Hitoshi Yano[1][a] and Ichiro Nishizaki[2][b]

[1]*Graduate School of Humanities and Social Sciences, Nagoya City University, Nagoya, 467-8501, Japan*

[2]*Graduate School of Advanced Science and Engineering, Hiroshima University, Higashi-Hiroshima, 739-8527, Japan*

Keywords: Bimatrix Games with Fuzzy Payoffs, Multiobjective Programming, Necessity Measure, Interactive Method, Weighted Tchebycheff Norm Method.

Abstract: In this paper, we propose an interactive algorithm for multiobjective bimatrix games with fuzzy payoffs. Using necessity measure and the weighted Tchebycheff norm method, an equilibrium solution concept is defined, which depends on weighting vectors specified by each player. Since it is very difficult to obtain such equilibrium solutions directly, instead of equilibrium conditions in the necessity measure space, equilibrium conditions in the expected payoff space are provided. Under the assumption that a player can estimate the opponent player's preference as the weighting vector of the weighted Tchebycheff norm method, the interactive algorithm is proposed to obtain a satisfactory solution of the player from among an equilibrium solution set by updating the weighting vector.

1 INTRODUCTION

Recently, various types of noncooperative games under uncertainty in strategic form have been investigated, and the corresponding equilibrium solution concepts have been proposed (Larbani, 2009). Campos (Campos, 1989) first formulated two-person zero-sum games with fuzzy payoffs. In her method, under the assumption that each element of a fuzzy payoff matrix is defined as a triangular fuzzy number (Dubois and Prade, 1980), such games are reduced to two kinds of linear programming problems by using Yager's method (Yager, 1981). Similarly, Li (Li, 1999) formulated two-person zero-sum games with triangular fuzzy numbers as two kinds of multiobjective programming problems, in which each objective function is corresponding to the extreme point of a triangular fuzzy number. Bector et al. (Bector et al., 2004) also formulated two-person zero-sum games with fuzzy payoffs as two kinds of optimization problems which depends on the defuzzification functions (Yager, 1981). Moreover, using the threshold values for the level sets (Dubois and Prade, 1980) and the ordering relation called the fuzzy max order, Maeda (Maeda, 2003) reduced two-person zero-sum games with triangular fuzzy numbers to two kinds of linear programming problems.

On the other hand, to deal with bimatrix games with triangular fuzzy numbers, Maeda (Maeda, 2000) defined an equilibrium solution concept using possibility measure and the threshold values for the level sets (Dubois and Prade, 1980). He formulated the corresponding mathematical programming problem to obtain such parametric equilibrium solutions. Mako et al. (Makó and Salamon, 2020) focused on bimatrix games with *LR* fuzzy numbers. Corresponding to the fuzzy Nash-equilibrium solution concept, they proposed the fuzzy correlated equilibrium solution concept, which is based on a joint distribution for mixed strategies of both players. Gao (Gao, 2013) introduced three kinds of uncertain equilibrium solution concepts based on uncertainty theory (Liu, 2007), which depend on the values of confidence levels. From a similar point of view based on uncertainty theory, Tang et al. (Tang and Li, 2020) proposed an uncertain equilibrium solution concept based on the Hurwicz criterion.

For multiobjective bimatrix games, Corley (Corley, 1985) first defined a Pareto equilibrium solution concept, and formulated quadratic programming problems to obtain Pareto equilibrium solutions through the Karush-Kuhn-Tucker conditions, in which multiobjective functions are scalarized by the

[a] https://orcid.org/0000-0002-4818-5695
[b] https://orcid.org/0000-0002-0060-4360

Yano, H. and Nishizaki, I.
Multiobjective Bimatrix Game with Fuzzy Payoffs and Its Solution Method using Necessity Measure and Weighted Tchebycheff Norm.
DOI: 10.5220/0010630700003063
In *Proceedings of the 13th International Joint Conference on Computational Intelligence (IJCCI 2021)*, pages 159-166
ISBN: 978-989-758-534-0; ISSN: 2184-3236

weighting coefficients. Nishizaki et al. (Nishizaki and Sakawa, 1995) formulated multiobjective bimatrix games incorporating fuzzy goals. They transformed multiobjective bimatrix games into usual bimatrix games by applying the weighting methods or the minimum operator (Sakawa, 1993), and defined the corresponding equilibrium solution concepts. They formulated the nonlinear programming problems to obtain such equilibrium solutions. Using dominance cones proposed by Yu (Yu, 1985), Nishizaki et al. (Nishizaki and Notsu, 2007) defined a nondominated equilibrium solution concept which is a generalization of Nash equilibrium solution concept, and formulate nonlinear programming problem to obtain nondominated equilibrium solutions by applying the Tamura and Miura conditions (Tamura and Miura, 1979).

In such a situation, in this paper, we propose an interactive algorithm for multiobjective bimatrix games with fuzzy payoffs. Using necessity measure (Dubois and Prade, 1980) and the weighted Tchebycheff norm method (Bowman, 1976), an equilibrium solution concept is introduced, which depends on weighting vectors of both players. To obtain such equilibrium solutions, the relationships between the equilibrium conditions in the membership function space and the equilibrium conditions in the expected payoff space. Under the assumption that a player can estimate the opponent player's weighting vector, the interactive algorithm is proposed to obtain a satisfactory solution of the player from among an equilibrium solution set by updating the weighting vector.

In section 2, we propose an interactive decision making method for multiobjective bimatrix games with fuzzy payoffs. Using necessity measure and the weighted Tchebycheff norm method, an equilibrium solution concept is introduced. An interactive algorithm is developed to obtain a satisfactory solution of a player from among a equilibrium solution set by updating the weighting vector. In section 3, a numerical example of two-objectve bimatrix games with fuzzy payoffs illustrates interactive processes under a hypothetical player to show the efficiency of the proposed method.

2 MULTIOBJECTIVE BIMATRIX GAMES WITH FUZZY PAYOFFS

In this section, we consider multiobjective bimatrix games with fuzzy payoffs. Let $i \in \{1,2,\cdots,m\}$ be a pure strategy of Player 1 and $j \in \{1,2,\cdots,n\}$ be a pure strategy of Player 2. $\tilde{A}_k = (\tilde{a}_{kij}), k = 1,\ldots,K$ are Player 1's $(m \times n)$-payoff matrices, and $\tilde{B}_l = (\tilde{b}_{lij}), l = 1,\cdots,L$ are Player 2's $(m \times n)$-payoff matrices. Elements $\tilde{a}_{kij}$ and $\tilde{b}_{lij}$ are *LR* fuzzy numbers (Dubois and Prade, 1980) whose membership functions are defined as follows.

$$\mu_{\tilde{a}_{kij}}(s) = \begin{cases} L_1\left(\frac{a_{kij}-s}{\alpha_{kij}}\right) & a_{kij} - \alpha_{kij} \leq s \leq a_{kij} \\ R_1\left(\frac{s-a_{kij}}{\beta_{kij}}\right) & a_{kij} \leq s \leq a_{kij} + \beta_{kij} \end{cases}$$

$$\mu_{\tilde{b}_{lij}}(t) = \begin{cases} L_2\left(\frac{b_{lij}-t}{\gamma_{lij}}\right) & b_{lij} - \gamma_{lij} \leq t \leq b_{lij} \\ R_2\left(\frac{t-b_{lij}}{\delta_{lij}}\right) & b_{lij} \leq t \leq b_{lij} + \delta_{lij} \end{cases}$$

where $L_1(\cdot)$ is a type function (Dubois and Prade, 1980) which is strictly monotone decreasing on the interval $[0,1]$, and $L_1(0) = 1, L_1(1) = 0$. For the other type functions $R_1(\cdot), L_2(\cdot)$ and $R_2(\cdot)$, similar conditions are satisfied. In the following, *LR* fuzzy numbers $\tilde{a}_{kij}$ and $\tilde{b}_{lij}$ are denoted as $(a_{kij}, \alpha_{kij}, \beta_{kij})_{LR}$ and $(b_{lij}, \gamma_{lij}, \delta_{lij})_{LR}$, respectively. Let

$$X \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathrm{R}^m \mid \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1,\cdots,m\}$$

$$Y \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathrm{R}^n \mid \sum_{j=1}^{n} y_j = 1, y_j \geq 0, j = 1,\cdots,n\}$$

be sets of mixed strategies for Players 1 and 2. Then, a multiobjective bimatrix game with fuzzy payoffs is formally expressed as follows.

P1

$$\underset{\mathbf{x} \in X}{\text{maximize}}\ (\mathbf{x}^T \tilde{A}_1 \mathbf{y}, \ldots, \mathbf{x}^T \tilde{A}_K \mathbf{y}) \tag{1a}$$

$$\underset{\mathbf{y} \in Y}{\text{maximize}}\ (\mathbf{x}^T \tilde{B}_1 \mathbf{y}, \ldots, \mathbf{x}^T \tilde{B}_L \mathbf{y}) \tag{1b}$$

It should be noted here that fuzzy expected payoffs $\mathbf{x}^T \tilde{A}_k \mathbf{y}, k = 1,\cdots,K$ and $\mathbf{x}^T \tilde{B}_l \mathbf{y}, l = 1,\cdots,L$ are expressed as *LR* fuzzy numbers because of the property (Dubois and Prade, 1980) of *LR* fuzzy numbers.

In P1, it is assumed that Players 1 and 2 have fuzzy goals $\tilde{G}_{1k}, k = 1,\cdots,K, \tilde{G}_{2l}, l = 1,\cdots,L$ for their expected payoffs, and the corresponding membership functions $\mu_{\tilde{G}_{1k}}(s), k = 1,\cdots,K,\ \mu_{\tilde{G}_{2l}}(t), l = 1,\cdots,L$ are linear or nonlinear functions, which are continuous and strictly increasing on the corresponding support for *LR* fuzzy numbers $\mathbf{x}^T \tilde{A}_k \mathbf{y}$ or $\mathbf{x}^T \tilde{B}_l \mathbf{y}$, *i.e.*,

$$S \stackrel{\text{def}}{=} \{s \in R^1 \mid \mu_{\mathbf{x}^T \tilde{A}_k \mathbf{y}}(s) > 0\}, \tag{2}$$

$$T \stackrel{\text{def}}{=} \{t \in R^1 \mid \mu_{\mathbf{x}^T \tilde{B}_l \mathbf{y}}(t) > 0\}. \tag{3}$$

To deal with fuzzy expected payoffs $\mathbf{x}^T \tilde{A}_k \mathbf{y}$ and $\mathbf{x}^T \tilde{B}_l \mathbf{y}$, we introduce concepts of the possibility measure and the necessity measure (Dubois and Prade,

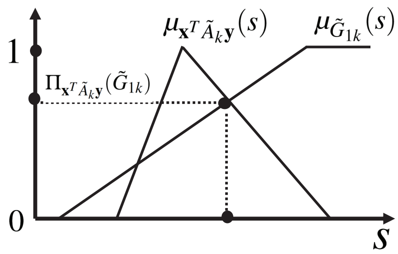

Figure 1: The possibility measure $\Pi_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(\tilde{G}_{1k})$.

1980). A possibility degree that fuzzy expected payoffs $\mathbf{x}^T\tilde{A}_k\mathbf{y}$ attains the fuzzy goal $\tilde{G}_{1k}$ can be expressed by using a possibility measure defined as follows.

$$\Pi_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(\tilde{G}_{1k}) \stackrel{\text{def}}{=} \sup_s \min\left(\mu_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(s), \mu_{\tilde{G}_{1k}}(s)\right)$$

Fig.1 shows the relationship between the possibility measure $\Pi_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(\tilde{G}_{1k})$ and the membership functions $\mu_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(u)$ and $\mu_{\tilde{G}_{1k}}(u)$. Based on the possibility measure, the following necessity measure (Dubois and Prade, 1980) is defined as a non-possibility degree that the "complement" of fuzzy expected payoffs $\mathbf{x}^T\tilde{A}_k\mathbf{y}$ attains the fuzzy goal $\tilde{G}_{1k}$.

$$N_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(\tilde{G}_{1k}) \stackrel{\text{def}}{=} \inf_s \max\left(1-\mu_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(s), \mu_{\tilde{G}_{1k}}(s)\right)$$

Dubois and Prade (1980) interpreted the above function as a necessity measure by analogy with modal logic where "a subset C is necessary" is equivalent to "non-C is not possible". Fig.2 shows the relationship between the necessity measure $N_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(\tilde{G}_{1k})$ and the membership functions $\mu_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(u)$ and $\mu_{\tilde{G}_{1k}}(u)$.

In this paper, we adopt the above mentioned necessity measure to deal with fuzzy expected payoffs. Then, P1 can be interpreted as the following problem.

P2

$$\underset{\mathbf{x}\in X}{\text{maximize}} \left(N_{\mathbf{x}^T\tilde{A}_1\mathbf{y}}(\tilde{G}_{11}), \ldots, N_{\mathbf{x}^T\tilde{A}_K\mathbf{y}}(\tilde{G}_{1K})\right) \quad (4a)$$

$$\underset{\mathbf{y}\in Y}{\text{maximize}} \left(N_{\mathbf{x}^T\tilde{B}_1\mathbf{y}}(\tilde{G}_{21}), \ldots, N_{\mathbf{x}^T\tilde{B}_L\mathbf{y}}(\tilde{G}_{2L})\right), \quad (4b)$$

where the necessity measures are defined as follows (see Fig.2 and Fig.3).

$$N_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(\tilde{G}_{1k}) \stackrel{\text{def}}{=} \inf_s \max\left(1-\mu_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(s), \mu_{\tilde{G}_{1k}}(s)\right),$$
$$k = 1, \ldots, K \quad (5)$$

$$N_{\mathbf{x}^T\tilde{B}_l\mathbf{y}}(\tilde{G}_{2l}) \stackrel{\text{def}}{=} \inf_t \max\left(1-\mu_{\mathbf{x}^T\tilde{B}_l\mathbf{y}}(t), \mu_{\tilde{G}_{2l}}(t)\right),$$
$$l = 1, \ldots, L \quad (6)$$

From the assumption of the membership functions $\mu_{\tilde{G}_{1k}}(s), k = 1, \ldots, K$, $\mu_{\tilde{G}_{2l}}(t), l = 1, \ldots, L$, the following relations always hold.

$$0 < N_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(\tilde{G}_{1k}) < 1,\ k = 1, \ldots, K, \forall\mathbf{x}\in X, \forall\mathbf{y}\in Y$$
$$0 < N_{\mathbf{x}^T\tilde{B}_l\mathbf{y}}(\tilde{G}_{2l}) < 1,\ l = 1, \ldots, L, \forall\mathbf{x}\in X, \forall\mathbf{y}\in Y$$

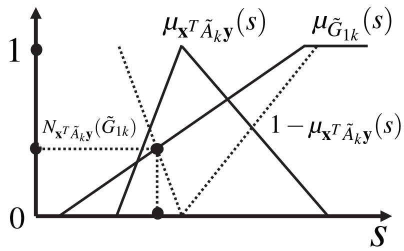

Figure 2: The necessity measure $N_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(\tilde{G}_{1k})$.

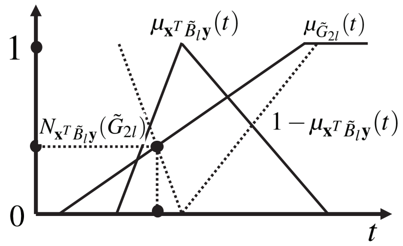

Figure 3: The necessity measure $N_{\mathbf{x}^T\tilde{B}_l\mathbf{y}}(\tilde{G}_{2l})$.

By applying the weighted Tchebycheff norm method, P2 can be transformed into a bimatrix game defined as follows.

P3($\mathbf{w}_1, \mathbf{w}_2$)

$$\underset{\mathbf{x}\in X}{\text{maximize}} \min_{k=1,\ldots,K} N_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(\tilde{G}_{1k})/w_{1k} \quad (7a)$$

$$\underset{\mathbf{y}\in Y}{\text{maximize}} \min_{l=1,\ldots,L} N_{\mathbf{x}^T\tilde{B}_l\mathbf{y}}(\tilde{G}_{2l})/w_{2l} \quad (7b)$$

where $\mathbf{w}_1 \stackrel{\text{def}}{=} (w_{11}, \cdots, w_{1K}) \in W_1$ and $\mathbf{w}_2 \stackrel{\text{def}}{=} (w_{21}, \cdots, w_{2L}) \in W_2$ are the weighting vectors for the necessity measures, and

$$W_1 \stackrel{\text{def}}{=} \{\mathbf{w}_1 \in R^K \mid \sum_{k=1}^{K} w_{1k} = 1, w_{1k} > 0, k = 1, \cdots, K\},$$

$$W_2 \stackrel{\text{def}}{=} \{\mathbf{w}_2 \in R^L \mid \sum_{l=1}^{L} w_{2l} = 1, w_{2l} > 0, l = 1, \cdots, L\}.$$

Now, we can introduce an equilibrium solution concept for P3($\mathbf{w}_1, \mathbf{w}_2$), which depends on the weighting vectors.

Definition 1. $(\mathbf{x}^*, \mathbf{y}^*)$ *is an equilibrium solution to P3($\mathbf{w}_1, \mathbf{w}_2$),if the following inequalities hold.*

$$\min_{k=1,\ldots,K} N_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k}$$
$$\geq \min_{k=1,\ldots,K} N_{\mathbf{x}^T\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k},\ \forall\mathbf{x}\in X \quad (8a)$$

$$\min_{l=1,\ldots,L} N_{\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}^*}(\tilde{G}_{2l})/w_{2l}$$
$$\geq \min_{l=1,\ldots,L} N_{\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}}(\tilde{G}_{2l})/w_{2l},\ \forall\mathbf{y}\in Y \quad (8b)$$

It is very difficult to obtain the equilibrium solution to P3($\mathbf{w}_1, \mathbf{w}_2$) from the computational aspect, because of the definition (5), (6), and the nonlinear membership functions $\mu_{\tilde{G}_{1k}}(s), k = 1,\ldots,K$, $\mu_{\tilde{G}_{2l}}(t), l = 1,\cdots,L$. To circumvent such a difficulty, at first, we consider the following bimatrix game, which is equivalent to P3($\mathbf{w}_1,\mathbf{w}_2$).

P4($\mathbf{w}_1,\mathbf{w}_2$)

$$\underset{\mathbf{x}\in X,\ v_1\in\mathrm{R}^1}{\text{maximize}}\ \ v_1$$
subject to
$$N_{\mathbf{x}^T\tilde{A}_k\mathbf{y}}(\tilde{G}_{1k})/w_{1k} \geq v_1,\ k = 1,\ldots,K \tag{9a}$$
$$\underset{\mathbf{y}\in Y,\ v_2\in\mathrm{R}^1}{\text{maximize}}\ \ v_2$$
subject to
$$N_{\mathbf{x}^T\tilde{B}_l\mathbf{y}}(\tilde{G}_{2l})/w_{2l} \geq v_2,\ l = 1,\ldots,L \tag{9b}$$

Let $(\mathbf{x}^*,\mathbf{y}^*,v_1^*,v_2^*)$ be an equilibrium solution to P4($\mathbf{w}_1,\mathbf{w}_2$). Then, the following equalities hold at $(\mathbf{x}^*,\mathbf{y}^*,v_1^*,v_2^*)$.

$$\min_{k=1,\ldots,K}\left(N_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k}\right) = v_1^* \tag{10a}$$
$$\min_{l=1,\ldots,L}\left(N_{\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}^*}(\tilde{G}_{2l})/w_{2l}\right) = v_2^* \tag{10b}$$

We denote α-level sets for *LR* fuzzy numbers $\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*$ and $\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}^*$ as the closed intervals :

$$(\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*)_\alpha \overset{\text{def}}{=} [\mathbf{x}^{*T}A^L_{k,\alpha}\mathbf{y}^*, \mathbf{x}^{*T}A^R_{k,\alpha}\mathbf{y}^*],$$
$$(\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}^*)_\alpha \overset{\text{def}}{=} [\mathbf{x}^{*T}B^L_{l,\alpha}\mathbf{y}^*, \mathbf{x}^{*T}B^R_{l,\alpha}\mathbf{y}^*],$$

respectively, where $A^L_{k,\alpha} \overset{\text{def}}{=} (a^L_{kij,\alpha})$, $A^R_{k,\alpha} \overset{\text{def}}{=} (a^R_{kij,\alpha})$, $B^L_{l,\alpha} \overset{\text{def}}{=} (b^L_{lij,\alpha})$ $B^R_{l,\alpha} \overset{\text{def}}{=} (b^R_{lij,\alpha})$ are $(m\times n)$-matrices. $a^L_{kij,\alpha}$ and $a^R_{lij,\alpha}$ mean the left and the right hand side extreme points of the α-level set for $\tilde{a}_{kij}$. Similarly, $b^L_{kij,\alpha}$ and $b^R_{lij,\alpha}$ mean the extreme points of the α-level set for $\tilde{b}_{kij}$.

Now, let us focus on the equality (10a). We denote the left hand side of the membership function $\mu_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(s)$ as $\mu^L_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(s)$. Since the membership function $\mu_{\tilde{G}_{1k}}(s)$ is strictly monotone increasing with respect to $s\in S$, the equality (10a) means that $1-\mu^L_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(s) \geq w_{1k}v_1^*$, $\mu_{\tilde{G}_{1k}}(s) \geq w_{1k}v_1^*$. Since $\mu^L_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(s)$ is strictly monotone increasing with respect to $s\in S$, it holds that $(\mu^L_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*})^{-1}(1-w_{1k}v_1^*) \geq s$ and $\mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*) \leq s$. As a result, we can obtain the following inequalities.

$$(\mu^L_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*})^{-1}(1-w_{1k}v_1^*) \geq \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*), k = 1,\cdots,K$$

It should be noted here that, from the property of the *LR* fuzzy number $\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*$, we can express the

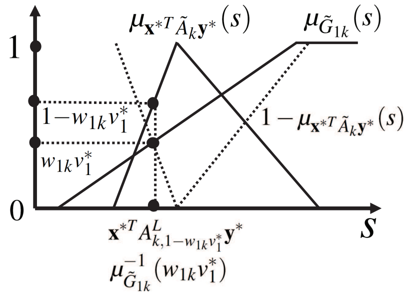

Figure 4: The relationship between $N_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})$ and $\mathbf{x}^{*T}A^L_{k,1-w_{1k}v_1^*}\mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*) = 0$.

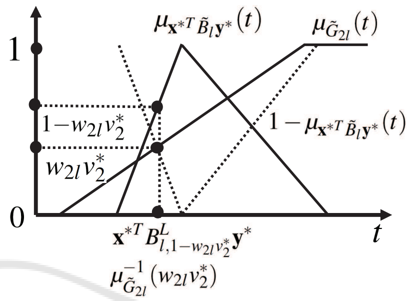

Figure 5: The relationship between $N_{\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}^*}(\tilde{G}_{2l})$ and $\mathbf{x}^{*T}B^L_{l,1-w_{2l}v_2^*}\mathbf{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*) = 0$.

above inequalities as follows, where $A^L_{k,1-w_{1k}v_1^*} \overset{\text{def}}{=} (a^L_{kij,1-w_{1k}v_1^*})$.

$$\mathbf{x}^{*T}A^L_{k,1-w_{1k}v_1^*}\mathbf{y}^* \geq \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*), k = 1,\cdots,K$$

Similarly, from the equality (10b), we can obtain the following inequalities, where $B^L_{l,1-w_{2l}v_2^*} \overset{\text{def}}{=} (b^L_{lij,1-w_{2l}v_2^*})$.

$$\mathbf{x}^{*T}B^L_{l,1-w_{2l}v_2^*}\mathbf{y}^* \geq \mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*), l = 1,\cdots,L$$

From such a point of view, the equalities (10a) and (10b) are equivalent to the following ones (see Fig.4 and Fig.5).

$$\min_{k=1,\ldots,K}\left(\mathbf{x}^{*T}A^L_{k,1-w_{1k}v_1^*}\mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right) = 0 \tag{11a}$$
$$\min_{l=1,\ldots,L}\left(\mathbf{x}^{*T}B^L_{l,1-w_{2l}v_2^*}\mathbf{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\right) = 0 \tag{11b}$$

Corresponding to (11a) and (11b), we consider the following bimatrix game, in which not only $(\mathbf{w}_1,\mathbf{w}_2)$ but also (v_1^*,v_2^*) are given as parameters.

P5($\mathbf{w}_1,\mathbf{w}_2;v_1^*,v_2^*$)

$$\underset{\mathbf{x}\in X}{\text{maximize}}\ \min_{k=1,\ldots,K}\{\mathbf{x}^TA^L_{k,1-w_{1k}v_1^*}\mathbf{y} - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\}$$
$$\underset{\mathbf{y}\in Y}{\text{maximize}}\ \min_{l=1,\ldots,L}\{\mathbf{x}^TB^L_{l,1-w_{2l}v_2^*}\mathbf{y} - \mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\}$$

For P5($\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$), we introduce an equilibrium solution concept.

Definition 2. $(\mathbf{x}^*, \mathbf{y}^*)$ *is an equilibrium solution to P5(*$\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$*), if the following inequalities hold.*

$$\begin{aligned}
&\min_{k=1,\ldots,K} \{\mathbf{x}^{*T} A^L_{k,1-w_{1k}v_1^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\} \\
&\geq \min_{k=1,\ldots,K} \{\mathbf{x}^T A^L_{k,1-w_{1k}v_1^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\},\ \forall \mathbf{x} \in X \\
&\min_{l=1,\ldots,L} \{\mathbf{x}^{*T} B^L_{l,1-w_{2l}v_2^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\} \\
&\geq \min_{l=1,\ldots,L} \{\mathbf{x}^{*T} B^L_{l,1-w_{2l}v_2^*} \mathbf{y} - \mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\},\ \forall \mathbf{y} \in Y
\end{aligned}$$

Then, the following relationships between equilibrium solutions to P5($\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$) and equilibrium solutions to P4($\mathbf{w}_1, \mathbf{w}_2$) hold.

Theorem 1. *If* $(\mathbf{x}^*, \mathbf{y}^*, v_1^*, v_2^*)$ *is an equilibrium solution to P4(*$\mathbf{w}_1, \mathbf{w}_2$*), then* $(\mathbf{x}^*, \mathbf{y}^*)$ *is an equilibrium solution to P5(*$\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$*).*

(Proof) : Assume that $(\mathbf{x}^*, \mathbf{y}^*)$ is not an equilibrium solution to P5($\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$). Then, there exists some $\mathbf{x} \in X$ such that

$$\begin{aligned}
&\min_{k=1,\ldots,K} \left\{\mathbf{x}^{*T} A^L_{k,1-w_{1k}v_1^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right\} < \\
&\min_{k=1,\ldots,K} \left\{\mathbf{x}^{T} A^L_{k,1-w_{1k}v_1^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right\}, \quad (14)
\end{aligned}$$

or, there exists some $\mathbf{y} \in Y$ such that

$$\begin{aligned}
&\min_{l=1,\ldots,L} \left\{\mathbf{x}^{*T} B^L_{l,1-w_{2l}v_2^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\right\} < \\
&\min_{l=1,\ldots,L} \left\{\mathbf{x}^{*T} B^L_{l,1-w_{2l}v_2^*} \mathbf{y} - \mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\right\}. \quad (15)
\end{aligned}$$

Assume that there exists some $\mathbf{x} \in X$ such that the inequality (14) is satisfied. Then, from (11a), the following relation holds.

$$\begin{aligned}
0 &= \min_{k=1,\cdots,K} \left(\mathbf{x}^{*T} A^L_{k,1-w_{1k}v_1^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right) \\
&< \min_{k=1,\cdots,K} \left(\mathbf{x}^{T} A^L_{k,1-w_{1k}v_1^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right)
\end{aligned}$$

Since (11a) is equivalent to (10a), the above relation is equivalent to the following inequality.

$$\begin{aligned}
v_1^* &= \min_{k=1,\cdots,K} \left(N_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k}\right) \\
&< \min_{k=1,\cdots,K} \left(N_{\mathbf{x}^{T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k}\right).
\end{aligned}$$

This contradicts the fact that $(\mathbf{x}^*, \mathbf{y}^*, v_1^*, v_2^*)$ is an equilibrium solution to P4($\mathbf{w}_1, \mathbf{w}_2$). Similarly, we can prove for the case that there exists $\mathbf{y} \in Y$ such that (15) is satisfied.

Theorem 2. *If* $(\mathbf{x}^*, \mathbf{y}^*)$ *is an equilibrium solution to P5(*$\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$*), where the following relations hold,*

$$\min_{k=1,\cdots,K} \left(\mathbf{x}^{*T} A^L_{k,1-w_{1k}v_1^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right) = 0, \quad (16)$$

$$\min_{l=1,\cdots,L} \left(\mathbf{x}^{*T} B^L_{l,1-w_{2l}v_2^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\right) = 0, \quad (17)$$

then, $(\mathbf{x}^*, \mathbf{y}^*, v_1^*, v_2^*)$ *is an equilibrium solution to P4(*$\mathbf{w}_1, \mathbf{w}_2$*).*

(Proof) : Assume that $(\mathbf{x}^*, \mathbf{y}^*, v_1^*, v_2^*)$ is not an equilibrium solution to P4($\mathbf{w}_1, \mathbf{w}_2$). Then, there exists some $\mathbf{x} \in X$ such that

$$\begin{aligned}
&\min_{k=1,\ldots,K} N_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k} \\
<\ &\min_{k=1,\ldots,K} N_{\mathbf{x}^{T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k}, \quad (18)
\end{aligned}$$

or, there exists some $\mathbf{y} \in Y$ such that

$$\begin{aligned}
&\min_{l=1,\ldots,L} N_{\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}^*}(\tilde{G}_{2l})/w_{2l} \\
<\ &\min_{l=1,\ldots,L} N_{\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}}(\tilde{G}_{2l})/w_{2l}. \quad (19)
\end{aligned}$$

Assume that there exists some $\mathbf{x} \in X$ such that (18) is satisfied. Then, from (16), it holds that

$$\begin{aligned}
v_1^* &= \min_{k=1,\ldots,K} \left(N_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k}\right) \\
&< \min_{k=1,\ldots,K} \left(N_{\mathbf{x}^{T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k}\right).
\end{aligned}$$

Since (10a) is equivalent to (11a), the following relation holds.

$$\begin{aligned}
0 &= \min_{k=1,\cdots,K} \left(\mathbf{x}^{*T} A^L_{k,1-w_{1k}v_1^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right) \\
&< \min_{k=1,\cdots,K} \left(\mathbf{x}^{T} A^L_{k,1-w_{1k}v_1^*} \mathbf{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right)
\end{aligned}$$

This contradicts the fact that $(\mathbf{x}^*, \mathbf{y}^*)$ is an equilibrium solution to P5($\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$). Similarly, we can prove for the case that there exists $\mathbf{y} \in Y$ such that (19) is satisfied.

From the above theorems, instead of solving P4($\mathbf{w}_1, \mathbf{w}_2$) directly, we can obtain an equilibrium solution to P4($\mathbf{w}_1, \mathbf{w}_2$) by solving P5($\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$), where (v_1^*, v_2^*) satisfies the equality conditions (16) and (17). On the other hand, an equilibrium solution to P5($\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$) is obtained by solving the following nonlinear programming problem (Nishizaki and Sakawa, 1995).

P6$(\mathbf{w}_1,\mathbf{w}_2;v_1^*,v_2^*)$

$$\underset{\mathbf{x}\in X,\ \mathbf{y}\in Y,\ p,\ q,\ \sigma_1,\ \sigma_2}{\text{maximize}} \quad \sigma_1+\sigma_2-p-q$$

subject to

$$A^L_{k,1-w_{1k}v_1^*}\mathbf{y}-\mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\mathbf{e}_1 \le p\mathbf{e}_1,\ k=1,\dots,K \tag{20a}$$

$$\mathbf{x}^T B^L_{l,1-w_{2l}v_2^*}-\mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\mathbf{e}_2 \le q\mathbf{e}_2,\ l=1,\dots,L \tag{20b}$$

$$\mathbf{x}^T A^L_{k,1-w_{1k}v_1^*}\mathbf{y}-\mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*) \ge \sigma_1,\ k=1,\dots,K \tag{20c}$$

$$\mathbf{x}^T B^L_{l,1-w_{2l}v_2^*}\mathbf{y}-\mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*) \ge \sigma_2,\ l=1,\dots,L, \tag{20d}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are $(m\times 1)$ and $(n\times 1)$ column vectors whose elements are all ones. It should be noted here that $p\ge\sigma_1$, $q\ge\sigma_2$, and $\sigma_1+\sigma_2-p-q\le 0$ always hold, because of the constraints in P6$(\mathbf{w}_1,\mathbf{w}_2;v_1^*,v_2^*)$.

The following theorem shows the relationship between an optimal solution to P6$(\mathbf{w}_1,\mathbf{w}_2;v_1^*,v_2^*)$ and an equilibrium solution to P4$(\mathbf{w}_1,\mathbf{w}_2)$.

Theorem 3. *Let* $(\mathbf{x}^*,\mathbf{y}^*,p^*,q^*,\sigma_1^*,\sigma_2^*)$ *be an optimal solution to P6*$(\mathbf{w}_1,\mathbf{w}_2;v_1^*,v_2^*)$*. If* $\sigma_1^*=p^*=0$, $\sigma_2^*=q^*=0$*, then* $(\mathbf{x}^*,\mathbf{y}^*)$ *is an equilibrium solution to P4*$(\mathbf{w}_1,\mathbf{w}_2)$*.*

(Proof) : Since $(\mathbf{x}^*,\ \mathbf{y}^*)$, $p^*=q^*=\sigma_1^*=\sigma_2^*=0$ is a feasible solution to P6$(\mathbf{w}_1,\mathbf{w}_2;v_1^*,v_2^*)$, the following inequalities hold.

$$A^L_{k,1-w_{1k}v_1^*}\mathbf{y}^*-\mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\mathbf{e}_1 \le \mathbf{0},\ k=1,\dots,K \tag{21a}$$

$$\mathbf{x}^{*T} B^L_{l,1-w_{2l}v_2^*}-\mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\mathbf{e}_2 \le \mathbf{0},\ l=1,\dots,L \tag{21b}$$

$$\mathbf{x}^{*T} A^L_{k,1-w_{1k}v_1^*}\mathbf{y}^*-\mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*) \ge 0,\ k=1,\dots,K \tag{21c}$$

$$\mathbf{x}^{*T} B^L_{l,1-w_{2l}v_2^*}\mathbf{y}^*-\mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*) \ge 0,\ l=1,\dots,L, \tag{21d}$$

From (21c) and (21d), it holds that

$$\min_{k=1,\cdots,K}\left(\mathbf{x}^* A^L_{k,1-w_{1k}v_1}\mathbf{y}^*-\mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right)=0,$$

$$\min_{l=1,\cdots,L}\left(\mathbf{x}^* B^L_{l,1-w_{2l}v_2}\mathbf{y}^*-\mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\right)=0.$$

This means that the following equalities hold.

$$\min_{k=1,\dots,K}\left(N_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k}\right)=v_1^* \tag{22a}$$

$$\min_{l=1,\dots,L}\left(N_{\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}^*}(\tilde{G}_{2l})/w_{2l}\right)=v_2^* \tag{22b}$$

On the other hand, from (21a) and (21c), the following inequality holds.

$$\begin{aligned}&\min_{k=1,\cdots,K}\left(\mathbf{x}^* A^L_{k,1-w_{1k}v_1}\mathbf{y}^*-\mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right)\\ &\ge \min_{k=1,\dots,K}\left(\mathbf{x}^T A^L_{k,1-w_{1k}v_1^*}\mathbf{y}^*-\mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)\right),\\ &\forall x\in X\end{aligned} \tag{23a}$$

From (21b) and (21d), the following inequality holds.

$$\begin{aligned}&\min_{l=1,\cdots,L}\left(\mathbf{x}^* B^L_{l,1-w_{2l}v_2}\mathbf{y}^*-\mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\right)\\ &\ge \min_{l=1,\dots,L}\left(\mathbf{x}^{*T} B^L_{l,1-w_{2l}v_2^*}\mathbf{y}-\mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)\right),\\ &\forall y\in Y\end{aligned} \tag{24a}$$

The above inequalities (22a), (22b), (23a) and (24a) can be equivalently expressed as follows.

$$\begin{aligned}v_1^* &= \min_{k=1,\dots,K}\{N_{\mathbf{x}^{*T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k}\}\\ &\ge \min_{k=1,\dots,K}\{N_{\mathbf{x}^{T}\tilde{A}_k\mathbf{y}^*}(\tilde{G}_{1k})/w_{1k}\},\forall x\in X\end{aligned}$$

$$\begin{aligned}v_2^* &= \min_{l=1,\dots,L}\{N_{\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}^*}(\tilde{G}_{2l})/w_{2l}\}\\ &\ge \min_{l=1,\dots,L}\{N_{\mathbf{x}^{*T}\tilde{B}_l\mathbf{y}}(\tilde{G}_{2l})/w_{2l}\},\forall y\in Y\end{aligned}$$

This means that an optimal solution to P6$(\mathbf{w}_1,\mathbf{w}_2;v_1^*,v_2^*)$ is an equilibrium solution to P4$(\mathbf{w}_1,\mathbf{w}_2)$.

3 AN INTERACTIVE ALGORITHM

From Theorem 3, if an optimal solution $(\mathbf{x}^*,\mathbf{y}^*,p^*,q^*,\sigma_1^*,\sigma_2^*)$ to P6$(\mathbf{w}_1,\mathbf{w}_2;v_1^*,v_2^*)$ satisfies the equality conditions (11a), (11b), then $(\mathbf{x}^*,\mathbf{y}^*,v_1^*,v_2^*)$ is an equilibrium solution to P4$(\mathbf{w}_1,\mathbf{w}_2)$. Unfortunately, we cannot specify such parameters (v_1^*,v_2^*) in advance. On the other hand, since the first term $\mathbf{x}^T A^L_{k,1-w_{1k}v_1^*}\mathbf{y}$ in the left hand of the inequality constraint (20c) is strictly monotone decreasing with respect to v^*, and the second term $\mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)$ in the left hand of the inequality constraint (20c) is strictly monotone increasing with respect to v_1^*, there exists some value of v_1^* such that $\mathbf{x}^{*T} A^L_{k,1-w_{1k}v_1^*}\mathbf{y}^*=\mu^{-1}_{\tilde{G}_{1k}}(w_{1k}v_1^*)$. At this value of v_1^*, $\sigma_1^*=0$ holds. In a similar way, we can find v_2^* such that $\sigma_2^*=0$, in which $\mathbf{x}^{*T} B^L_{l,1-w_{2l}v_2^*}\mathbf{y}^*=\mu^{-1}_{\tilde{G}_{2l}}(w_{2l}v_2^*)$.

From such a point of view, we can develop the algorithm to find the values of (v_1^*,v_2^*) such that $\sigma_1^*=0,\sigma_2^*=0$ by repeatedly updating (v_1^*,v_2^*), in which the conditions (16), (17) are satisfied. From the property of the level sets for fuzzy numbers, the parameters (v_1^*, v_2^*) must satisfy the inequalities $0 \leq w_{1k} v_1^* \leq 1,\ k = 1, \ldots, K,\ 0 \leq w_{2l} v_2^* \leq 1,\ l = 1, \ldots, L$. From the above conditions, the search range of (v_1, v_2) is expressed as $0 \leq v_1^* \leq \min_{k=1,\ldots,K} 1/w_{1k},\ 0 \leq v_2^* \leq \min_{l=1,\ldots,L} 1/w_{2l}$. Using the bisection method with respect to (v_1^*, v_2^*), we can find the values of (v_1^*, v_2^*) such that $\sigma_1^* = \sigma_2^* = 0$.

According to the above discussions, we propose an interactive algorithm to obtain a satisfactory solution of Player 1 from among an equilibrium solution set under the assumption that Player 1 can estimate Player 2's weighting vector $\mathbf{w}_2 \in W_2$.

Interactive Algorithm.

Step 1. Each player elicits his/her membership functions $\mu_{\tilde{G}_{1k}}(s), k = 1, \cdots, K,\ \mu_{\tilde{G}_{2l}}(t), l = 1, \cdots, L$, for fuzzy expected payoffs $\mathbf{x}^T \tilde{A}_k \mathbf{y}, k = 1, \cdots, K$, $\mathbf{x}^T \tilde{B}_l \mathbf{y}, l = 1, \cdots, L$, which are strictly monotone increasing with respect to $s \in S$ or $t \in T$.

Step 2. Set Player 1's weighting vector as $\mathbf{w}_1 = (1/K, \cdots, 1/K)$. Estimate Player 2's weighting vector $\mathbf{w}_2 \in W_2$.

Step 3. Set the initial values of the parameters (v_1^*, v_2^*) as $v_1^* \leftarrow \{\min_{k=1,\ldots,K} 1/w_{1k}\}/2$, $v_2^* \leftarrow \{\min_{l=1,\ldots,L} 1/w_{2l}\}/2$.

Step 4. Solve P6($\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$), and obtain the optimal solution $(\mathbf{x}^*, \mathbf{y}^*, p^*, q^*, \sigma_1^*, \sigma_2^*)$.

Step 5. If $\sigma_1^* > \varepsilon$, then $v_1^* \leftarrow v_1^* + \Delta v_1$, else if $\sigma_1^* < -\varepsilon$, then $v_1^* \leftarrow v_1^* - \Delta v_1$. If $\sigma_2^* > \varepsilon$, then $v_2^* \leftarrow v_2^* + \Delta v_2$, else if $\sigma_2^* < -\varepsilon$, $v_2^* \leftarrow v_2^* - \Delta v_2$, where Δv_1, Δv_2 and ε are sufficiently small positive constants, and return to Step 4. If $|\sigma_1^*| \leq \varepsilon$ and $|\sigma_2^*| \leq \varepsilon$, then go to Step 6, where ε is a sufficiently small positive constant.

Step 6. If Player 1 is not satisfied with the current values of the membership functions $\mu_{\tilde{G}_{1k}}(\mathbf{x}^{*T} A_k \mathbf{y}^*),\ k = 1, \ldots, K$, then update the weighting vector $w_{1k},\ k = 1, \ldots, K$ and return to Step 3. Otherwise, stop.

It should be noted here that ε should be set as an appropriate value corresponding to the step size values Δv_1 and Δv_2.

4 A NUMERICAL EXAMPLE

To show the efficiency of the proposed algorithm, consider a situation in which two competing firms plan to release a new product (Gao, 2013). Assume that each firm has only two marketing alternatives. A mixed strategy determines their budget among two marketing alternatives. Because of the lack of past statistical data about the demands, suppose that two kinds of indexes with respect to the demands are expressed as the fuzzy payoff matrices. Each element of the matrices are *LR* fuzzy numbers (Dubois and Prade, 1980) denoted as $\tilde{a}_{kij} \overset{\text{def}}{=} (a_{kij}, \alpha_{kij}, \beta_{kij})_{LR}$ and $\tilde{b}_{lij} \overset{\text{def}}{=} (b_{lij}, \gamma_{lij}, \delta_{lij})_{LR}$, respectively, where $(a_{111}, \alpha_{111}, \beta_{111}) = (120, 40, 40)$, $(a_{112}, \alpha_{112}, \beta_{112}) = (216, 50, 50)$, $(a_{121}, \alpha_{121}, \beta_{121}) = (192, 42, 42)$, $(a_{122}, \alpha_{122}, \beta_{122}) = (96, 21, 21)$, $(a_{211}, \alpha_{211}, \beta_{211}) = (50, 20, 20)$, $(a_{212}, \alpha_{212}, \beta_{212}) = (90, 30, 30)$, $(a_{221}, \alpha_{221}, \beta_{221}) = (32, 15, 15)$, $(a_{222}, \alpha_{222}, \beta_{222}) = (100, 40, 40)$, $(b_{111}, \gamma_{111}, \delta_{111}) = (120, 30, 30)$, $(b_{112}, \gamma_{112}, \delta_{112}) = (24, 10, 10)$, $(b_{121}, \gamma_{121}, \delta_{121}) = (48, 20, 20)$, $(b_{122}, \gamma_{122}, \delta_{122}) = (96, 25, 25)$, $(b_{211}, \gamma_{211}, \delta_{211}) = (50, 20, 20)$, $(b_{212}, \gamma_{212}, \delta_{212}) = (77, 25, 25)$, $(b_{221}, \gamma_{221}, \delta_{221}) = (30, 10, 10)$, $(b_{222}, \gamma_{222}, \delta_{222}) = (15, 5, 5)$. $L(x)$ and $R(x)$ are set as $\max(0, 1 - x), x \in [0, 1]$. Assume that hypothetical players set his/her membership functions as follows (Step 1).

$$\mu_{\tilde{G}_{1k}}(s) = \frac{s - E_{k10}}{E_{k11} - E_{k10}},\ k = 1, 2$$

$$\mu_{\tilde{G}_{2l}}(t) = \frac{t - E_{l20}}{E_{l21} - E_{l20}},\ l = 1, 2$$

where $E_{111} = 230, E_{110} = 0, E_{211} = 110, E_{210} = 0$, $E_{121} = 150, E_{120} = 0, E_{221} = 90, E_{220} = 0$. Then, the corresponding necessity measures can be expressed as the bilinear fractional functions.

$$N_{\mathbf{x}^T \tilde{A}_k \mathbf{y}}(\tilde{G}_{1k}) = \frac{\sum_{i=1}^2 \sum_{j=1}^2 a_{kij} x_i y_j - E_{k10}}{E_{k11} - E_{k10} + \sum_{i=1}^2 \sum_{j=1}^2 \beta_{kij} x_i y_j},$$
$$k = 1, 2$$

$$N_{\mathbf{x}^T \tilde{B}_l \mathbf{y}}(\tilde{G}_{2l}) = \frac{\sum_{i=1}^2 \sum_{j=1}^2 b_{lij} x_i y_j - E_{l20}}{E_{l21} - E_{l20} + \sum_{i=1}^2 \sum_{j=1}^2 \delta_{lij} x_i y_j},$$
$$l = 1, 2$$

At Step 2, Player 1 sets his/her initial weighting vector as $\mathbf{w}_1 = (w_{11}, w_{12}) = (0.5, 0.5)$ and estimates Player 2's weighting vector as $\mathbf{w}_2 = (w_{21}, w_{22}) = (0.5, 0.5)$. At Steps 4 and 5, P6($\mathbf{w}_1, \mathbf{w}_2; v_1^*, v_2^*$) is solved repeatedly until the inequality conditions $|\sigma_1^*| \leq \varepsilon$ and $|\sigma_2^*| \leq \varepsilon$ are satisfied, in which the step sizes are set as $\Delta v_1 = \Delta v_2 = 0.001$, and the parameter of the termination condition is set as $\varepsilon = 0.05$. In this example, at the third iteration, the satisfactory solution of Player 1 is obtained, which is an approximate equilibrium solution to P3($\mathbf{w}_1, \mathbf{w}_2$). It should be emphasized here that any equilibrium solution to P3($\mathbf{w}_1, \mathbf{w}_2$) cannot be obtained by applying the other methods which have been proposed until now, because the corresponding necessity measures are bilinear fractional functions.

Table 1: Interactive processes.

Iteration	1	2	3
w_{11}	0.5	0.6	0.7
w_{12}	0.5	0.4	0.3
w_{21}	0.5	0.5	0.5
w_{22}	0.5	0.5	0.5
$\mathbf{x}^{*T}A^L_{1,1-w_{11}v_1^*}\mathbf{y}^*$	138.26	136.38	140.16
$\mathbf{x}^{*T}A^L_{2,1-w_{12}v_1^*}\mathbf{y}^*$	50.244	45.123	38.314
$\mathbf{x}^{*T}B^L_{1,1-w_{21}v_2^*}\mathbf{y}^*$	64.294	64.283	64.294
$\mathbf{x}^{*T}B^L_{2,1-w_{22}v_2^*}\mathbf{y}^*$	32.036	32.039	32.058
$\mu_{\tilde{G}_{11}}(\mathbf{x}^{*T}A^L_{1,1-w_{11}v_1^*}\mathbf{y}^*)$	0.6011	0.5929	0.6093
$\mu_{\tilde{G}_{12}}(\mathbf{x}^{*T}A^L_{2,1-w_{12}v_1^*}\mathbf{y}^*)$	0.4567	0.4102	0.3483
$\mu_{\tilde{G}_{21}}(\mathbf{x}^{*T}B^L_{1,1-w_{21}v_2^*}\mathbf{y}^*)$	0.4286	0.4285	0.4286
$\mu_{\tilde{G}_{22}}(\mathbf{x}^{*T}B^L_{2,1-w_{22}v_2^*}\mathbf{y}^*)$	0.3559	0.3559	0.3562
x_1^*	0.3421	0.3421	0.3421
x_2^*	0.6578	0.6578	0.6578
y_1^*	0.6002	0.7316	0.9150
y_2^*	0.3997	0.2683	0.0849

5 CONCLUSION

In this paper, an interactive algorithm for multiobjective bimatrix games with fuzzy payoffs has been proposed to obtain a satisfactory solution from among an equilibrium solution set by updating the weighting vector of the weighted Tchebycheff norm method. We cannot directly obtain the equilibrium solution based on necessity measure for multiobjective bimatrix games with fuzzy payoffs, because of the definition of necessity measure or the nonlinearity of the membership functions. To circumvent such computational difficulties, the equilibrium conditions in the membership function space are replaced by the equilibrium conditions in the expected payoff space. As a result, it is possible to obtain the corresponding equilibrium solution, even if membership functions are nonlinear. However, in the computational aspect, it is important to set the parameters Δv_1, Δv_2 and ε in Step 5 of the proposed algorithm as sufficiently small positive constants. Especially, it should be noted here that the values of Δv_1, Δv_2 deeply depend on the value of ε which is the termination condition of the proposed algorithm.

REFERENCES

Bector, C. R., Chandra, S., and Vijay, V. (2004). Duality in linear programming with fuzzy parameters and matrix games with fuzzy payoffs. *Fuzzy Sets and Systems*, 146:253–269.

Bowman, V. J. (1976). *On the relationship of the Tchebycheff norm and the efficient frontier of multi-criteria objectives, in Multiple Criteria Decision Making (H. Thiriez and S. Zionts, eds.)*. Springer-Verlag.

Campos, L. (1989). Fuzzy linear programming models to solve fuzzy matrix games. *Fuzzy Sets and Systems*, 32:275–289.

Corley, H. W. (1985). Games with vector payoffs. *Journal of Optimization Theory and Applications*, 47:491–498.

Dubois, D. and Prade, H. (1980). *Fuzzy Sets and Systems : Theory and Applications*. Academic Press.

Gao, J. (2013). Uncertain bimatrix game with applications. *Fuzzy Optimization and Decision Making*, 12:65–78.

Larbani, M. (2009). Non cooperative fuzzy games in normal form : A survey. *Fuzzy Sets and Systems*, 160:3184–3210.

Li, D. F. (1999). A fuzzy multi-objective approach to solve fuzzy matrix games. *The Journal of Fuzzy Mathematics*, 7:907–912.

Liu, B. (2007). *Uncertainty Theory (2nd ed.)*. Springer, Berlin.

Maeda, T. (2000). Characterization of the equilibrium strategy of the bimatrix game with fuzzy payoff. *Journal of Mathematical Analysis and Applications*, 251:885–896.

Maeda, T. (2003). On characterization of equilibrium strategy of two person zero-sum game with fuzzy payoffs. *Fuzzy Sets and Systems*, 139:283–296.

Makó, Z. and Salamon, J. (2020). Correlated equilibrium of games in fuzzy environment. *Fuzzy Sets and Systems*, 398:112–127.

Nishizaki, I. and Notsu, T. (2007). Nondominated equilibrium solutions of a multiobjective two-person nonzero-sum game and corresponding mathematical programming problem. *Journal of Optimization Theory and Applications*, 135:217–239.

Nishizaki, I. and Sakawa, M. (1995). Equilibrium solutions for multiobjective bimatrix games incorporating fuzzy goals. *Journal of Optimization Theory and Applications*, 86:433–458.

Sakawa, M. (1993). *Fuzzy sets and interactive multiobjective optimization*. Plenum Press, New York.

Tamura, K. and Miura, S. (1979). Necessary and sufficient conditions for local and global nondominated solutions in decision problems with multi-objectives. *Journal of Optimization Theory and Applications*, 28:501–523.

Tang, M. and Li, Z. (2020). A novel uncertain bimatrix game with hurwicz criterion. *Soft Computing*, 24:2441–2446.

Yager, R. R. (1981). A procedure for ordering fuzzy numbers in the unit interval. *Information Sciences*, 24:143–161.

Yu, P. L. (1985). *Multiple-Criteria Decision Making, Concepts, Techniques, and Extensions*. Plenum Press, New York.