# Reduced Order Modeling for Thermal Problems with Temperature-dependent Conductivities using Matrix Interpolation

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Abstract: In this paper model order reduction of thermal problems with temperature dependent material parameters is considered. It is assumed that the full thermal problem is set up by a commercial solver where the user has only limited access to internal datastructures. For the full problem an approximation based on matrix interpolation is proposed which is applicable to commercial solvers like Simcenter Thermal Flow where system matrices can be extracted for given temperature fields. Model order reduction for the approximated problem is achieved by POD and DEIM.

## Nomenclature

| | |
|---|---|
| $A$ | conductance matrix, $A \in \mathcal{R}^{n\times n}$ |
| $A_{lin}$ | linear approximation of conductance matrix |
| $B$ | input matrix, $B \in \mathcal{R}^{n\times m}$ |
| $c_p$ | specific heat capacity |
| $C$ | output matrix, $A \in \mathcal{R}^{p\times n}$ |
| $E$ | mass matrix, $E \in \mathcal{R}^{n\times n}$ |
| $h$ | volume heat load |
| $h_f$ | boundary heat flux |
| $k$ | region index |
| $m$ | dimension of input vector |
| $n$ | dimension of discretized temperature vector |
| $n_r$ | reduced dimension |
| $\vec{n}$ | normal at the boundary of $\Omega$ |
| $P_{deim}$ | projection matrix of DEIM Dofs, $P_{deim} \in \mathcal{R}^{n\times n_{deim}}$ |
| $P_{co\_deim}$ | projection matrix of DOFs coupled with DEIM Dofs,$P_{co\_deim} \in \mathcal{R}^{n\times n_{co\_deim}}$ |
| $\vec{q}$ | volume heat flux |
| $t$ | time |
| $T$ | temperature |
| $[T_{min}, T_{max}]$ | temperature interval |
| $T_{amb}$ | ambient temperature |
| $u$ | input (load) |
| $U$ | DEIM matrix, $U \in \mathcal{R}^{n\times n_{deim}}$ |
| $V$ | POD projection matrix, $V \in \mathcal{R}^{n\times n_r}$ |
| $\vec{x}$ | discretized temperature vector |
| $\vec{x}_0$ | discretized temperature vector at $T = 0$ |
| $\vec{x}_r$ | reduced temperature vector |
| $\vec{y}$ | output vector |
| $\alpha$ | convection coefficient |
| $\varepsilon_{abs}$ | absolute error |
| $\varepsilon_{rel}$ | relative error |
| $\mu$ | heat conductivity |
| $\mu_{lin}$ | linearized heat conductivity |
| $\Omega$ | domain |
| $\Omega_N$ | Neumann boundary of $\Omega$ |
| $\Omega_R$ | convection boundary of $\Omega$ |
| $\rho$ | density |
| $\xi$ | normalized temperature vector |

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# 1 INTRODUCTION

Subject of this paper is model order reduction of thermal problems with temperature dependent conductivities e.g. for the development of virtual temperature sensors. There are methods proposed in literature, see e.g. (Fritzen et al., 2018), but a constraint here is that Simcenter Thermal Flow (NX) (Anderl and Binde, 2018) should be used as solver for the thermal problem. The user does not have full access to the internal datastructures of the commercial solver, from outside it is only possible to extract matrices (capacitance, conductance, convection) of the discretized system for given temperature fields. This plugin has been realized by a special subroutine, see (Benner et al., 2021), chapter 12 ”Use case - Virtual Sensors”. In the current version, the user specifies the temperature field for which the matrices are extracted. So from a mathematical viewpoint, temperature dependent coefficients are approximated by constant ones. For a small temperature range this approximation will be accurate enough. But for a wider temperature range or for higher accuracy a better approximation would be desirable. Here we will discuss strategies based on multi-linear matrix interpolation to improve the constant approximation without restricting the generality of the method in terms of number of regions and materials.

In the next section the thermal model is set up, in sect. 3 the approximation of conductance matrix is discussed. Subject of sect. 4 is reduced order modelling by POD-DEIM, and in sect. 5 the method is applied to a thermal model of a motor. The paper concludes with a summary and possible extensions of the method.

# 2 THERMAL MODEL

The starting point is the thermal energy equation which reads for heat conduction with Fourier's Law $\vec{q} = -\mu\nabla T$, (S. R. de Groot, 1969) for a computational domain $\Omega$ as

$$\begin{aligned} \rho c_p \partial_t (T) + \nabla \cdot (-\mu(T)\nabla T) &= h && \text{in } \Omega \\ \vec{q}\cdot\vec{n} &= h_f && \text{on } \Gamma_N \\ \vec{q}\cdot\vec{n} &= \alpha(T - T_{amb}) && \text{on } \Gamma_R \end{aligned} \tag{1}$$

Here, $T$ is the temperature field, $T_{amb}$ the ambient temperature, $\rho$ the density, $c_p$ the specific heat capacity, $\mu$ the heat conductivity, and $\alpha$ the convection coefficient (L. Landau, 1975). The thermal losses are captured by the volume heat load $h$ or the heat fluxes $h_f$ at the boundary.

The equation is discretized and written as state-space system of the form

$$E\,\dot{\vec{x}} = A(\vec{x})\,\vec{x} + B\,\vec{u}, \quad \vec{x}(0) = \vec{x}_0 \tag{2}$$

$$y = Cx \tag{3}$$

where $\vec{x}$ is the temperature vector:

$$\vec{x} = (x_1, ..., x_n)^T \tag{4}$$

$\vec{u}$ the input driving the system:

$$\vec{u} = (u_1, ..., u_m)^T \tag{5}$$

and $\vec{y}$ the output:

$$\vec{y} = (y_1, ..., y_p)^T \tag{6}$$

Component $x_i$ of $\vec{x}$ corresponds to the temperature of node $i$ in region $k_i$ of the discretized domain. $E$ is the capacitance matrix and $A(\vec{x})$ the conductance matrix , respectively. For given $\vec{x}$, matrices $E$, $A(\vec{x})$ and $B$ may be extracted from NX using a special subroutine. $E$ has diagonal form:

$$E = (e_{i,i}), \quad e_{i,i} = \rho c_p^{(k_i)} \bar{e}_{i,i} \tag{7}$$

or

$$E = \rho\, diag((c_p^{(k_1)}, ..., c_p^{(k_n)})\,\overline{E} \tag{8}$$

with

$$diag((c_p^{(k_1)}, ..., c_p^{(k_n)}) = \begin{pmatrix} c_p^{(k_1)} & & \\ & \ddots & \\ & & c_p^{(k_n)} \end{pmatrix} \tag{9}$$

The system is stable, $E$ has positive diagonal elements and $A(\vec{x})$ is symmetric and negative definite.

# 3 MATRIX INTERPOLATION

In this section, an improved approximation $\tilde{A}(\vec{x})$ compared to the constant approximation for $A(\vec{x})$ in (2) is constructed. As the exact form of the conductance matrix in (2) is not available and only conductance matrices for given temperature fields may be extracted from Simcenter Thermal Flow, it is proposed to construct a higher order approximation by matrix interpolation. For this purpose extract conductance matrices for $T = T_{min}$ and $T = T_{max}$:

$$A_{min} = A(T_{min}), \quad A_{max} = A(T_{max}) \tag{10}$$

and interpolate between these matrices. In the interior of region $k$, $A_{min}$ and $A_{max}$ have the form

$$A_{min} = \mu^{(k)}(T_{min})\bar{A}^{(k)}, \quad A_{max} = \mu^{(k)}(T_{max})\bar{A}^{(k)} \tag{11}$$

respectively, with

$$\bar{A}^{(k)} = (\bar{a}_{i,j}^{(k)}) \tag{12}$$

Important properties of the conductance matrix are that it is symmetric and the sum of columns / rows is zero:

$$A = (a_{i,j}), \quad a_{i,j} = a_{j,i}, \quad \sum_{j=1}^{n} a_{i,j} = 0 \tag{13}$$

Consider the following candidates for approximation

$$A^{(1)}(\vec{x}) := A_{min} + diag(\vec{\xi}(\vec{x}))\,\Delta A \tag{14}$$

$$A^{(2)}(\vec{x}) := A_{min} + \Delta A\, diag(\vec{\xi}(\vec{x})) \tag{15}$$

$$A^{(3)}(\vec{x}) := A_{min} + \frac{1}{2}\Big(diag(\vec{\xi}(\vec{x}))\,\Delta A + \Delta A\, diag(\vec{\xi}(\vec{x}))\Big) \tag{16}$$

where

$$\Delta A = A_{max} - A_{min} \tag{17}$$

and $\vec{\xi}(\vec{x})$ is defined by

$$\vec{\xi}(\vec{x}) = \frac{\vec{x} - T_{min}}{T_{max} - T_{min}} \tag{18}$$

Inside regions with nonlinear conductivities, $\vec{\xi}$ maybe replaced by:

$$\vec{\xi}^{(k)}(\vec{x}^{(k)}) = \frac{\mu^{(k)}(\vec{x}^{(k)}) - \mu^{(k)}(T_{min})}{\mu^{(k)}(T_{max}) - \mu^{(k)}(T_{min})} \tag{19}$$

For conductivities depending linearly on the temperature, (18) and (19) are equivalent. (18) has the advantage that it is independent of the region. So if not otherwise stated, (18) is used in the following. $A^{(1)}$ interpolates rowwise between $A_{min}$ and $A_{max}$, $A^{(2)}$ columnwise and $A^{(3)}$ both rowwise and columnwise. $A^{(3)}$ is symmetrical, but the sum of rows is not zero in general. This can be corrected by modification of the diagonal:

$$A^{(4)}(\vec{x}) := A_{min} + \frac{1}{2}\Big(diag(\vec{\xi}(\vec{x}))\,\Delta A + \Delta A\, diag(\vec{\xi}(\vec{x})) - diag(\Delta A \vec{\xi}(\vec{x}))\Big) \tag{20}$$

Since only $A^{(4)}$ fulfils both conditions in (13), we will concentrate on this approximation.

In the following, further characteristics of $A^{(4)}$ are discussed. Let:

$$A^{(4)} = \left(a^{(4)}_{i,j}\right) \tag{21}$$

For elements in the interior of region $k$ it holds

$$a^{(4)}_{i,i} = \mu^{(k)}_{lin}(x_i)\bar{a}^{(k)}_{i,i} - 0.5\sum_j \mu^{(k)}_{lin}(x_j)\bar{a}^{(k)}_{i,j} \tag{22}$$

and

$$a^{(4)}_{i,j} = 0.5(\mu^{(k)}_{lin}(x_i) + \mu^{(k)}_{lin}(x_j))\bar{a}^{(k)}_{i,j} \quad i \neq j \tag{23}$$

where $\mu^{(k)}_{lin}$ is a linearization of $\mu^{(k)}$:

$$\mu^{(k)}_{lin}(x) = \mu^{(k)}(T_{min}) + (x - T_{min}) \frac{\mu^{(k)}(T_{max}) - \mu^{(k)}(T_{min})}{T_{max} - T_{min}} \tag{24}$$

So off-diagonal elements $a_{i,j}$ (row $i$ and column $j$) get the arithmetic mean of $\mu^{(k)}_{lin}(x_i)$ and $\mu^{(k)}_{lin}(x_j)$ as weight, whereas all surrounding nodes contribute to the weight of a diagonal element. So this construction maybe considered as a simplified discretization.

# 4 REDUCED ORDER MODELING

For reduced order modelling of system (2), several methods are possible, e.g. quadratic-bilinear Krylov (Ahmad et al., 2016; Cao et al., 2018). Here we apply a combination of POD (Proper Orthogonal Decomposition) and DEIM (Discrete Empirical Interpolation Method)(Chaturantabut and Sorensen, 2010). For the conductance matrix, approximation $A^{(4)}$ in (21) is used. With $\xi$ in (18), $A^{(4)}$ consists of a constant and a linear part:

$$A^{(4)}(\vec{x}) = A_{min} + A_{lin}(\vec{x}) \tag{25}$$

with

$$A_{lin}(\vec{x}) = \frac{1}{2}\Big(diag(\vec{\xi}(\vec{x}))\,\Delta A + \Delta A\, diag(\vec{\xi}(\vec{x})) - diag(\Delta A \vec{\xi}(\vec{x}))\Big) \tag{26}$$

The general procedure is as follows:

- Generate snapshots $\vec{x}$ and training data $E^{-1}A_{lin}(\vec{x})\vec{x}$
- Compute POD projection matrix $V$ with dimensions $n \times n_r$
- Compute DEIM matrices $U, P_{deim}, P_{co\_deim}$ with dimensions $n \times n_{deim}$ and $n \times n_{co\_deim}$.

$P_{deim}, P_{co\_deim}$ are projection matrices of the form

$$P_{deim} = [e_{i_1}, ..., e_{i_{n_{deim}}}] \tag{27}$$

$$P_{co\_deim} = [e_{j_1}, ..., e_{j_{n_{co\_deim}}}] \tag{28}$$

where $i_1, ..., i_{n_{deim}}$ are the indices of the "DEIM Dofs" and $j_1, ..., j_{n_{co\_deim}}$ are the DOFs coupled with DEIM DOFs. The model reduced only by POD would have the form:

$$\dot{\vec{x}}_r = A_r\vec{x}_r + V^T E^{-1} A_{lin}(V\vec{x}_r) V\vec{x}_r + V^T Bu \tag{29}$$

where

$$\vec{x}_r = (\tilde{x}_1, \ldots, \tilde{x}_{n_r})^T \tag{30}$$

is the reduced solution vector, and

$$A_r = V^T E^{-1} A_{min} V \tag{31}$$

Since the nonlinear term $A_{lin}(V\vec{x}_r)V\vec{x}_r$ still depends on the original problem size $n$, further reduction is necessary which is achieved by DEIM. The POD+DEIM reduced model has the form:

$$\begin{aligned} \dot{\vec{x}}_r = A_r\vec{x}_r + \frac{1}{2}V^T M_{deim}\Big[ & \\ & diag\big(\vec{\xi}(\vec{x}_{deim})\big)\Delta A_{deim}\vec{x}_{co\_deim} \\ & +\Delta A_{deim}\, diag\big(\vec{\xi}(\vec{x}_{co\_deim})\big)\vec{x}_{co\_deim} \\ - \; & diag\big(\Delta A_{deim}\vec{\xi}(\vec{x}_{co\_deim})\big)\vec{x}_{deim}\Big] + V^T Bu \end{aligned} \tag{32}$$

with

$$M_{deim} = U(P_{deim}^T U)^{-1} \tag{33}$$

$$\Delta A_{deim} = P_{deim}^T E^{-1} \Delta A\, P_{co\_deim} \tag{34}$$

$$\vec{x}_{deim} = P_{deim}^T V \vec{x}_r \tag{35}$$

$$\vec{x}_{co\_deim} = P_{co\_deim}^T V \vec{x}_r \tag{36}$$

# 5 APPLICATION: HEATING OF A MOTOR

The proposed method is applied to the thermal model of a motor. Fig. 1 shows the motor components and materials, Fig. 2 the sensor positions. The thermal conductivity is constant in rotor and stator regions and temperature-dependent in windings, circuit rings and shaft, see Fig. 3. For construction of a higher order approximation in (21), conductance matrices $A_{min} = A(T_{min})$ and $A_{max} = A(T_{max})$ are extracted from NX for $T_{min} = 20°C$ and $T_{max} = 300°C$. $\vec{\xi}$ in (21) is defined by (19) in windings and circuit rings regions, and by (18) otherwise. The problem dimensions are shown in Table 1.

Table 1: Dimensions of full and reduced problem.

| | |
|---|---|
| $n$ | 186527 |
| $n_r$ | 60 |
| $n_{deim}$ | 60 |
| $n_{co\_deim}$ | 332 |

The loads are applied to the windings. Two test cases are considered:

- Constant load
- Time-dependent load

In each test case, the initial condition is $T(0) = 22°C$.

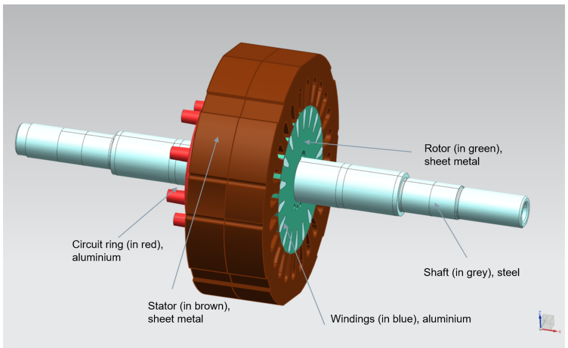



Figure 1: Components of the motor.

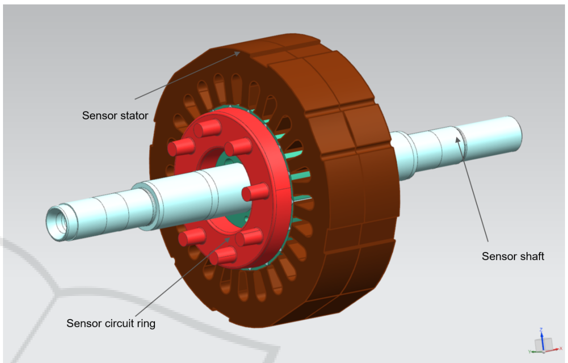



Figure 2: Sensor positions.

## 5.1 Constant Load

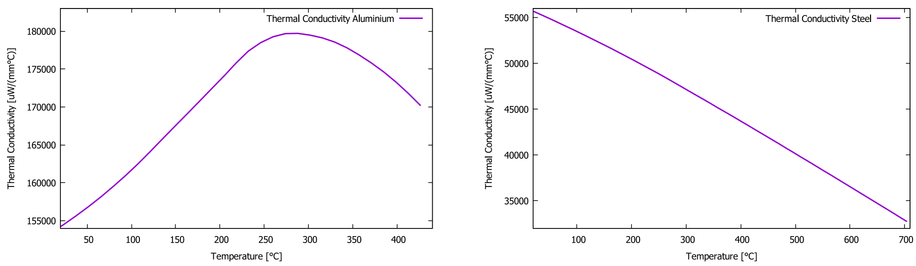
In the first test case, a constant load of $u = 2e7\ W/m^3$ is applied to the windings. Fig. 4 shows a comparison between NX and the reduced linear model, with a constant conductance matrix extracted for $T = 20°$. Figs. 5 and 6 show results for the full and the reduced nonlinear model, respectively. With both models, higher accuracies are achieved compared to the reduced linear model, whereby there are only minor differences between the full and the reduced nonlinear model. The absolute and relative errors are listed in Table 2.

Table 2: Maximum absolute and relative errors between the listed models.

| Model 1 | Model 2 | $\varepsilon_{abs}$ | $\varepsilon_{rel}$ |
|---|---|---|---|
| NX | MOR(nonlin) | $0.9°C$ | 0.8% |
| NX | MOR(lin) | $5.6°C$ | 2.9% |

## 5.2 Time-dependent Load

In the second test case, a time-dependent load is applied (Fig. 7). Again the full and the reduced nonlinear models achieve higher accuracies than the reduced linear model, see Figs. 8-10 and Table 3.




(a) Aluminium (b) Steel

Figure 3: Thermal conductivities of the materials under consideration.

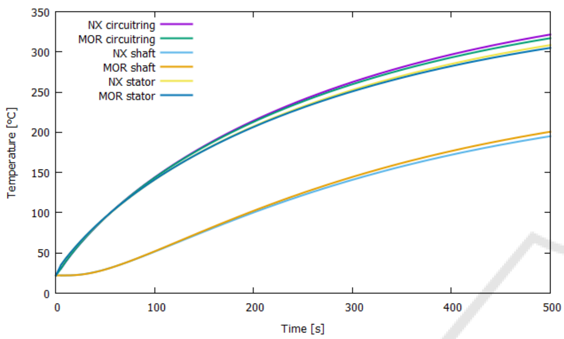



Figure 4: Comparison between NX and reduced linear model, conductance matrix extracted for T=20°$C$.

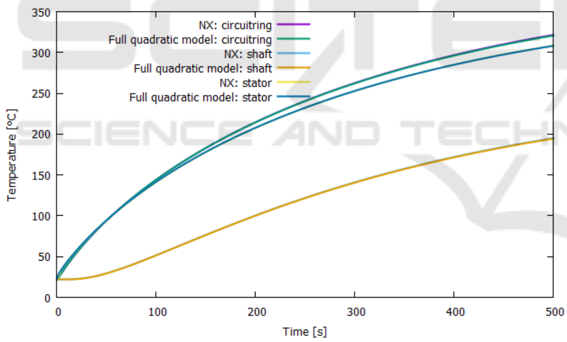



Figure 5: Comparison between NX and full nonlinear model, conductance matrices extracted for T=20°$C$ and T=300°$C$.

Table 3: Maximum absolute and relative errors between the listed models.

| Model 1 | Model 2 | $\varepsilon_{abs}$ | $\varepsilon_{rel}$ |
|---|---|---|---|
| NX | MOR(nonlin) | 1.6°$C$ | 2.8% |
| NX | MOR(lin) | 6.1°$C$ | 3.0% |

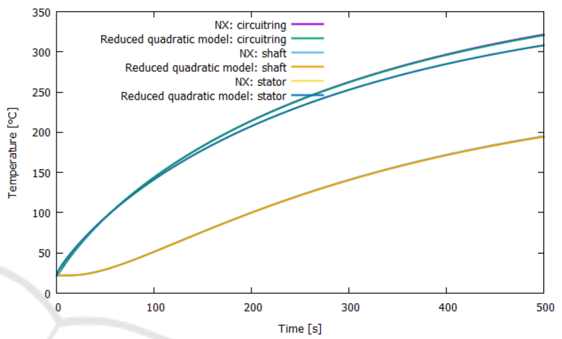



Figure 6: Comparison between NX and reduced nonlinear model, conductance matrices extracted for T=20°$C$ and T=300°$C$.

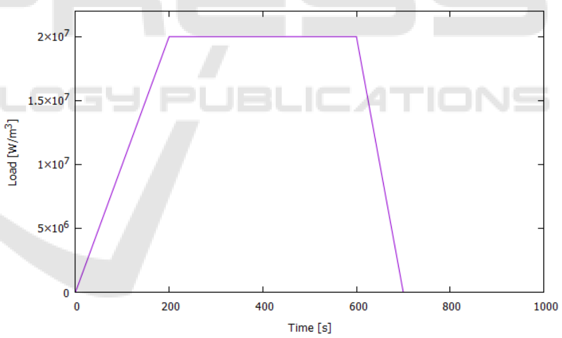



Figure 7: Time-dependent load case.

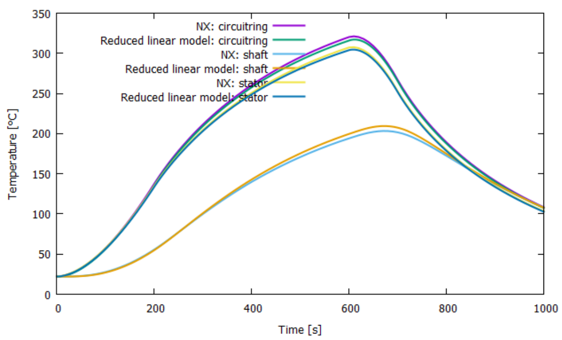



Figure 8: Comparison between NX and reduced linear model, conductance matrix extracted for T=20°$C$.

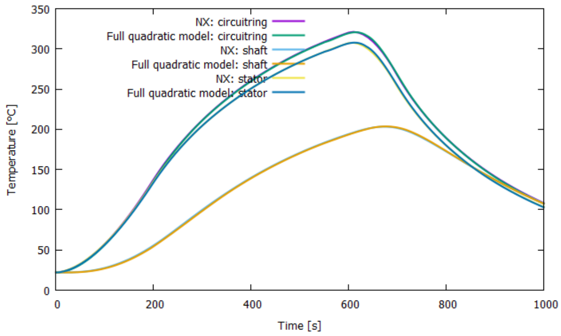



Figure 9: Comparison between NX and full nonlinear model, conductance matrices extracted for T=20°$C$ and T=300°$C$.

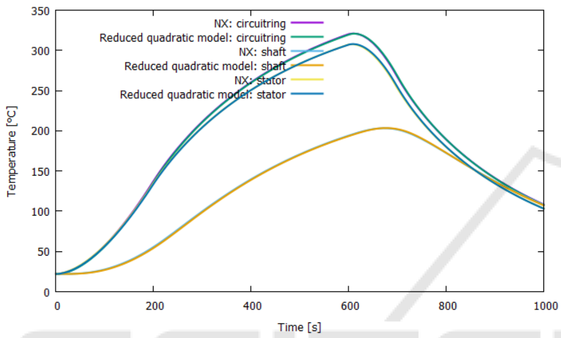



Figure 10: Comparison between NX and reduced nonlinear model, conductance matrices extracted for T=20°$C$ and T=300°$C$.

# 6 CONCLUSIONS

In this paper, model order reduction of thermal problems with temperature-dependent conductivities has been considered, with the constraint that a commercial solver is used for the full problem where only matrices for given temperature fields can be extracted. It has been proposed to approximate the conductance matrix by multi-linear matrix interpolation which only slightly complicates the solution workflow. In the selected examples, the reduced model of this approximation achieves higher accuracies compared to a model based on a constant approximation of the conductance matrix. Further improvements maybe achieved by algorithms of deep learning (Löhner et al., 2021) which will be the subject of future investigations.

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