Soft Directional Substitutable based Decompositions for MOVCSP

Maher Helaoui[1][a] and Wady Naanaa[2]

[1]LIMTIC Laboratory, Higher Institute of Business Administration, University of Gafsa, Tunisia

[2]LIMTIC Laboratory, National Engineering School of Tunis, Tunis, Tunisia

Keywords: Multi-objective Valued Constraint Satisfaction Problems, Tractable Class, Directional Substitutable Valuation Functions, Decomposition Scheme for General MOVCSP.

Abstract: To better model several artificial intelligence and combinatorial problems, classical Constraint Satisfaction Problems (CSP) have been extended by considering soft constraints in addition to crisp ones. This gave rise to a Valued Constraint Satisfaction Problems (VCSP). Several real-world artificial intelligence and combinatorial problems require more than one single objective function. In order to present a more appropriate formulation for these real-world problems, a generalization of the VCSP framework called Multi-Objective Valued Constraint Satisfaction Problems (MOVCSP) has been proposed.

This paper addresses combinatorial optimization problems that can be expressed as MOVCSP. Despite the NP-hardness of general MOVCSP, we can present tractable versions by forcing the allowable valuation functions to have specific mathematical properties. This is the case for MOVCSP whose dual is a binary *MOVCSP* with crisp binary valuation functions only and with a weak form of Neighbourhood Substitutable Valuation Functions called Directional Substitutable Valuation Functions.

1 INTRODUCTION

Constraint Satisfaction Problems (CSP) provide a general and convenient framework to model and solve numerous combinatorial problems including temporal reasoning (van Beek and Manchak, 1996), computer vision (Schlesinger, 2007)...In the standard CSP framework, the constraints are defined by crisp relations, which specify the consistent combinations of values. With these relations, one can force some pairs of intervals to overlap, and any plan that does not meet this requirement is considered as inconsistent even though the intervals are very close.

However, one may need to express various degrees of consistency in order to reflect the specificity of the problem at hand. The valued constraint satisfaction problems (VCSPs) approach (Schiex et al., 1995) is intended to model such situations. A VCSP consists of a set of variables taking values in discrete sets called *domains*. A valued constraint is defined through the use of a valuation function. The role of a valuation function is to associate a degree of desirability to each combination of values. The problem is to find an assignment of values to variables from their respective domains with an optimal cost.

[a] https://orcid.org/0000-0002-4748-4773

The computational complexity of finding the optimal solution to a VCSP has been largely studied in many works and several classes of tractable VCSPs, that is, VCSPs that are solvable in polynomial time, have been identified and solved. Tractability is obtained by limiting the set of allowed valuation functions and or by detecting some desirable properties exhibited by the problem structure (Cohen et al., 2008a; Cohen et al., 2008b; Greco and Scarcello, 2011; Cohen et al., 2012; Cooper and Zivný, 2011; Cooper and Zivný, 2012; Helaoui and Naanaa, 2013; Helaoui et al., 2013; Cooper et al., 2016; Carbonnel and Cooper, 2016).

However, in real-world situations like the discrete time/cost trade-off problem (Vanhoucke, 2005; Debels and Vanhoucke, 2007; Tavana et al., 2014), one may need to express multiple objectives to optimize in order to reflect the specificity of the problem at hand (Greco and Scarcello, 2013).

Incorporating conflicting objective functions divide the solution set into dominated and non-dominated solutions. With reference to Pareto, Non Dominated Solutions (NDS) are solutions where we cannot improve further the attainability of one objective without degrading the attainability of another, which means that a compromise should be found.

Helaoui, M. and Naanaa, W.
Soft Directional Substitutable based Decompositions for MOVCSP.
DOI: 10.5220/0010271802180225
In *Proceedings of the 10th International Conference on Operations Research and Enterprise Systems (ICORES 2021)*, pages 218-225
ISBN: 978-989-758-485-5

Table 1: The Π project.

Tasks	Prede-cessors	choice 1	choice 2	choice 3
T_A	–	(15,10)	(9,25)	(3,50)
T_B	T_A	(15,10)	(12,30)	(6,90)
T_C	T_A	(15,10)	(9,35)	(6,60)
T_D	T_A	(30,20)	(24,50)	(21,80)
T_E	T_B,T_C	(15,10)	(9,30)	(3,60)
T_F	T_D,T_E	(15,10)	(12,58)	(6,250)

Example 1. DISCRETE TIME COST TRADE OFF PROBLEM (DE ET AL., 1997).
Let Π be a project defined as follows:
Π is comprised of 6 tasks: T_A, T_B, T_C, T_D, T_E and T_F. The predecessors of each task are defined by column "Predecessor" of Table 1.
The various options of the executions times and the relatives costs of each tasks are given in columns 3, 4 and 5. For instance, Task T_A could be executed in 15 time units with cost 10 or in 9 time units with cost 25 or even in 3 time unit but the cost rise to 50. Solving the problem amounts to finding, for each task, one choice such that both global costs and global makespan are optimized and the precedence constraints are satisfied.

The multi-objectives valued constraint satisfaction problems (MOVCSP) presented in (Ali et al., 2019) is intended to model such situations. A Multi-objectives VCSP consists in a VCSP where the goal is to find an assignment of values to variables, from their respective domains, with an optimal multi-objectives valuations. Solving a problem with several multi-objective functions is commonly referred to as multi-objective problem. The goal is to compute the best set of compromise solutions called *Pareto borders.*

Furthermore, interchangeability and substitutability are two techniques that have been initially introduced for CSP (Freuder, 1991). In (Lecoutre et al., 2012), Neighbourhood Substitutability has been extended to VCSP. A decomposition directional substitutability algorithm that applies when the studied problem does not satisfy the conditions of interchangeability or substitutability has been proposed in (Naanaa, 2008; Naanaa et al., 2009) respectively for CSP and CSOP : a VCSP with Crisp binary Constraint.

In this paper, we present tractable versions of MOVCSP by forcing the allowable valuation functions to have specific mathematical properties. This is the case for MOVSCP whose dual is a binary crisp *MOVCSP* with crisp binary valuation functions only and with Directional Substitutable Valuation Functions. We denote this MOVCSP class by $\mathcal{L}(MODS)$. We also take advantage of the discovered tractable class to conceive a decomposition scheme for general MOVCSP.

The paper is organized as follows: the next Section introduces MOVCSP. In Section 3 we study Soft Directional Substitutable MOVCSP. We conclude in Section 4.

2 MULTI-OBJECTIVE VCSP

In a MOVCSP, and as for a VCSP (Schiex et al., 1995), for each objective $j = 1,2,\ldots,k$, we assume a set E_j of possible valuations which is a totally ordered with a minimal element $\perp_j$ and a maximal element $\top_j$. In addition, we need k monotone operators $\oplus_j$, $j : 1,\ldots,k$. These components can be gathered in k valuation structures each of which can be specified as follows:

Definition 1. *A valuation structure S_j is the triple $S_j = (E_j, \oplus_j, \preceq_j)$, where*

- *E_j is a set of valuations for the objective function j;*
- *$\preceq_j$ is a total order on E_j;*
- *$\oplus_j$ is commutative, associative and monotone binary operator.*

Once the valuation structure S is specified, the multi-objective valued constraint satisfaction problem (MOVCSP) can be defined as follows:

Definition 2. *A multi-objective valued constraint satisfaction problem denoted (MOVCSP) is defined by the tuple $(X, \mathcal{D}, C, \mathcal{S})$ such as:*

- *X is a finite set of variables;*
- *$\mathcal{D}$ is a finite set of value domain, such that $D_x \in \mathcal{D}$ denotes the domain of $x \in X$.*
- *$\mathcal{S} = (S_1,\ldots,S_k)$, where each S_j is a valuation structure of objective j;*
- *C is a set of valued constraints. Each constraint is an ordered pair (σ, Φ), where $\sigma \subseteq X$ is the scope of the constraint and Φ is a k-functions vector $\langle \phi_1,\ldots,\phi_k \rangle$, where each function ϕ_j is from $\Pi_{x\in\sigma} D_x$ to E_j.*

The arity of a multi-objective valued constraint is the size of its scope. The arity of a MOVCSP is the maximum over the arities of all its constraints.

To simplify the notation, and if there is no confusion, we will denote each $\oplus_j$ by $\oplus$, each $\preceq_j$ by $\preceq$ and each ϕ_j by ϕ.

The valuation of an assignment t that assigns values to a subset of variables $V \subseteq X$ is obtained by

$$\Phi(t) = \left(\bigoplus_{(\sigma,\phi_1)\in C,\sigma\subseteq V} \phi_1(t\downarrow\sigma),\ldots, \bigoplus_{(\sigma,\phi_k)\,\in\, C,\sigma\subseteq V} \phi_k(t\downarrow\sigma) \right) \quad (1)$$

Where $t \downarrow \sigma$ denotes the projection of t on the variables of σ. Hence, an optimal solution of a MOVCSP on n variables is a n-tuple t such that $\Phi(t)$ is optimal over all possible n-tuples.

In order to simplify the notation we denote $v_1,\ldots,v_{|\sigma|}$ by v_σ.

Definition 3. *Let t_1 and t_2 two solutions.*

- *We say that solution t_1 dominates solution t_2 ($t_1 \preceq_D t_2$) if, for each objective j, we have*

$$\bigoplus_{(\sigma,\Phi)\in C} \phi_j(t_1\downarrow\sigma) \preceq \bigoplus_{(\sigma,\Phi)\in C} \phi_j(t_2\downarrow\sigma)$$

  with at least one objective, we have a strict inequality.
- We say that t_1 and t_2 are two Non Dominated Solutions (NDS) if there are two objectives j and j' $\in k$, such that

$$\bigoplus_{(\sigma,\Phi)\in C} \phi_j(t_1\downarrow\sigma) \preceq \bigoplus_{(\sigma,\Phi)\in C} \phi_j(t_2\downarrow\sigma) \wedge$$

$$\bigoplus_{(\sigma,\Phi)\in C} \phi_{j'}(t_2\downarrow\sigma) \succ \bigoplus_{(\sigma,\Phi)\in C} \phi_{j'}(t_2\downarrow\sigma)$$

This allows us to define a partial order between the dominated solutions.

Lemma 1. *$\preceq_D$ is transitive.*

Proof Lemma: According to the Definition 3, we have:

- (i) $t_1 \preceq_D t_2$ if and only if for each objective j

$$\bigoplus_{(\sigma,\Phi)\in C} \phi_j(t_1\downarrow\sigma) \preceq \bigoplus_{(\sigma,\Phi)\in C} \phi_j(t_2\downarrow\sigma)$$

- (ii) $t_2 \preceq_D t_3$ if and only if for each objective j

$$\bigoplus_{(\sigma,\Phi)\in C} \phi_j(t_2\downarrow\sigma) \preceq \bigoplus_{(\sigma,\Phi)\in C} \phi_j(t_3\downarrow\sigma)$$

- (i) and (ii) imply that for each objective j

$$\bigoplus_{(\sigma,\Phi)\in C} \phi_j(t_1\downarrow\sigma) \preceq \bigoplus_{(\sigma,\Phi)\in C} \phi_j(t_3\downarrow\sigma)$$

Hence $t_1 \preceq_D t_3$. □

Example 2. *We will return to the same DTCT project* Π *presented in Example 1. This project* Π *can be modelled as a bi-objectives VCSP $\mathcal{P}_1$ defined such that:*

1. *X is a finite set of variables such that each x_i is a task i;*
2. *$\mathcal{D} = \{v_1, v_2, v_3\}$ is a set of finite domains, where $v_{choice} \in \mathcal{D}$ denotes the value v_{choice} of the variable x_i;*
3. *$\mathcal{S} = (E, \oplus, \preceq)$ is a fair valuation structure, where $\oplus$ is the sum and E the set of integers.*
4. *C is a set of valued constraints. Each unary valued constraint C^1 is an ordered pair $(\langle x_i\rangle, \Phi(v_{choice}) = \langle \phi_d(v_{choice}), \phi_c(v_{choice})\rangle)$.*

We get $P_1 =$

$$(X, \mathcal{D}, \mathcal{S}, (\langle x_i\rangle, \Phi))$$

The predecessors[2] *of each task are defined in the second column of Table 2. The valuation functions of the bi-objectives VCSP $\mathcal{P}_1$ are given in columns* 3, 4 *and* 5.

Table 2: The bi-objectives VCSP $\mathcal{P}_1 = \Pi$.

X	Predecessors	$\Phi(v_1)$	$\Phi(v_2)$	$\Phi(v_3)$
x_A	–	(15,10)	(9,25)	(3,50)
x_B	x_A	(15,10)	(12,30)	(6,90)
x_C	x_A	(15,10)	(9,35)	(6,60)
x_D	x_A	(30,20)	(24,50)	(21,80)
x_E	x_B,x_C	(15,10)	(9,30)	(3,60)
x_F	x_D,x_E	(15,10)	(12,58)	(6,250)

3 TRACTABLE CLASS FOR MOVCSPs

If it is more easy to generalize tractability from VCSP to MOVCSP with Soft Neighbourhood Substitutable Valuation Functions only. What about MOVCSP with a weak form of Soft Neighbourhood Substitutable called Soft Directional Substitutable Valuation Functions? In this section, we present tractable class for MOVCSPs that take advantage of Soft Directional Substitutable Valuation Functions.

3.1 The Power of a Binary Crisp *MOVCSP*

We present the power of a binary *MOVCSP* with crisp binary valuation functions only motivated by the fact that

1. Any binary MOVCSP with only modular binary functions and any crisp binary functions can be transformed in polynomial time to an equivalent

[2]In graph theory, the precedence constraints of $\mathcal{P}_1$ are satisfied by using RANK algorithm in order to give the variables order of execution.

binary *MOVCSP* with crisp binary valuation functions only.

2. The dual problem of any MOVCSP is a binary *MOVCSP* with crisp binary valuation functions only.

We define a binary CSOP as a binary VCSP such that binary valuations are only in $\{\bot, \top\}$. A binary MOCSOP is a binary MOVCSP such that binary valuations are only in $\{\bot_j, \top_j\}$ for each objective j.

Let, for each objective j, $\phi_j : D \times D' \longrightarrow E_j$ be a binary function which is not necessarily modular. In the following, we show that restricting the first argument of ϕ_j to specific subsets of D yields a family of modular binary functions.

Definition 4. *Let, for any objective j, $\phi_j : D \times D' \longrightarrow E_j$ be a binary function and let a, b be in D. We say that a and b are modular with regard to all ϕ_j, $a \sim_\Phi b$, if and only if the restriction of all ϕ_j to $\{a, b\} \times D'$ is modular.*

We note the class of MOVCSP with only modular binary functions and any crisp binary functions $\mathcal{L}_2(M)$. Note that

$$\Phi \in \mathcal{L}_2(M) \quad \Leftrightarrow \quad a \sim_\Phi b, \quad \forall a, b \in D \wedge \forall\, j \in k \tag{2}$$

Lemma 2. *Let $\mathcal{P}$ a binary MOVCSP. If $\mathcal{P} \in \mathcal{L}_2(M)$ then it exists a polynomial transformation ρ such that $\rho(\mathcal{P}) =$ binary MOCSOP*

Proof Lemma: By applying DECOMPOSE algorithm presented in (Helaoui and Naanaa, 2013) for each objective j to any binary MOVCSP with only modular binary functions and any crisp binary functions we get a binary MOCSOP. Since DECOMPOSE algorithm run on $O(ed)$ (where e is the number of constraints, and d is the size of the largest value domain) and it must be called for each objective j. Then ρ can be done on $O(ked)$. We can conclude that ρ is a polynomial transformation. □

The dual problem of a MOVCSP is a reformulation of the problem that expresses each constraint of the original problem as a variable. The dual problems contain only unary and binary constraints and therefore are binary problems. Therefore, it is possible to apply the known algorithms for such problems.

The dual problem of a MOVCSP is a reformulation of the latter which considers each constraint of the original problem as a variable. The unary constraint associated with such variable specifies unary and binary costs and costs given by the unary and binary constraints of the original problem.

Binary constraints of the dual problem express the fact that the variables common to two constraints must have the same value.

Table 3: The valuation functions of the primal problem $\mathcal{P}_2$.

Φ	1	2
1	(α_1, α_2)	$(\bot_1, \bot_2)$
2	$(\top_1, \top_2)$	(β_1, β_2)

Definition 5. *Let $\mathcal{P} = (X, \mathcal{D}, C, S)$ a MOVCSP. The dual of $\mathcal{P}$, denoted $\mathcal{P}^*$ is defined by $(X^*, \mathcal{D}^*, C^*, S^*)$ such that:*

- $X^* = C$;
- $\mathcal{D}^*$ *is such that* $D^*_c = \{t \in D_x \times D_y \mid \sigma(c) = \langle x, y \rangle\}$;
- $S^* = S$;
- $C^* = C^{*1} \cup C^{*2}$ *où*
  - $C^{*1} = \{(\langle c \rangle, \phi) \mid c = (\langle x, y \rangle, \Phi) \in C\}$ *with* $\Phi(t) = \Phi(t \downarrow x) \oplus \Phi(t)$;
  - $C^{*2} = \{(\langle c_1, c_2 \rangle, \Phi) \mid c_1, c_2 \in C \wedge \sigma(c_1) \cap \sigma(c_2) \neq \varnothing\}$ *with*

$$\Phi(t, t') = \begin{cases} (\bot_1, \ldots, \bot_k) \textit{if}(t \downarrow \sigma(c_1) \cap \sigma(c_2) \\ = t' \downarrow \sigma(c_1) \cap \sigma(c_2)) \\ (\top_1, \ldots, \top_k) \textit{else} \end{cases}$$

where $\sigma(c)$ designates the scope of the constraint c. As can be seen from the above definition, all binary constraints of the dual of a MOVCSP are binary crisp constraints. Hence the following Lemma

Lemma 3. *The dual of a MOVCSP is a binary MOCSOP.*

Example 3. Let $\mathcal{P}_2$ is a binary bi-Objectives VCSP composed of three variables x_1, x_2 and x_3. (see Figure 1). The domain D_i of each variable x_i is formed

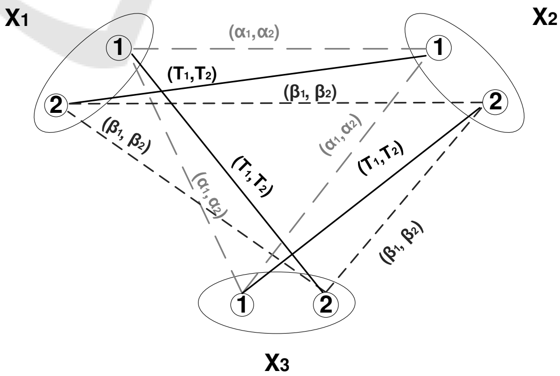

Figure 1: $\mathcal{P}_2$ the Primal bi-objectives VCSP.

of two values $D_i = \{1, 2\}$. This bi-Objectives VCSP has three constraints $c_1 = (\langle x_1, x_2 \rangle, \Phi)$, $c_2 = (\langle x_2, x_3 \rangle, \Phi)$ and $c_3 = (\langle x_3, x_1 \rangle, \Phi)$ whose valuations functions Φ are defined in Table 3. Referring to Definition 5, the dual of $\mathcal{P}_2$ is a problem $\mathcal{P}^*_2$ defined as follows

Table 4: Unary cost valuation functions of the dual problem $\mathcal{P}_2^*$.

Values	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
Unary-cost Φ	(α_1,α_2)	$(\bot_1,\bot_2)$	$(\top_1,\top_2)$	(β_1,β_2)

- $\mathcal{P}_2^*$ is a bi-objectives VCSP composed of three variables (the constraint of $\mathcal{P}_2$): c_1, c_2 and c_3. (See Figure 2).
- The domain D_j of each variable c_j is formed of four tuple values $D_j = \{(1,1),(1,2),(2,1),(2,2)\}$.
- Binary constraints in $\mathcal{P}_2$ become unary constraints in $\mathcal{P}_2^*$:
  - for each variable c_j, the value $(1,1)$ has a unary cost (α_1,α_2), the value $(1,2)$ has a unary cost $(\bot_1,\bot_2)$, the value $(2,1)$ has a unary cost $(\top_1,\top_2)$ and it will be filtered by applying an arc-consistency algorithm, and the value $(2,2)$ has a unary cost (β_1,β_2).
- binary constraints in $\mathcal{P}_2^*$ should be added Prohibiting the choice of two values of the same variable in $\mathcal{P}_2$. These binary constraints are crisp as they prohibit impossible solutions for the primal problem.

As can be seen in Table 4 and Figure 2 $\mathcal{P}_2^*$ is a bi-objectives CSOP.

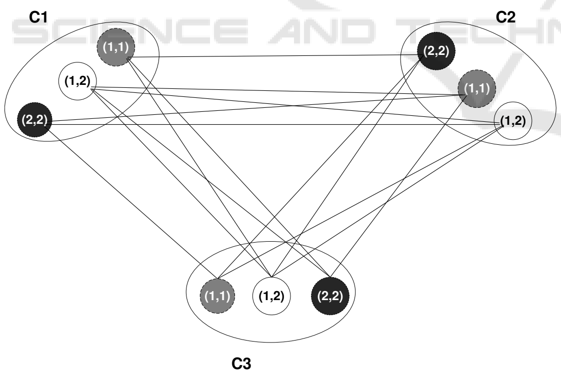

Figure 2: $\mathcal{P}_2^*$ the Dual of the bi-objectives VCSP.

3.2 Directional Substitutability for a Binary MOCSOP

The Directional Substitutability (Naanaa, 2008) is a weak form of Neighbourhood Substitutability (Freuder, 1991). Initially, this concept has been defined for binary CSPs. In this paper, we generalize the concept of directional substitutability to reflect multi-objective unary cost functions involved in binary MOCSOP.

Definition 6. *The* multi objective inconsistency graph *of a binary MOCSOP $\mathcal{P}$ is a simple graph $GI(\mathcal{P})$ in which the vertices correspond to the k values of the variables and edges connecting pairs of vertices representing for each objective incompatible values.*

Multi Objective Directional substitution for binary MOCSOP, in this paper, is defined using, as a reference, an orientation of the multi objective inconsistency graph of a binary MOCSOP.

Definition 7. *An orientation A_j of multi objective inconsistency graph of binary MOCSOPs is an assignment for each objective j of a direction to each edge $\{\phi_j(a),\phi_j(b)\}$ of graph resulting to the arc $(\phi_j(a),\phi_j(b))$ or arc $(\phi_j(b),\phi_j(a))$.*

The concept of multi-objective directional substitutability is then a binary relation defined, in this paper, as follows

Definition 8. *Let $\mathcal{P} = (X,\mathcal{D},C,S)$ is a binary MOCSOP, $x \in X$ and $a,b \in D_x$. A value a is said Multi Objective Directionally Substitutable (MODS) to b with reference to A_j (notation $a \preceq_{A_j} b$) if for each objective $j = 1,\ldots,k$:*

1. *for all $(\langle x\rangle,\phi) \in C^1$, we have*

$$\phi_j(a) \preceq_j \phi_j(b)$$

2. *for all $(\langle x,y\rangle,\phi) \in C^2$ et $a' \in D_y$, we have*

$$(\phi_j(a),\phi_j(a')) \in A_j \Rightarrow (\phi_j(b),\phi_j(a')) \in A_j \wedge$$

$$\phi_j(a,a') \preceq_j \phi_j(b,a')$$

With reference to this definition we can define Multi-Objective Directional Substitutable Valued Constraint Satisfaction Problem with Crisp binary Constraints as follows

Definition 9. *Let $\mathcal{P}$ a binary MOVCSP. $\mathcal{P}$ is a binary Multi-Objective Directional Substitutable Valued Constraint Satisfaction Problem with only Crisp binary Constraints $\mathcal{P} \in \mathcal{L}(MODS)$ if and only if for each variable x of $\mathcal{P}$ and for each $a,b \in D_x$: a is MODS to b or b is MODS to a $a \preceq_{A_j} b \vee b \preceq_{A_j} a$*

Given an orientation A_j for each objective j of multi objective inconsistency graph, the relation $\preceq_{A_j}$ defines a total order on the domain of each variable.

Lemma 4. *$\preceq_{A_j}$ is a total order on D_x.*

Proof Lemma: $\preceq_{A_j}$ is trivially reflexive. We prove that $\preceq_{A_j}$ is also transitive. To this end, assume the opposite and proceed to obtain a contradiction. Let $u,v,w \in D_x$ such that for each objective j $u \preceq_{A_j} v \wedge v \preceq_{A_j} w$ This means that $u \preceq_{A_j} v \wedge v \preceq_{A_j} w$

but $\exists\ j$ such that $u \not\preceq_{A_j} w$. This means that, for all $u' \in D_y$, we have for each objective j

$$\phi_j(u) \preceq_j \phi_j(v) \wedge [(\phi_j(u),\phi_j(u')) \in A_j \Rightarrow (\phi_j(v),\phi_j(u')) \in A_j \wedge \phi_j(u,u') \preceq_j \phi_j(v,u')] \quad (3)$$

$$\phi_j(v) \preceq_j \phi_j(w) \wedge [(\phi_j(v),\phi_j(u')) \in A_j \Rightarrow (\phi_j(w),\phi_j(u')) \in A_j \wedge \phi_j(v,u') \preceq_j \phi_j(w,u')] \quad (4)$$

as $\exists\ j$ such that $u \not\preceq_{A_j} w$ and $\preceq$ is a total order, it must exists $u' \in D_y$ such that it must exists j where:

$$\phi_j(u) \succ_j \phi_j(w) \vee [(\phi_j(u),\phi_j(u')) \in A_j \wedge (\phi_j(w),\phi_j(u')) \notin A_j \vee \phi_j(u,u') \succ_j \phi_j(w,u')] \quad (5)$$

From (3) and (4) we get for each objective j $\phi_j(u) \preceq_j \phi_j(w) \wedge [(\phi_j(u),\phi_j(u')) \notin A_j \vee (\phi_j(w),\phi_j(u')) \in A_j \wedge \phi_j(w,u') \succ_j \phi_j(u,u')]$ which contradicts (5).

Thus, $\preceq_{A_j}$ is transitive. □

The binary relation $\sim_{A_j}$ is defined on D_x as follows: $u \sim_{A_j} v$ if and only if $u \preceq_{A_j} v$ and $v \preceq_{A_j} u$.

Lemma 5. *$\sim_{A_j}$ is an equivalence relation on D_x.*

Thus, each domain D_x can be divided into subsets $D_x = D_{x,1} \cup \ldots \cup D_{x,s}$ such that the elements of each $D_{x,Q}$, $k = 1,\ldots,s$ they are all comparable, that is to say, that for any $a,b \in D_{x,Q}$, we have $a \preceq_{A_j} b$ or $b \preceq_{A_j} a$. Each $D_{x,Q}$ is a chain of value totally ordered by $\preceq_{A_j}$.

In each chain $D_{x,Q}$, we can distinguish the subset of directional dominant elements denoted by $D^+_{x,Q}$.

$$D^+_{x,Q} = \{a \in D_{x,Q} \mid \forall\, b \in D_{x,Q}, a \preceq_{A_j} b\} \quad (6)$$

3.3 Tractability of the Directional Substitutable MOCSOP Class

In what follows, we study the tractability of binary MOCSOP.

The following approach identifies a tractable class of binary MOCSOP which is based on the Directional Substitutable functions.

We denote that if $\mathcal{P}$ is in $\mathcal{L}(MODS)$ then it is with Directional Substitutable Valuation Functions only.

Theorem 1. *The class of binary Multi-Objective Directional Substitutable Valued Constraint Satisfaction Problem with only crisp binary constraint ($\mathcal{L}(MODS)$) is tractable.*

Proof Theorem 1. *First we will make $\mathcal{P}$ arc-consistent. Then, we show that by selecting a dominant element (see (6)) of each domain value, an optimal solution is obtained. This means that any n-tuple $t \in \Pi_{x\in X}\, D^+_x$ is an optimal solution. Referring to the Definition 8 and the fact that the function Φ is computable in polynomial time, this selection can be done in polynomial time.*

Suppose that $t \in \Pi_{x\in X}\, D^+_x$ is not an optimal solution. This means that t is inconsistent or that $\Phi(t)$ is not dominate.

Suppose that t is inconsistent. Therefore t must include, at least, a pair of incompatible values. Let $a \in D^+_x$ and $b \in D^+_y$ such a pair. Since a and b are inconsistent, for each objective j such that $(\phi_j(a),\phi_j(b)) \in A_j$ $\{\phi_j(a),\phi_j(b)\}$ must be an arc of GI($\mathcal{P}$). As a result, we must have for each objective j $(\phi_j(a),\phi_j(b)) \in A_j$ or $(\phi_j(b),\phi_j(a)) \in A_j$.

Assume without loss of generality that $(\phi_j(a),\phi_j(b)) \in A_j$, (otherwise we can reason on $\phi_j(b)$ rather than $\phi_j(a)$ and obtain the same result). It follows that $(\forall\ j)\ \phi_j(a,b) = \top_j$, and since $a \in D^+_x$ then for all $a' \in D_x$, we must have $(\forall\ j)\ \phi_j(a',b) = \top_j$. This means that $(\forall\ j)$ b has no support in D_x and so that $\mathcal{P}$ is not arc-consistent, hence a contradiction.

Suppose now that $\Phi(t)$ is not dominant, therefore $t' \in \bigoplus_{x\in X} D_x$ such that $\Phi(t') \prec \Phi(t)$. Since the values of the function $\Phi(t)$ are obtained from those of $\phi(t \downarrow x)$ using a monotone operator there must be $x \in X$ such that $t \downarrow x = a \wedge t' \downarrow x = a'$ and $\exists\ j$ such that $\phi_j(a') \prec_j \phi_j(a)$. It follows that $a \notin D^+_x$), hence a contradiction. □

3.4 Usefulness of Directional Substitutable MOVCSP Class

Given a MOVCSP $\mathcal{P}$ not in a Directional Substitutable MOVCSP class ($\mathcal{L}(MODS)$), is-it possible to use $\mathcal{L}(MODS)$ class to solve $\mathcal{P}$? A problem decom-

Function ORDER$^+(\phi, D^+_x, v, A_j) : \bar{v}$
$D^+_x \leftarrow D^+_x \setminus v$
$\bar{v} \leftarrow \varnothing$
while D^+_x **do**
  $u \leftarrow$ MINCOST(ϕ, D^+_x)
  $D^+_x \leftarrow D^+_x \setminus u$
  Order $\leftarrow$ *true*
  for $u' \in \mathcal{D}$ **do**
    for $v \in \bar{v}$ **do**
      for $j \in k$ **do**
        if
        $\phi_j(v,u') \preceq_{A_j} \phi_j(u,u') \wedge \phi_j(v) \succ_j$
        $\phi_j(u) \vee \phi_j(v,u') \succ_{A_j}$
        $\phi_j(u,u') \wedge \phi_j(v) \preceq_j \phi_j(u)$ **then**
          Order $\leftarrow$ *false*
          break
  if Order **then** $\bar{v} \leftarrow \bar{v} \cup \{u\}$

position scheme for MOVCSPs that takes advantage of Directional Substitutable Valuation Functions even when the studied problem is not limited to these Functions can be solved within a backtrack-based search. The **Algorithm** DS–MOEDAC

1. computes $\mathcal{P}'$ the dual of $\mathcal{P}$ (line 1)
2. computes the subset of directional dominant elements denoted by D^+ (line 2)
3. in order to update the Pareto set of non dominated solution s^*, according to Definition 3 and Lemma 1, if a solution s dominates one solution s_i from s^*, deletes s_i from s^* and adds s to s^* (lines 3, 4).
4. calls the Function ORDER$^+$ to identify in $O(ked^3)$ (where k the number of objectives and e is the number of constraints) a tractable sub-problem $\mathcal{P}''$ of $\mathcal{P}'$ such that $\mathcal{P}''$ is in $\mathcal{L}(MODS)$. (line 5) For a value v there may be more than one $\bar{v}$ partitions. As a result, we can use any partition strategy. For example, the partition that promotes values which minimize cost.
5. calls the Function MOEDAC* presented in (Ali et al., 2019) to compute Pareto-based Soft Arc Consistency.

This decomposition scheme can be distinguished by the possibility of instantiating variables by assigning to each one of them a subset of values in $\mathcal{L}(MODS)$ instead of single values for the $\mathcal{P}'$.

Example 4. Let $\mathcal{P}_3$ is a bi-Objectives binary VCSP composed of three variables x_1, x_2 and x_3. (see Figure 3). The domain D_i of each variable x_i is formed

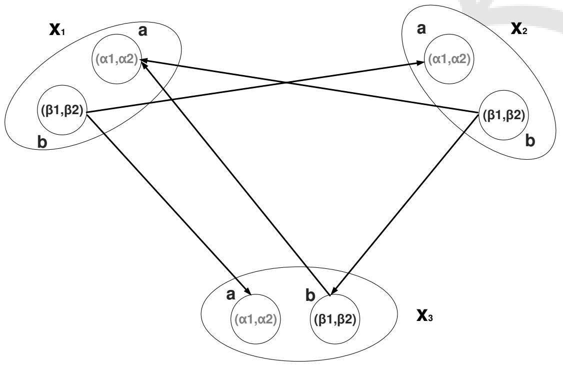

Figure 3: $\mathcal{P}_3$ in a bi-objectives $\mathcal{L}(MODS)$ class.

of two values $D_i = \{a,b\}$. This bi-Objectives VCSP has three constraints $c_1 = (\langle x_1,x_2\rangle,\Phi)$, $c_2 = (\langle x_2,x_3\rangle,\Phi)$ and $c_3 = (\langle x_1,x_3\rangle,\Phi)$ whose valuations functions Φ_c are defined in Table 5. We suppose that for the both objectives $j = 1,2$ $\alpha_j \preceq \beta_j$ Referring to the Definition 9 $\mathcal{P}_3$ is in a bi-objectives $\mathcal{L}(MODS)$. Referring to the orientation A given in Figure 3 and referring to the proof of Theorem 1, if we affect the

Algorithm 1: DS–MOEDAC$^*(P,Y,s^*) : s^*$.

```
1  P' ← DUAL(P)
   if Y' = ∅ then
2      s ← D⁺(P')
       ADD ← FALSE
       for ∀s_i ∈ s* do
           if s ≺_D s_i then
3              s* ← s* \ s_i
               ADD ← TRUE
       if ADD then
4          s* ← s* ∪ s
   else
       x ← SELECT(Y')
       while true do
           v ← MINCOST(φ, D⁺_x)
5          v̄ ← ORDER⁺(φ, D⁺_x, v, A_j)
           D⁺_x ← D⁺_x \ v̄
           P'' ← P'
6          P'' ← MOEDAC*(P'')
           if ∅ ∈ 𝒟 then break
           else s* ← DS–MOEDAC*(P'', Y \ {x}, s*)
```

Table 5: The valuation functions of $\mathcal{P}_3$.

$\phi(v_1,v_2)=\phi(v_1,v_3)$	a	b
a	$(\bot_1,\bot_2)$	$(\top_1,\top_2)$
b	$(\top_1,\top_2)$	$(\bot_1,\bot_2)$

$\phi(v_2,v_3)$	a	b
a	$(\bot_1,\bot_2)$	$(\bot_1,\bot_2)$
b	$(\bot_1,\bot_2)$	$(\top_1,\top_2)$

$\phi(v_{i=1,2,3})$	a	b
	(α_1,α_2)	(β_1,β_2)

value a to each variable x_i we get the optimal solution of $\mathcal{P}_3$ since for each variable x_i: $a \preceq_{A_j} b$.

Let $\mathcal{P}$ a MOVCSP. We deduce the tractability of MOVCSPs through their duals.

Corollary 1. *The MOVCSP $\mathcal{P}$ is tractable if*

$$Dual(MOVCSP) = \mathcal{P}' \in \mathcal{L}(MODS)$$

Proof Corollary 1. *From Lemma 3 the $Dual(MOVCSP)$ is MOCSOP. By Theorem 1 we have that $\mathcal{L}(MODS)$ is a tractable class of MOCSOP. This means that the MOVCSP $\mathcal{P}$ such that $Dual(\mathcal{P}) = \mathcal{P}' \in \mathcal{L}(MODS)$ is tractable.* □

4 CONCLUSION

In this paper we have proposed a Soft Directional Substitutable based Decompositions for MOVCSP. Despite the NP-hardness of MOVCSP we have presented a tractable classes by forcing the allowable valuation functions to have specific mathematical properties. This is the case of MOVCSP classes MOVCSPs with Directional Substitutable Valuation Functions only ($\mathcal{L}(MODS)$). As usefulness of $\mathcal{L}(MODS)$ MOVCSP class even when the studied problem is not limited to these functions, we have proposed a Directional Substitutable decomposition algorithm. As a natural extension of this work, in order to validate the practical use of $\mathcal{L}(MODS)$, we will compare a problem decomposition scheme for MOVCSPs that uses Pareto-based Soft Arc Consistency only with a decomposition scheme witch, in addition, takes advantage of $\mathcal{L}(MODS)$.

REFERENCES

Ali, L. B., Helaoui, M., and Naanaa, W. (2019). Pareto-based soft arc consistency for multi-objective valued csps. In Rocha, A. P., Steels, L., and van den Herik, H. J., editors, *Proceedings of the 11th International Conference on Agents and Artificial Intelligence, ICAART 2019, Volume 2, Prague, Czech Republic, February 19-21, 2019*, pages 294–305. SciTePress.

Carbonnel, C. and Cooper, M. C. (2016). Tractability in constraint satisfaction problems: a survey. *Constraints An Int. J.*, 21(2):115–144.

Cohen, D. A., Cooper, M. C., Creed, P., Marx, D., and Salamon, A. Z. (2012). The tractability of CSP classes defined by forbidden patterns. *J. Artif. Intell. Res.*, 45:47–78.

Cohen, D. A., Cooper, M. C., and Jeavons, P. (2008a). Generalising submodularity and horn clauses: Tractable optimization problems defined by tournament pair multimorphisms. *Theor. Comput. Sci.*, 401(1-3):36–51.

Cohen, D. A., Jeavons, P., and Gyssens, M. (2008b). A unified theory of structural tractability for constraint satisfaction problems. *J. Comput. Syst. Sci.*, 74(5):721–743.

Cooper, M. C., Duchein, A., Mouelhi, A. E., Escamocher, G., Terrioux, C., and Zanuttini, B. (2016). Broken triangles: From value merging to a tractable class of general-arity constraint satisfaction problems. *Artif. Intell.*, 234:196–218.

Cooper, M. C. and Zivný, S. (2011). Hybrid tractability of valued constraint problems. *Artif. Intell.*, 175(9-10):1555–1569.

Cooper, M. C. and Zivný, S. (2012). Tractable triangles and cross-free convexity in discrete optimisation. *J. Artif. Intell. Res.*, 44:455–490.

De, P., Dunne, E. J., Ghosh, J. B., and Wells, C. E. (1997). Complexity of the discrete time-cost tradeoff problem for project networks. *Oper. Res.*, 45(2):302–306.

Debels, D. and Vanhoucke, M. (2007). A decomposition-based genetic algorithm for the resource-constrained project-scheduling problem. *Oper. Res.*, 55(3):457–469.

Freuder, E. C. (1991). Eliminating interchangeable values in constraint satisfaction problems. In Dean, T. L. and McKeown, K. R., editors, *Proceedings of the 9th National Conference on Artificial Intelligence, Anaheim, CA, USA, July 14-19, 1991, Volume 1*, pages 227–233. AAAI Press / The MIT Press.

Greco, G. and Scarcello, F. (2011). Structural tractability of constraint optimization. In Lee, J. H., editor, *Principles and Practice of Constraint Programming - CP 2011 - 17th International Conference, CP 2011, Perugia, Italy, September 12-16, 2011. Proceedings*, volume 6876 of *Lecture Notes in Computer Science*, pages 340–355. Springer.

Greco, G. and Scarcello, F. (2013). Constraint satisfaction and fair multi-objective optimization problems: Foundations, complexity, and islands of tractability. In Rossi, F., editor, *IJCAI 2013, Proceedings of the 23rd International Joint Conference on Artificial Intelligence, Beijing, China, August 3-9, 2013*, pages 545–551. IJCAI/AAAI.

Helaoui, M. and Naanaa, W. (2013). Modularity-based decompositions for valued CSP. *Ann. Math. Artif. Intell.*, 67(2):165–187.

Helaoui, M., Naanaa, W., and Ayeb, B. (2013). Submodularity-based decomposing for valued CSP. *Int. J. Artif. Intell. Tools*, 22(2).

Lecoutre, C., Roussel, O., and Dehani, D. E. (2012). WCSP integration of soft neighborhood substitutability. In Milano, M., editor, *Principles and Practice of Constraint Programming - 18th International Conference, CP 2012, Québec City, QC, Canada, October 8-12, 2012. Proceedings*, volume 7514 of *Lecture Notes in Computer Science*, pages 406–421. Springer.

Naanaa, W. (2008). A domain decomposition algorithm for constraint satisfaction. *ACM J. Exp. Algorithmics*, 13.

Naanaa, W., Helaoui, M., and Ayeb, B. (2009). A domain decomposition algorithm for constraint satisfaction and optimization. In *JFPC 2009 - Actes des Cinquiemes Journees Francophones de Programmation par Contraintes*.

Schiex, T., Fargier, H., and Verfaillie, G. (1995). Valued constraint satisfaction problems: hard and easy problems. In *In Proceedings of the* 14^{th} *IJCAI, 631–637*, Montréal, Canada.

Schlesinger, D. (2007). Exact solution of permuted submodular minsum problems. In Yuille, A. L., Zhu, S. C., Cremers, D., and Wang, Y., editors, *Energy Minimization Methods in Computer Vision and Pattern Recognition, 6th International Conference, EMMCVPR 2007, Ezhou, China, August 27-29, 2007, Proceedings*, volume 4679 of *Lecture Notes in Computer Science*, pages 28–38. Springer.

Tavana, M., Abtahi, A., and Damghani, K. K. (2014). A new multi-objective multi-mode model for solving preemptive time-cost-quality trade-off project scheduling problems. *Expert Syst. Appl.*, 41(4):1830–1846.

van Beek, P. and Manchak, D. W. (1996). The design and experimental analysis of algorithms for temporal reasoning. *J. Artif. Intell. Res.*, 4:1–18.

Vanhoucke, M. (2005). New computational results for the discrete time/cost trade-off problem with time-switch constraints. *Eur. J. Oper. Res.*, 165(2):359–374.