Improved Output Feedback Control of Constrained Linear Systems using Invariant Sets

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Keywords: Linear Systems, Invariant Sets, Output Feedback, Constraints.

Abstract: We propose an improved design method for output feedback control of discrete-time linear systems subject to state and control constraints, additive disturbances and measurement noise. Output Feedback Controlled-Invariant polyhedral sets are used to ensure that state and input constraints are satisfied all time. The control strategy seeks to enforce the set of states consistent with the measured output into a closed ball around the origin. The control input is computed through the solution of Linear Programming (LP) problems, whose goal is to minimize the size of the ball one step ahead. Then, we use the optimization results to reduce the set of admissible states, steering the state to a smaller ball around the origin. The improvement provided by the proposed strategy is illustrated by numerical examples.

1 INTRODUCTION

The theory of positively invariant sets is an important tool to deal with constrained control systems (Blanchini and Miani, 2015). Constraints arise naturally in real-life control problems from physical limitations on state, control and output variables, which can be represented as convex polyhedral sets, in general.

A set is said to be positively invariant with respect to a given system if any trajectory originated from this set does not leave it. When considering state feedback control, if there exists a control action based on the measured state that keeps the state trajectory in a given set, such a set is said to be controlled-invariant (Blanchini and Miani, 2015). Most known techniques assume that the system state can be fully measured, however this may not be possible in some applications. One then has to consider invariance under output feedback.

In (Artstein and Raković, 2011) the notion of invariance with respect to output feedback under nonparametric disturbances was proposed within a set dynamics approach, in a more conceptual and general framework. In (Dórea, 2009) an output feedback structure was studied and conditions were defined to check if a given polyhedral set can be made invariant under output feedback. Such sets were said to be Output Feedback Controlled-Invariant (OFCI). If a set is OFCI, then, a suitable sequence of control inputs can be computed, based on the knowledge of the measured output, in order to confine the state trajectory therein. Using this concept, the computation of a picewise affine (PWA) law using multiparametric programming was proposed in (Dantas et al., 2018).

Model Predictive Control (MPC) techniques have also been used to solve constrained problems via output feedback as in (Lee and Kouvaritakis, 2000), (Mayne et al., 2006), (Goulart and Kerrigan, 2007), (Løvaas et al., 2008), (Subramanian et al., 2017). Typically, a stabilizing state feedback gain and a full-order linear observer to estimate the state are designed, and a Robust Controlled-invariant (RPI) set with respect to the error dynamics is computed.

Based on OFCI polyhedral sets, in a recent paper (Almeida and Dorea, 2020) the authors proposed an output feedback strategy for the regulation problem in discrete-time linear systems subject to state and control constraints, and unknown-but-bounded disturbances and measurement noise. Given an OFCI polyhedron included in the set of state constraints, a Linear Programming (LP) problem was set up in order to compute a control action that enforces constraints satisfaction and minimizes, one step ahead, a guaranteed distance from the admissible states to the origin. A dynamic control strategy was also proposed, for which an OFCI polyhedron is obtained for an augmented system that comprises the system and compensator states. By using the dynamic controller, the

Mancini, A., Almeida, T. and Dórea, C.
Improved Output Feedback Control of Constrained Linear Systems using Invariant Sets.
DOI: 10.5220/0010556705660573
In *Proceedings of the 18th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2021)*, pages 566-573
ISBN: 978-989-758-522-7

uncertainty on the state is progressively reduced using information about the contraction of an invariant set defined in the estimation error space.

In this paper, starting from the strategy proposed in (Almeida and Dorea, 2020), we show that the uncertainty on the state can be further reduced by using information given by the solution of the LP problem. By doing so, we can achieve faster convergence of the state trajectory to a ball around the origin, which is smaller than that obtained by (Almeida and Dorea, 2020), specially in the static output feedback case. The improvement provided by the proposed strategy is illustrated by numerical examples.

2 INVARIANT SETS

Consider the linear, time-invariant, discrete-time system described by:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ed(k), \\ y(k) &= Cx(k) + \eta(k), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $d \in \mathbb{R}^r$ is the disturbance, $y \in \mathbb{R}^p$ is the measured output, $\eta \in \mathbb{R}^p$ is the measurement noise and $k \in \mathbb{N}$ is the sampling time. The disturbance and the measurement noise are assumed to be unknown but bounded to C-sets $\mathcal{D} \subset \mathbb{R}^r$ and $\mathcal{N} \subset \mathbb{R}^p$, respectively. Moreover, the system is subject to state and control constraints: $x \in \Omega_x$ and $u \in \mathcal{U}$, where $\Omega_x \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$ are also C-sets. A C-set is a convex and compact (closed and bounded) set containing the origin.

The constraints on the state variables and control inputs, and the bounds on disturbance and measurement noise are given by the following convex polyhedral sets containing the origin:

$$\begin{aligned} \Omega_x = \{x : G_x x \leq \bar{1}\}, \quad \mathcal{U} = \{u : Uu \leq \bar{1}\}, \\ \mathcal{D} = \{d : Dd \leq \bar{1}\}, \quad \mathcal{N} = \{\eta : N\eta \leq \bar{1}\}, \end{aligned} \tag{2}$$

with $G_x \in \mathbb{R}^{g_x \times n}$, $U \in \mathbb{R}^{v \times m}$, $D \in \mathbb{R}^{s \times r}$, $N \in \mathbb{R}^{q \times p}$.

We now present some important definitions to characterise invariant sets and invariance under output feedback control.

Definition 2.1. Given λ, $0 \leq \lambda < 1$, the set $\Omega \in \mathbb{R}^n$ is said to be controlled-invariant with contraction rate λ with respect to system (1) if $\forall x \in \Omega, \exists u \in \mathcal{U} : Ax + Bu + Ed \in \lambda\Omega, \forall d \in \mathcal{D}$ (Blanchini, 1994).

If Ω is controlled-invariant then, for any initial condition $x(0) \in \Omega$, there exists a state feedback law $u(x(k))$ satisfying the control constraints which is able to keep the state trajectory of the controlled system within $\lambda\Omega, \forall k \geq 0$, for all admissible disturbances $d \in \mathcal{D}$.

We now consider to accomplish constraints enforcement through output feedback control. Even though the state of the system is not known exactly, each measurement y carries information about its location. Consider the set $\mathcal{Y}(\Omega) \in \mathbb{R}^p$, which contains all admissible outputs y that can be associated to $x \in \Omega$:

$$\mathcal{Y}(\Omega) = \{y : y = Cx + \eta \text{ for } x \in \Omega, \eta \in \mathcal{N}\}. \tag{3}$$

Consider also the set $\mathcal{C}(y(k))$, which represents the set of states compatible with each measurement $y(k) \in \mathbb{R}^p$:

$$\mathcal{C}(y) = \{x : Cx = y - \eta, \text{ for } \eta \in \mathcal{N}\}. \tag{4}$$

Set-invariance under output feedback can be characterized by the following definition (Dórea, 2009):

Definition 2.2. The set Ω is said to be Output Feedback Controlled-Invariant (OFCI) with contraction rate λ, $0 \leq \lambda < 1$, with respect to system (1) if $\forall y \in \mathcal{Y}(\Omega), \exists u \in \mathcal{U} : Ax + Bu + Ed \in \lambda\Omega, \forall d \in \mathcal{D}$ and $\forall x \in \Omega, \eta \in \mathcal{N}$ such that $Cx = y - \eta$.

If Ω is OFCI with contraction rate λ, if $x(k) \in \Omega$, then there exists a control $u(y(k)) \in \mathcal{U}$, computed from the measured output at time k, such that $x(k+1) \in \lambda\Omega, \forall k$, in spite of the disturbance $d(k) \in \mathcal{D}$ and noise $\eta \in \mathcal{N}$.

In (Dórea, 2009), necessary and sufficient conditions were established to check if a polyhedral set Ω is OFCI with contraction rate λ.

The dynamic output feedback control strategy used here employs state observers. The possibility of confining the related estimation error into an invariant set can be characterized by conditioned-invariant sets, defined as follows:

Definition 2.3. (Dórea, 2009) The set Ω is said to be conditioned-invariant with contraction rate λ, $0 \leq \lambda < 1$, with respect to system (1) if $\forall y \in \mathcal{Y}(\Omega), \exists v : Ax + v + Ed \in \lambda\Omega, \forall d \in \mathcal{D}$ and $\forall x \in \Omega, \eta \in \mathcal{N}$ such that $Cx = y - \eta$.

In what follows, the invariant sets defined in this section will be used to build an online optimization strategy to compute an output feedback control able to enforce state and control constraints and steer the state trajectory to a as small as possible ball around the origin.

3 OUTPUT FEEDBACK CONTROLLERS

In this section, we describe the online optimization strategy to compute static and dynamic output feedback constrained controllers proposed in (Almeida and Dorea, 2020), on which our approach is based.

3.1 Static Controller

Let $\Omega = \{x : Gx \leq \bar{1}\} \subset \Omega_x,\ G \in \mathbb{R}^{g \times n}$ be an OFCI polyhedral set. We consider the solution to the following output regulation problem under constraints:

Based on the measurements $y(k)$, $\forall x(0) \in \Omega \subset \Omega_x$, compute $u(k) \in \mathcal{U}$ such that $x(k) \in \Omega, \forall k \geq 0, \forall d \in \mathcal{D}, \forall \eta \in \mathcal{N}$, and $x(k)$ converges to a small ball around the origin $\mathcal{B}(\varepsilon)$.

From Definition 2.2, the design goal can be achieved if $x(0)$ belongs to the OFCI set Ω. From Definition 2.2 and the set of admissible outputs (3), Ω is OFCI with contraction rate λ if, and only if (Dórea, 2009),(Almeida and Dorea, 2020):

$$\begin{gathered}\forall y \in \mathcal{Y}(\Omega), \exists u : G(Ax + Bu + Ed) \leq \lambda \bar{1} \\ \mathcal{U}u \leq \bar{1} \\ \forall x, \eta, d : y - Cx = \eta,\ Gx \leq \bar{1},\ N\eta \leq \bar{1},\ Dd \leq \bar{1}.\end{gathered} \tag{5}$$

Since the disturbances acting on the system are unknown, at a given step k, the input signal $u(k)$ must enforce the constraints for all $d \in \mathcal{D}$. Also, as the state at step k is not available, $u(k)$ must enforce the constraints for all $x \in \Omega$ consistent with the output $y(k)$. That can be achieved by considering the worst case row-by-row of G as follows (Dórea, 2009).

Let the elements of the vector $\phi(y) \in \mathbb{R}^g$, which depend on the current output, be defined by the solution of the following LP problems:

$$\begin{aligned}\phi_j(y) &= \max_x G_j A x, \\ \text{s.t. } & Gx \leq \bar{1},\ -NCx \leq -Ny(k) + \bar{1}\end{aligned} \tag{6}$$

with $j = 1, \ldots, g$, and let the elements of vector $\delta \in \mathbb{R}^g$ be given by the solution of the following LP problems:

$$\begin{aligned}\delta_j &= \max_d G_j E d, \\ \text{s.t. } & Dd \leq \bar{1},\end{aligned} \tag{7}$$

with $j = 1, \ldots, g$.

Considering (6) and (7) condition (5) can be rewritten as:

$$\forall y \in \mathcal{Y}(\Omega), \exists u \in \mathcal{U} : \begin{bmatrix}\phi(y) \\ \bar{0}\end{bmatrix} + \begin{bmatrix}GB \\ U\end{bmatrix} u \leq \begin{bmatrix}\lambda\bar{1} - \delta \\ \bar{1}\end{bmatrix}. \tag{8}$$

Besides constraints satisfaction, we also seek to steer the state x to the smallest ball $\mathcal{B}(\varepsilon)$ around the origin. To this end, we use the strategy of enforcing the one-step evolution of the set of states consistent with the measurement y(k) into the smallest ball $\mathcal{B}(\varepsilon) = \{x : Hx \leq \varepsilon\bar{1}\}$, where $H = \begin{bmatrix} I & -I \end{bmatrix}^T$. Then, $\forall y \in \mathcal{Y}(\Omega)$, we must enforce:

$$\begin{gathered}H(Ax + Bu + Ed) \leq \varepsilon\bar{1}, \\ \forall x, \eta, d : y - Cx = \eta,\ Gx \leq \bar{1},\ N\eta \leq \bar{1},\ Dd \leq \bar{1}.\end{gathered} \tag{9}$$

Let $\varphi(y) \in \mathbb{R}^{2n}$ be a vector whose components are given by the solution of the following LP problems:

$$\begin{aligned}\varphi_j(y) &= \max_x H_j A x, \\ \text{s.t. } & Gx \leq \bar{1},\ -NCx \leq -Ny(k) + \bar{1}\end{aligned} \tag{10}$$

with $j = 1, \ldots, 2n$ and the vector $\gamma \in \mathbb{R}^{2n}$ whose components are given by:

$$\begin{aligned}\gamma_j &= \max_d H_j E d, \\ \text{s.t. } & Dd \leq \bar{1}\end{aligned} \tag{11}$$

with $j = 1, \ldots, 2n$.

Then, condition (9) can be then rewritten as:

$$\forall y \in \mathcal{Y}(\Omega) : HBu - \varepsilon\bar{1} \leq -\varphi(y) - \gamma. \tag{12}$$

Now it is possible to compute a control action that simultaneously satisfies state and control constraints, ensuring output feedback invariance, and drives the states consistent with the measured output to the smallest closed ball around the origin. This control action can be computed online from the solution of the following LP problem:

$$\begin{gathered}u(y(k)) = \arg \min_{u,\varepsilon} \varepsilon \\ \text{s.t} \begin{bmatrix} GB & \bar{0} \\ U & \bar{0} \\ HB & -\bar{1} \end{bmatrix} \begin{bmatrix} u \\ \varepsilon \end{bmatrix} \leq \begin{bmatrix} \bar{1} - \phi(y(k)) - \delta \\ \bar{1} \\ -\varphi(y(k)) - \gamma \end{bmatrix}.\end{gathered} \tag{13}$$

For the online solution of problem (13), at each time step k, from the current measured output $y(k)$ the vectors $\phi(y)$ and $\varphi(y)$ are computed through the solutions of (6) and (10). The vectors δ and γ do not depend on the system evolution and should be computed only once.

3.2 Dynamic Controller

Achieving output feedback invariance of a polyhedral set is far from being an easy task. This is mainly due to the fact that the set of states consistent with a single measurement $y(k)$ is, in general, very large.

Hence, a control action $u(k)$ able to enforce the one-step evolution of this set of states into the polyhedron does not always exist. In (Dórea, 2009) a state observer structure was proposed to tackle this problem. Once one has an estimate of the state $x(k)$, and the estimation error is bounded to another polyhedral set, then, the set of admissible states can be significantly reduced, making it easier to achieve output feedback invariance.

In (Dórea, 2009), (Almeida and Dorea, 2020) the following compensator structure is used:

$$\begin{aligned} z(k+1) &= \nu(z(k), y(k)), \\ u(k) &= \kappa(z(k), y(k)). \end{aligned} \tag{14}$$

System (1) under this compensator (14) can be represented in an extended state space formulation:

$$\begin{aligned} \xi(k+1) &= \hat{A}\xi(k) + \hat{B}\omega(k) + \hat{E}d(k) \\ \zeta(k) &= \hat{C}\xi(k) + \hat{\eta}(k) \end{aligned} \tag{15}$$

where $\xi = \begin{bmatrix} x \\ z \end{bmatrix}$ is the extended state vector, $\omega = \begin{bmatrix} u \\ v \end{bmatrix}$ is the extended input vector, $\zeta = \begin{bmatrix} y \\ z \end{bmatrix}$ is the extended output vector, $\hat{\eta} = \begin{bmatrix} \eta \\ 0 \end{bmatrix}$ is the extended noise vector and $\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}$, $\hat{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}$, $\hat{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$. Moreover, $u(k)$ and $v(k)$ are functions of the extended output vector $\begin{bmatrix} y \\ z \end{bmatrix}$ as expressed in (14).

Control constraints and bounds on the measurement noise for the extended system can be defined over the extended space accordingly. Since (15) is a linear system with the same structure as (1) the input computation method illustrated previously can be equally applied.

Consider now a pair of compact convex polyhedral sets $(\mathcal{S}, \mathcal{V})$, represented by: $\mathcal{S} = \{x : G_s x \leq \bar{1}\}$, $\mathcal{V} = \{x : G_v x \leq \bar{1}\}$ and satisfying the assumption: $\mathcal{S} \subset \mathcal{V} \subset \Omega_x$, $\mathcal{S}$ is conditioned-invariant and $\mathcal{V}$ is controlled-invariant.

It turns out that the polyhedral set $\hat{\Omega}$ below satisfies a necessary condition for output feedback invariance with respect to the augmented system, being simultaneously controlled and conditioned-invariant (Dórea, 2009).

$$\hat{\Omega} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} : \underbrace{\begin{bmatrix} G_v & \bar{0} \\ G_s & -G_s \end{bmatrix}}_{\hat{G}} \begin{bmatrix} x \\ z \end{bmatrix} \leq \begin{bmatrix} \bar{1} \\ \bar{1} \end{bmatrix} \right\} \tag{16}$$

where $G_v \in \mathbb{R}^{g_v \times n}$, $G_s \in \mathbb{R}^{g_s \times n}$ and $\hat{G} \in \mathbb{R}^{(g_v+g_s)\times 2n}$.

If we interpret the compensator state $z(k)$ as an estimate of the system state $x(k)$, then the estimation error is bounded by the conditioned-invariant set $\mathcal{S}$, for $G_s(x-z) \leq \bar{1}$.

The compensator structure (14) is quite general, allowing the design of nonlinear observers. However, as discussed in (Almeida and Dorea, 2020), there is no evidence that such a nonlinear observer would perform better than a linear one. Then, a linear observer has been adopted as follows:

$$z(k+1) = Az(k) + Bu(k) + L[y(k) - \hat{y}(k)] \tag{17}$$

where $\hat{y} = Cz(k)$ is the estimated output and the observer gain $L \in \mathbb{R}^{n\times p}$ is a parameter to be designed so that the eigenvalues of $(A - LC)$ lie inside the complex unit disc.

The estimation error dynamics $e(k) = x(k) - z(k)$ is given by:

$$e(k+1) = A_e e(k) + E_e d_e(k) \tag{18}$$

where $A_e = A - LC$, $E_e = \begin{bmatrix} E & -L \end{bmatrix}$ and $d_e(k) = \begin{bmatrix} d(k) \\ \eta(k) \end{bmatrix}$. Bounds on the additive disturbance d_e can be easily obtained combining the bounds on d and η.

Given the stabilizing observer gain L, one must compute an invariant polyhedron $\mathcal{S}$ with respect to (18). A natural choice is the minimal Robust Positively Invariant (mRPI) set (Rakovic et al., 2005), which is the smallest invariant set of (18) complying with the disturbances. Let, then $\mathcal{S}_m$ be the mRPI of (18) with contraction rate λ_m, which can be computed using the algorithm proposed in (Rakovic et al., 2005). It will be used to compute a pair $(\mathcal{S}_m, \mathcal{V})$ that composes an OFCI polyhedron $\hat{\Omega}$ (16) .

A natural choice for $\mathcal{V}$ is the maximal controlled-invariant set contained in Ω_x, which can be computed using the algorithm proposed in (Dórea and Hennet, 1999).

The set of admissible initial states is given by the projection of $\hat{\Omega}$ onto the state space. With the aim of enlarging this set, $\mathcal{S}$ is scaled up to $\alpha^*\mathcal{S}$, with $\alpha^* = \max_{\alpha\geq 1} \alpha$ such that $\hat{\Omega}$ remains OFCI.

In (Almeida and Dorea, 2020) it is also shown that it is possible to compute offline a strictly decreasing sequence $\{\bar{\alpha}(k)\}$, $k = 0, 1, \ldots, \bar{k}_m$, starting from $\bar{\alpha}(0) = \alpha_m$, where $\bar{k}_m$ is the smallest value of k such that $\bar{\alpha}(k) \leq 1$. Hence, we have that $e(k) \in \bar{\alpha}(k)\mathcal{S}_m$ for $k < \bar{k}_m$ and $e(k) \in \mathcal{S}_m$ for $k \geq \bar{k}_m$. This information is used in the control action computation in order to progressively reduce the uncertainty on the state $x(k)$.

Then, as long as the pair $(\alpha_m\mathcal{S}_m, \mathcal{V})$ forms an OFCI polyhedron, $u(k)$ can be computed similarly as in (13), but with G replaced by G_v and $\phi(k)$ and $\varphi(k)$ given by:

$$\begin{cases} \phi_j(k) = \max_x G_{vj}Ax, \ j = 1,\ldots,g_v \\ \varphi_j(k) = \max_x H_jAx, \ j = 1,\ldots,2n \end{cases} \quad (19)$$

$$\text{s.t.} \begin{bmatrix} G_v \\ G_s \end{bmatrix} x \leq \begin{bmatrix} \bar{1} \\ \bar{\alpha}(k)\bar{1} + G_s z(k) \end{bmatrix}, \ -NCx \leq -Ny(k) + \bar{1}.$$

It has been shown through numerical examples in (Almeida and Dorea, 2020) that the strategy described above is, in general, less conservative than MPC approaches based on linear observers, in terms of obtaining larger sets of admissible initial states. However, no performance assessment was presented. In the next section we propose an improvement of the strategy aiming at speeding up the convergence of the state trajectories to a guaranteed smaller ball around the origin.

4 IMPROVED STATE TRAJECTORY OPTIMIZATION

The difficulty in optimizing performance via output feedback under constraints lies in the fact that a single control action must cope with constraint satisfaction of a set of states consistent with the measurement. This is specially difficult in the static feedback case, where only the present output measurement is available. The optimization strategy described in the previous section minimizes one step ahead the worst case distance from the set of states consistent with the measurement to the origin. Here, we propose to use the solution of the optimization problem in order to further reduce the set of possible states and, as a consequence, improve the convergence of the states to a smaller ball around the origin.

Let $\varepsilon(k+1)$ be the optimal solution of the LP problem (13). Then, from (9), one can see that:

$$x(k) \in \mathcal{B}(\varepsilon(k)) \quad (20)$$

This information can now be added to the computation of the vectors $\phi(y)$ and $\varphi(y)$ to further reduce the uncertainty on $x(k)$, as follows:

$$\begin{cases} \phi_j(k) = \max_x G_jAx, \ j = 1,\ldots,g \\ \varphi_j(k) = \max_x H_jAx, \ j = 1,\ldots,2n \end{cases} \quad (21)$$

$$\text{s.t. } Gx \leq \bar{1}, \ Hx \leq \varepsilon(k)\bar{1}, \quad -NCx \leq -Ny(k) + \bar{1}.$$

For the dynamic output feedback control the vectors $\phi(y)$ and $\varphi(y)$ with the additional constraint are given by:

$$\begin{cases} \phi_j(k) = \max_x G_{vj}Ax, \ j = 1,\ldots,g_v \\ \varphi_j(k) = \max_x H_jAx, \ j = 1,\ldots,2n \end{cases} \quad (22)$$

$$\text{s.t.} \begin{bmatrix} G_v \\ G_s \\ H \end{bmatrix} x \leq \begin{bmatrix} \bar{1} \\ \bar{\alpha}(k)\bar{1} + G_s z(k) \\ \varepsilon(k)\bar{1} \end{bmatrix}, \quad -NCx \leq -Ny(k) + \bar{1}.$$

This way, our proposed optimization problem guarantees that the set of states $x(k)$ consistent with the measured output $y(k)$ belongs to both the controlled-invariant set, by forcing $Gx \leq \bar{1}$ in (21) (if Ω is OFCI) and $G_vx \leq \bar{1}$ in (22) (if the par $(\mathcal{V}, \alpha_m\mathcal{S}_m)$ is OFCI), and the closed ball $\mathcal{B}(\varepsilon(k))$. As a result, the set of states consistent with the measurements becomes smaller improving, therefore, the performance of the controller.

It is worth mentioning that, for the dynamic feedback case, we are able to compute offline the decreasing sequence $\bar{\alpha}$, which defines the contraction rate of the invariant sets related to the estimation error, based on λ_m. On the other hand, we do not have the same previous information for the state $x(k)$. We have to compute $\varepsilon(k)$ online because the closed ball $\mathcal{B}(\varepsilon)$ is not an invariant set, then, there is no defined contraction rate.

5 NUMERICAL EXAMPLES

Example 5.1. (Static feedback) Consider the linearized discrete-time system (1) obtained for a two coupled-tank system (Martins et al., 2014), that can be seen in Figure 1, for which:

$$A = \begin{bmatrix} 0.9448 & 0 \\ 0.0537 & 0.9448 \end{bmatrix}, B = \begin{bmatrix} 0.1357 \\ 0.0028 \end{bmatrix}; \quad E = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad (23)$$

State and control constraints are given respectively by:

$$\Omega_x = \{x : |x_i| \leq 15, i = 1,2\}, \ \mathcal{U} = \{u : |u| \leq 4\} \quad (24)$$

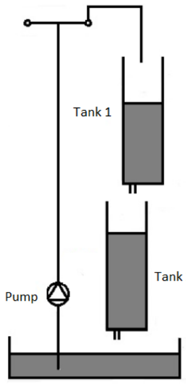

Figure 1: Coupled-tank system.

The measurement noise is bounded by: $\mathcal{N} = \{\eta : |\eta| \leq 0.05\}$. The system is not subjected to disturbances.

A λ_v-contractive controlled-invariant set $\mathcal{V}$ contained in Ω_x with a contraction rate of $\lambda_v = 0.95$ was computed. It was verified that $\mathcal{V}$ is also OFCI with contraction rate $\lambda = 0.9876$. Then, a static output feedback can be computed to enforce state and control consraints.

State vector trajectories resulting from the static controller with and without the additional constraints (13) are shown in Figure 2. It is possible to see that the state trajectory resulting from the control action using the additional constraint reaches a smaller ball around the origin and is faster than the one resulting from the controller proposed in (Almeida and Dorea, 2020).

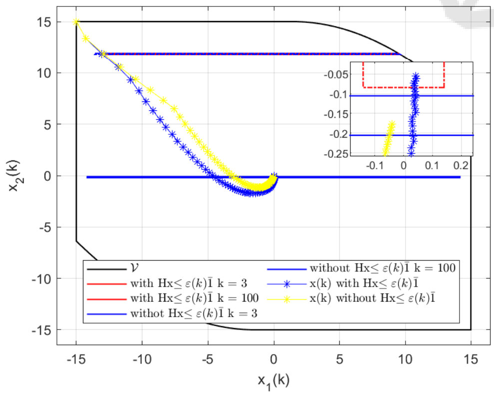

Figure 2: State vector trajectories inside $Hx \leq \varepsilon(k)\bar{1}$ for $k = 3$ and $k = 100$.

In Figure 3 it is also possible to see that when considering the additional constraint the state trajectory is associated to a sequence with smaller values of $\varepsilon(k)$. We can also see that, as the system is not affected by disturbances, then $\varepsilon(k)$ tends to 0.

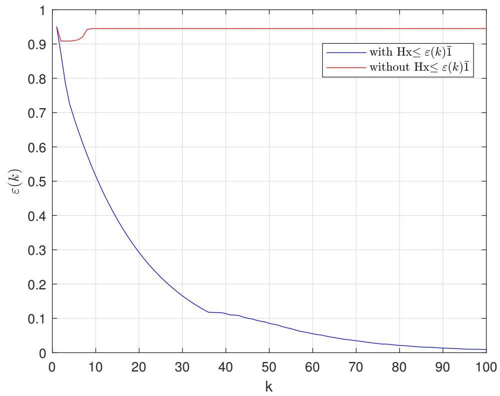

Figure 3: Evaluation of $\varepsilon(k)$.

Example 5.2. (Dynamic controller) Consider the discrete-time system (1) for which (Almeida and Dorea, 2020):

$$A = \begin{bmatrix} 0.745 & -0.002 \\ 5.61 & 0.780 \end{bmatrix}, B = \begin{bmatrix} -5.6 \cdot 10^{-4} \\ 0.464 \end{bmatrix}, E = I_2, \\ C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{25}$$

State and control constraints are given respectively by:

$$\Omega_x = \{x : |x_1| \leq 0.4, \ |x_2| \leq 15\}, \ \mathcal{U} = \{u : |u| \leq 10\} \tag{26}$$

Bounds for disturbance and measurement noise are given by: $\mathcal{D} = \{d : |d_1| \leq 0.02, \ |d_2| \leq 0.4\}$ and $\mathcal{N} = \{\eta : |\eta| \leq 0.1\}$.

A λ_v-contractive controlled-invariant set $\mathcal{V}$ contained in Ω_x with a contraction rate of $\lambda_v = 0.99$ was computed. The gain $L = \begin{bmatrix} 0.728 & 5.610 \end{bmatrix}^T$ was designed for the observer, to result in the eigenvalues of $A - LC$ at 0.017 and 0.780. An approximation of the mRPI $\mathcal{S}_m$ was then computed with $\lambda_m = 0.99$. It was checked that the pair $(\mathcal{V}, \mathcal{S}_m)$ forms an OFCI polyhedron with respect to the extended system (15).

State trajectories resulting from the dynamic controller with and without the additional constraints (22), starting from $x(0) = \begin{bmatrix} 0.4 & 15 \end{bmatrix}$, with $z(0) = \begin{bmatrix} 0.1 & 10 \end{bmatrix}$, under random disturbance and noise, are shown in Figure 4 illustrating that state constraints are satisfied. It also shows the closed balls $Hx \leq \varepsilon(k)\bar{1}$ for $k = 3$ and $k = 50$. We can see that the final set is smaller in the trajectory considering the additional constraint.

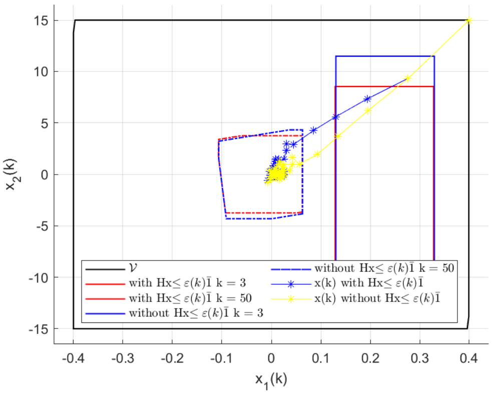

Figure 4: State vector trajectories inside $Hx \leq \varepsilon(k)\bar{1}$ for $k = 3$ and $k = 50$.

The property of keeping the state trajectory in smaller sets for each sample k can also be seen in Figure 5. When considering the additional constraint the state trajectory is associated to a sequence of smaller values of $\varepsilon(k)$.

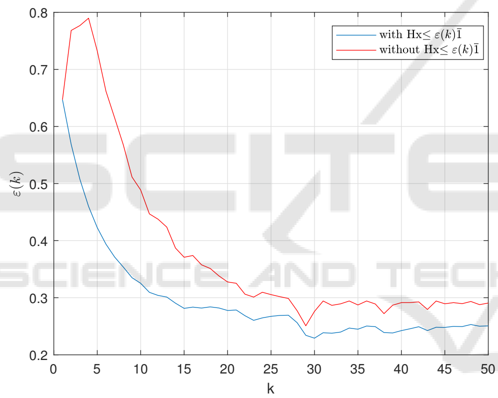

Figure 5: Evaluation of $\varepsilon(k)$.

6 CONCLUSIONS

In this work, an improved design method for output feedback control for constrained, linear, discrete-time systems subject to persistent disturbances was proposed using the concept of Output Feedback Controlled-Invariance (OFCI) sets.

A modification was proposed in the algorithm of (Almeida and Dorea, 2020) in order to further reduce the set of states consistent with the measurement, by taking into account the results of an optimization problem solved in the previous step.

Both static and dynamic controllers were considered. The performance improvement of the modified controller was illustrated through numerical examples, being more remarkable in the static case.

Future work must consider using a set which is homothetic with respect to the controlled invariant set as a target, in order to guarantee ultimate boundedness of the state trajectories.

ACKNOWLEDGEMENTS

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brazil (CAPES) – Finance Code 001, and by the National Council for Scientific and Technological Development - Brazil (CNPq), grant #309862/2019-1.

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