The Impact of Information Geometry on the Analysis of the Stable M/G/1 Queue Manifold

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Abstract: Information geometry (IG) provides the characterization of the structure of statistical models from a differential geometric point of view. By considering families of probability distributions as manifolds with coordinate charts determined by the parameters of each individual model, the tools of differential geometry, such as divergences and metric tensors, provide effective means of studying their characteristics. The research undertaken in this paper presents a novel approach to the modelling study of information geometrics of a queueing system. In this context, the manifold of stable M/G/1queue is characterised from the viewpoint of IG, the Kullback's divergence (KD) and J-divergence (JD) are determined. Also, it is revealed that the stable M/G/1 queue manifold has a zero 0 -Gaussian curvature a non-zero Ricci Curvature Tensor (RCT). Unifying IG with Queueing Theory enables the study of dynamics of queueing system from a novel Riemannian Geometry (RG) point of view, leading to the analysis of the stable M/G/1 queue, based on Theory of Relativity (TR).

1 INTRODUCTION

Information geometry (IG) has been widely applied in many research fields such as statistical inference, stochastic control and neural networks (c.f., Amari, 1985) In other words, IG aims to apply the techniques of differential geometry (DG) to statistics. This means that IG's main idea is to apply methods and techniques of non-Euclidean geometry to stochastic processes and probability theories. IG indicates that the use of an Euclidian geometry technique is useful to think of a family of probability distributions as a statistical manifold (SM). Moreover, IG has been adopted for the study of statistical manifolds (SMs), where the geometric metrics gave a new description of the probability density function which plays an important role in SM and can be regarded as the coordinate system.

A manifold (c.f., Škoda, 2019) is a topological finite dimensional Cartesian space, $\mathbb{R}^n$, where one has an infinite-dimensional manifold. $\mathbb{R}^n$ could be described merely as topological space (may be defined as a set of points, along with a set of neighbourhoods for each point, satisfying a set of axioms relating points and neighbourhoods). In addition, IG supports reasoning intuitively the description of SMs. Note that although figures can be visualised (i.e., plotted in coordinate charts), they should be thought of as purely abstract figures, namely, geometric figures. One may have a higher level of appreciation of the significant importance of IG (c.f., Nielsen, 2018). In Figure 1, the parameter inference $\theta^\wedge$ of a model from data can be interpreted as a decision-making problem: One has to decide which parameter of a family of models $M = \{m_\theta\}_{\theta\epsilon\Theta}$ suits "best'' the data, where Θ is the set of parameters $\{\theta_1, \theta_2, .., \theta_n\}$ of the probability density function of the distribution of the geometric manifold. IG provides a differential-geometric manifold structure M that is useful for developing decision rules. In (Amari, 1985), the exponential distribution families were investigated whilst (Dodson, 1999) studied some special exponential distributions such as the bivariate normal distribution, the Gamma distribution, the McKay distribution and the Frund distribution and revealed their geometric structures.

In this paper, a study is undertaken of the geometric structure of the stable M/G/1 queue

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manifold (QM) as well as finding its information matrix exponential (IME). The (IME) is a matrix on square matrices analogous to the ordinary exponential function.

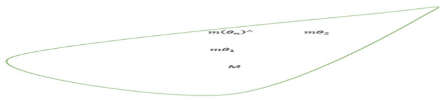

Figure1: Parametrization of a SM (c.f., Nielsen, 2018).

It is used to solve systems of linear differential equations. In addition to that, the matrix exponential plays a crucial role in the theory of Lie groups (c.f., Hall, 2015). To our knowledge, there is only one research paper (c.f., Nakagawa, 2002), which studied the IG of a stable M/D/1 queues, where a geometric structure was introduced on the set of M/D/1 queues by employing the properties of queue length paths. This point of view motivated the novel track of the research of this paper linking IG with information matrix theories towards a new re-interpretation of the stable M/G/1 queue. In this context, by analogy to information theory (IT), the geometric approach adopted in this paper enables the study of invariance and equivariance of figures in a coordinate-free approach (n.b., by equivariance as a concept, it is meant when there is a group acting on a pair of spaces and there is a map from functions on one to the functions on the other (c.f., Kondor and Trivedi, 2018). In the context of this paper, Ricci curvature (c.f., Nielsen, 2020) measures the deviation of the Riemannian metric (RM) from the standard Euclidean metric (EM) and how scalar curvature measures the deviation in the volume of a geodesic ball from the volume of an Euclidean ball of the same radius (c.f., Figure 2).

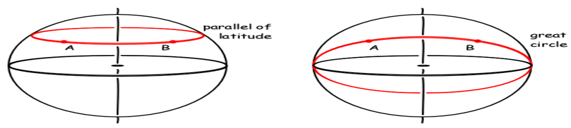

Figure 2: Geometric representation of geodesics on curved surfaces (c.f., Norton, 2020).

In IG, the Fisher information metric (FIM) is a particular Riemannian metric (RM), which can be defined on a smooth statistical manifold (i.e., a smooth manifold whose points are probability measures defined on a common probability space). It can be used to calculate the informational difference between measurements. The FIM measures closeness of the shape between two distribution functions, it is also proportional to the amount of information that the distribution function contains about the parameter of the probability density function of the SM. The focus of this work is foundational with the following list of its contributions:

i) The FIM and its inverse as well as the FIM for the stable M/G/1 QM are introduced.

ii) A novel α (or $\nabla^{(\alpha)}$)-connection (the $\alpha-$connection (c.f., Dodson, 2005), maps each co ordinate θ_i of $\theta, i = 1,2,3,\ldots,n$ to a value. In particular, the 1-connection (or, 'exponential connection') and the (-1) – connection (or, 'mixture connection') of a stable M/G/1 queue manifold are devised. iii) The KD and the JD of a stable M/G/1 queue are determined. iv)The stable M/G/1 queue's manifold could be considered to be incompressible or solenoidal, in which case any closed surface has no net flux across it (n.b., A flux is a vector quantity, describing the magnitude and direction of the flow of a substance or property c.f., Divergence Theorem by (c.f.,MIT Open Course Ware, 2010), which is the second 3-dimensional analogue of Green's Theorem stating that 'If F is a vector field with continuous derivatives defined on a region D $\subseteq R^2$ with boundary curve C, then, the flux of F across C is equal to the integral of the divergence over its interior'). v) The exponential of the FIM for the stable M/G/1 queue is shown to be a solution of a differential equation of the form $\frac{dx}{dt} = Ax$, where x is an *n*-dimensional vector and *A* is an nxn matrix. This paper is a major extension of a short paper by (Mageed and Kouvatsos, 2019) with the following contributions:

- The determination of the Kullback Divergence and the J-divergence of the stable M/G/1 QM.
- The proof that the exponential of the Fisher information matrix of the stable M/G/1 QM is a solution of a differential equation of the form $\frac{dx}{dt} =$ A*x*.

The main original contributions of this paper are described below.

- The inclusion of the definitions of Gaussian and Ricci curvatures and their physical interpretations;
- The proposed novel approach for the pioneer visualization of queueing systems via computational information geometry;
- The development of a new quantitative approach (which hadn't been discovered at the time we presented our UKPEW 2019;
- The determination of new important links between classical queueing theory and other mathematical disciplines, such as IG, matrix theory Riemannian geometry and the Theory of

Relativity by providing for first time i) The full detailed derivations of the Gaussian curvature ii) The Ricci curvature tensor and iii) The full physical as well as the geometric interpretation of these new results;

- The provision of a novel link between Ricci Curvature (RCT) and the stability analysis of the stable M/G/1 QM.

The rest of this paper is organised as follows: Section 2 presents preliminary definitions associated with (IG). The FIM and its inverse as well as the Fisher information metric for a stable M/G/1 queue manifold are introduced in Section 3. The α(or $\nabla^{(\alpha)}$)-*connection* of a stable M/G/1 queue manifold is obtained in Section 4. The KD and JD (c.f., Peng, Sun and Jiu, 2007) of a stable M/G/1 QM are obtained in Section 5. The structured proofs that the stable M/G/1 queue manifold has a non-zero Ricci Curvature Tensor (RCT) is devised in Section 6. The exponential matrix analysis of a stable M/G/1 queue is obtained in Section 7. Conclusions and future research directions are included in Section 8.

2 MAIN DEFINITIONS IN IG

Definition 2.1: Statistical Manifold (SM). $M = \{p(x,\theta)|\theta\epsilon\Theta\}$ is called an SM (c.f., Li, Sun, Tao and Jiu, 2007) if x is a random variable in sample space X and $p(x,\theta)$ is the probability density function, which satisfies certain regular conditions. Here, $\theta = (\theta_1, \theta_2, .., \theta_n)\epsilon\Theta$ is an n-dimensional vector in some open subset $\Theta \subset \mathbb{R}^n$, and θ can be viewed as the coordinates on manifold *M*.

Definition 2.2: Potential Function. The potential function $\Psi(\theta)$ (c.f., (2.1)) (c.f., Li, Sun, Tao and Jiu, 2007) is the distinguished negative function of the coordinates alone of ($\mathcal{L}(x;\theta) = ln(p(x;\theta))$) and in a sequel, it will appear in the information geometric analysis of the M/G/1 queue manifold.

Definition 2.3: Fisher's Information Matrix (FIM). The FIM (or, Fisher's metric) $[g_{ij}]$ (c.f., Dodson, 2005) is given by the Hessian (the nxn matrix of the partial derivatives of the potential function $\Psi(\theta)$ with respect to the coordinates) i.e.,

$$[g_{ij}] = \left[\frac{\partial^2}{\partial\theta^i\,\partial\theta^j}(\Psi(\theta))\right], i,j = 1,2,..,n \qquad (2.1)$$

with respect to natural coordinates.

Definition 2.4: Inverse Matrix of Fisher's Information Matrix (FIM). Given the FIM, the inverse matrix of $[g_{ij}]$ is defined by (c.f., Dodson, 2005).

$$[g^{ij}]= ([g_{ij}])^{-1} = \frac{adj[g_{ij}]}{\Delta}, \Delta = \det[g_{ij}] \qquad (2.2)$$

The FIM for the manifold M is given in θ coordinates by the arc length function.

$$(ds)^2 = \sum_{i,j=1}^{n} g_{ij}\,(d\theta^i)(d\theta^j) \qquad (2.3)$$

Definition 2.5: α-Connection. For each $\alpha\epsilon\mathbb{R}$, the α(or $\nabla^{(\alpha)}$)-*connection* (c.f., Dodson, 2005) is the torsion-free affine connection with components:

$$\Gamma_{ij,k}^{(\alpha)} = (\tfrac{1-\alpha}{2})(\partial_i\partial_j\partial_k(\Psi(\theta))) \qquad (2.4)$$

where $\Psi(\theta)$ *is the potential function and* $\partial_i = \frac{\partial}{\partial\theta_i}$.

Definition 2.6: Kullback's Divergence (KD), $\boldsymbol{K(p,q)}$. Assume $p(x;\theta_p)$ and $q(x;\theta_q)$ are two points on the manifold *M*, the Kullback's divergence $K(p,q)$ (c.f., Li, Sun, Tao and Jiu, 2007) is defined by

$$K(p,q) = E_{\theta_p}\left[ln\left(\frac{p(x;\theta_p)}{q(x;\theta_q)}\right)\right] = \int p(x;\theta_p)\,ln\left(\frac{p(x;\theta_p)}{q(x;\theta_q)}\right)dx \qquad (2.5)$$

where E_{θ_p} stands for the expected value and the J-divergence is defined by

$$J(p,q) = \int ln\left(\frac{p(x;\theta_p)}{q(x;\theta_q)}\right)^{(p(x;\theta_p)-q(x;\theta_q))} dx \qquad (2.6)$$

When the two $p(x;\theta_p)$ and $q(x;\theta_q)$ are close enough and by using Taylor's formula, the following analytic (c.f., Li, Sun, Tao and Jiu, 2007) result holds:

$$K(\theta,\theta+d\theta) = J(\theta,\theta+d\theta) = \frac{1}{2}(ds)^2$$

where $(ds)^2$ stands for the square of the arc length of the manifold.

Definition 2.8:

1. Under the θ coordinate system, the $\alpha-$ curvature Riemannian Tensors, $R_{ijkl}^{(\alpha)}$ (c.f., Li, Sun, Tao and Jiu, 2007) are defined by

$$R_{ijkl}^{(\alpha)} = [(\partial_j\Gamma_{ik}^{s(\alpha)} - \partial_i\Gamma_{jk}^{s(\alpha)})g_{sl} + (\Gamma_{jt,l}^{(\alpha)}\Gamma_{ik}^{t(\alpha)} - \Gamma_{it,l}^{(\alpha)}\Gamma_{jk}^{t(\alpha)})], i,j,k,l,s,t = 1,2,3,\ldots,n \qquad (2.7)$$

where $\Gamma_{ij}^{k(\alpha)} = \Gamma_{ij,s}^{(\alpha)}g^{sk}$, *i,j,k,s = 1,2,...,n*

2. The $\alpha-$Ricci curvatures (Ricci Tensors) $R_{ik}^{(\alpha)}$ are determined by (c.f., Li, Sun, Tao and Jiu, 2007).

$$R_{ik}^{(\alpha)} = R_{ijkl}^{(\alpha)}g^{jl}, i,j,k,l = 1,2,3,\ldots,n \qquad (2.8)$$

3. The $\alpha-$sectional curvatures $K_{ijij}^{(\alpha)}$ are defined by (c.f., Li, Sun, Tao and Jiu, 2007).

$$K_{ijij}^{(\alpha)} = \frac{R_{ijij}^{(\alpha)}}{(g_{ii})(g_{jj})-(g_{ij})^2}, i,j = 1,2,\dots,n \qquad (2.9)$$

Specifically, if $n = 2$, the $\alpha-$sectional curvature $K_{1212}^{(\alpha)} = K^{(\alpha)}$ is called $\alpha-$Gaussian curvature and is given by (c.f., Li, Sun, Tao and Jiu, 2007).

$$K^{(\alpha)} = \frac{R_{1212}^{(\alpha)}}{det(g_{ij})} \qquad (2.10)$$

4. The Ricci Tensor (c.f., Loveridge, 2016) is simply a contraction of the Riemannian Tensor (c.f., Li, Sun, Tao and Jiu, 2007).
5. The Ricci curvature Tensor (RCT) (c.f., Rudelius, 2012) of an oriented Riemannian Manifold M means the extent to which the volume of a geodesic ball on the surface differs from the volume of a geodesic ball in Euclidean space.
6. The Ricci curvature (RCT) (c.f., Ollivier, 2010) contracts the evolution of volumes under the geodesic flow. When Ricci curvature is positive, then according to the Bonnet Myers theorem (c.f., Ollivier, 2010) the Riemannian manifold is more positively curved than a sphere and the diameter of the manifold is smaller.

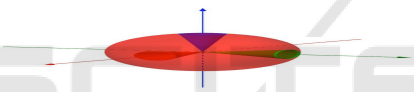

Figure 3: (RCT) describes how conical regions in the manifold differ in volume from the equivalent conical regions in Euclidean space (c.f., Thomas, 2015).

Definition 2.9:
1. Considering the linear system of differential equations

$$\frac{dx}{dt} = Ax \qquad (2.11)$$

with x is an n-dimensional vector and A is an nxn matrix. It can be shown that (Gunawardena, 2006) the matrix exponential

$$e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = I + A + \frac{A^2}{2!} + \dots + \frac{A^k}{k!} + \dots. \qquad (2.12)$$

is the solution of (2.11).

2. If the characteristic polynomial of A is defined by

$$\Phi(\delta) = \det(A - \delta I) \qquad (2.13)$$

then, the set of eigen values of A will is defined to be (c.f., Gunawardena, 2006) the set of all the roots of the equation

$$\Phi(\delta) = (\delta) = \det(A - \delta I) = 0 \qquad (2.14)$$

and corresponding eigen vectors x assigned to each eigen value δ are defined to satisfy the equation:

$$Ax = \delta x \qquad (2.15)$$

Another way to represent e^A will be

$$e^A = T\, e^D\, T^{-1} \qquad (2.16)$$

where D is the diagonal matrix of eigen values of A, and T is matrix having of the corresponding eigen vectors of A as its columns (c.f., Gunawardena, 2006).

3 THE FIM AND ITS INVERSE FOR THE STABLE M/G/1 QM

According to (El-Affendi and Kouvatsos, 1983), the maximum entropy (ME) state probability of the generalized geometric solution of a stable M/G/1 queue (c.f., Figure. 4), subject to normalisation, mean queue length (MQL), L and server utilisation, $\rho(<1)$ is given by

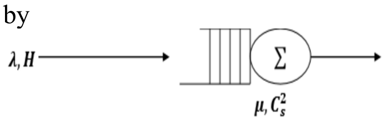

Figure 4: A Stable M/G/1 queue.

$$p(n) = \begin{cases} 1-\rho, & n = 0 \\ (1-\rho)gx^n, & n \geq 1 \end{cases} \qquad (3.1)$$

where $g = \frac{\rho^2}{(L-\rho)(1-\rho)}$, $x = \frac{L-\rho}{L}$ and $L = \frac{\rho}{2}\left(1 + \frac{1+\rho C_s^2}{1-\rho}\right)$ (MQL of Pollaczeck-Khinchin Formula of a stable M/G/1 queue), $\rho = 1 - p(0)$ (server utilisation) and C_s^2 (SCV of the service times).

It clearly follows that $p(n)$ of (3.1) can be rewritten as

$$p(n) = \begin{cases} 1-\rho, & n = 0 \\ \dfrac{2\rho(\frac{1+\rho\beta}{1-\rho} - 1)^{n-1}}{(\frac{1+\rho\beta}{1-\rho} + 1)^n}, & n \geq 0, \text{ with } \beta = C_s^2 \end{cases} \qquad (3.2)$$

Theorem 3.1. For the stable M/G/1 queue manifold, it holds that

(i) The FIM is given by

$$[g_{ij}] = \begin{pmatrix} \frac{1}{(1-\rho)^2} & 0 \\ 0 & \frac{-1}{(\beta+1)^2} \end{pmatrix} \qquad (3.3)$$

(ii) The square of the arc length (i.e., Fisher Information Metric) is determined by

$$(ds)^2 = \left(\frac{1}{(1-\rho)^2}\right)(d\rho)^2 - \frac{1}{(\beta+1)^2}(d\beta)^2 \qquad (3.4)$$

(iii) The inverse of Fisher Information Matrix is given by

$$[g^{ij}] = \frac{adj[g_{ij}]}{\Delta} = \begin{pmatrix} (1-\rho)^2 & 0 \\ 0 & -(\beta+1)^2 \end{pmatrix} \qquad (3.5)$$

Proof. Following (3.2), two cases would arise.

Case I: For $n = 0$, $p(n) = 1 - \rho$. Hence, the coordinate system is one dimensional satisfying

$$\mathcal{L}(x;\theta) = ln\big(p(x;\theta)\big) = ln(1-\rho), \quad \theta = \theta_1 = \rho \tag{3.6}$$

The potential function $\Psi(\theta)$will be the standalone part of (- $\mathcal{L}(x;\theta)$)involving the coordinates, i.e.,

$$\Psi(\theta) = -ln(1-\rho) \tag{3.7}$$

Thus,

$$\partial_1 = \frac{\partial\Psi}{\partial\rho} = \frac{1}{1-\rho} \tag{3.8}$$

$$\partial_1\partial_1 = \frac{\partial^2\Psi}{\partial\rho^2} = \frac{1}{(1-\rho)^2} \tag{3.9}$$

FIM is given by

$$[g_{ij}] = \left[\frac{\partial^2\Psi}{\partial\rho^2}\right] = \left[\frac{1}{(1-\rho)^2}\right] \tag{3.10}$$

The inverse of the FIM is determined by

$$[g^{ij}] = [g_{ij}]^{-1} = [(1-\rho)^2] \tag{3.11}$$

Moreover,

$$\Gamma_{11,1}^{(\alpha)} = \left(\frac{1-\alpha}{(1-\rho)^3}\right), \Gamma_{11}^{1(\alpha)} = \frac{1-\alpha}{(1-\rho)}, \Gamma_{11}^{1(0)} = \frac{1}{(1-\rho)} \tag{3.12}$$

Following the same argument, the proofs of (ii) and (iii) follow.

4 The α(OR $\nabla^{(\alpha)}$)-CONNECTION OF THE M/G/1 QM

By definition (2.8), we have

$$\Gamma_{11,1}^{(\alpha)} = \frac{(1-\alpha)}{(1-\rho)^3} \tag{4.1}$$

Similarly, the remaining components are devised. Furthermore, after some lengthy calculations

$$\Gamma_{11}^{1(\alpha)} = \frac{1-\alpha}{(1-\rho)}, \Gamma_{11}^{1(0)} = \frac{1}{(1-\rho)} \tag{4.2}$$

$$\Gamma_{22}^{2(\alpha)} = -\frac{1-\alpha}{(1+\beta)}, \Gamma_{22}^{2(0)} = -\frac{1}{(1+\beta)} \tag{4.3}$$

The remaining components could be computed as above. Using the above derivations, the Ricci curvature of the stable M/G/1 QM can be devised.

5 THE KD AND THE J-D OF STABLE M/G/1 QM

Following (2.6), KD is expressed by

$$K(p,q) = E_{\theta_p}\left[ln\left(\frac{p(x;\theta_p)}{q(x;\theta_q)}\right)\right] =$$

$$\begin{cases} \boldsymbol{ln}\left(\frac{1-\rho_p}{1-\rho_q}\right), & \boldsymbol{n=0} \\ \boldsymbol{ln}\left(\frac{1-\rho_p}{1-\rho_q}\right)\left(\frac{1+\beta_q}{1+\beta_p}\right)\left[\left(\frac{\rho_q(2+\rho_q(\beta_q-1))}{\rho_p(2+\rho_p(\beta_p-1))}\right)\left(\frac{1+\beta_q}{1+\beta_p}\right)\right]^L, & \boldsymbol{n \geq 0} \end{cases} \tag{5.1}$$

where L is MQL of Pollaczeck-Khinchin Formula of a stable M/G/1 QM. (5.1)

Moreover, in a similar fashion, it could be seen that

$$J(p,q) = K(p,q) + K(q,p) = 0 \tag{5.2}$$

Equation (5.2) presents a great contribution as it shows that the stable M/G/1 QM is incompressible or non-solenoidal, in which case any closed surface has no net flux across it.

6 THE STABLE M/G/1 QM HAS A NON-ZERO RICCI CURVATURE (RCT) TENSOR

In this section, it is revealed that the stable M/G/1 QM is developable (can be mapped onto the plane surface without distortion of curves: any curve from such a surface drawn onto the flat plane remains the same) and has a non-zero Ricci curvature, shortly written as (RCT) tensor (the M/G/1 QM is more positively curved than a sphere and the diameter of the manifold is smaller).

Theorem 6.1. The stable M/G/1 QM.

i) Has a zero 0-Gaussian curvature ii) Has a non-zero Ricci tensor

Proof. Case i), by definition (2.10), part i), it is enough to show that the α −Gaussian curvature

$$K^{(0)} = \frac{R_{1212}^{(0)}}{det(g_{ij})} = 0 \tag{6.1}$$

It could be verified that,

$$R_{1212}^{(\alpha)} = 0 \tag{6.2}$$

$det(g_{ij}) = -\frac{1}{(\beta+1)^2(1-\rho)^2} \neq 0$. Hence, $K^{(0)} = \frac{R_{1212}^{(0)}}{det(g_{ij})} = 0$, which proves the developability Case i) of stable M/G/1 QM.

Case ii) To prove that the stable M/G/1 (QM) Ricci tensor is non-zero, one needs to show that the α − RCs, $R_{ik}^{(\alpha)}$ are given by(c.f., definition 2.8, part 2)

$$R_{ik}^{(\alpha)} = R_{ijkl}^{(\alpha)} g^{jl}, i,j,k,l = 1,2,3,\ldots,n$$

is non zero, which means that at least one of its components is non-zero. By (6.1), $R_{11}^{(\alpha)}$ equals

$$R_{1212}^{(\alpha)} g^{11} + R_{1112}^{(\alpha)} g^{12} + R_{1211}^{(\alpha)} g^{21} + R_{1212}^{(\alpha)} g^{22}$$

Engaging the same procedure as in (6.1), we have

$$R_{11}^{(\alpha)} = R_{12}^{(\alpha)} = R_{22}^{(\alpha)} = 0 \tag{6.3}$$

$$R_{21}^{(\alpha)} = -\frac{1}{(1-\rho)^2} \tag{6.4}$$

Hence,

$$R_{21}^{(\alpha)} \neq 0 \tag{6.5}$$

This proves Case ii).

As $\rho \to 1$, $R_{21}^{(0)} \to -\infty$. This shows the significant impact of instability of the two dimensional M/G/1 QM. This presents a novel link between Ricci Curvature (RCT) and the stability analysis of Queueing Systems. It is clear that $R_{21}^{(0)}$ is a server utilization dependent function. To experiment more closely the impact of the server utilization, ρ and the behaviour of the (RCT). It is observed by Figure 5 that the stability phase of M/G/1 QM enforces (RCT) to be a decreasing function in ρ, whereas in Figure 6, it can be seen that instability phase of M/G/1 QM enforces (RCT) to be an increasing in ρ.

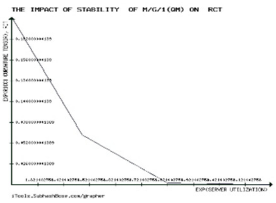

Figure 5.

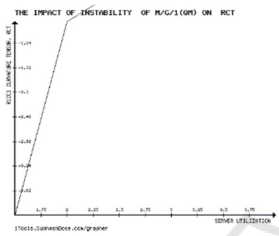

Figure 6.

7 THE EXPONENTIAL MATRIX OF FIM OF STABLE M/G/1 QM

Theorem 7.1. The exponential matrix of the Fisher information of the stable M/G/1 QM is a solution of a differential equation of the form $\frac{dx}{dt}$ = Ax.

Proof. It has been proved earlier (c.f. Theorem 3.1) that the FIM of the stable M/G/1 queue manifold $[g_{ij}], i, j = 1,2$ is given by

$$[g_{ij}]=\begin{pmatrix} \frac{1}{(1-\rho)^2} & 0 \\ 0 & \frac{-1}{(\beta+1)^2} \end{pmatrix} \quad (7.1)$$

We write

$$[g_{ij}] = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a = \left(\frac{1}{(1-\rho)^2}\right), b = \frac{-1}{(\beta+1)^2} \quad (7.2)$$

It follows that $\Phi(\delta) = (\delta) = \det([g_{ij}] - \delta I) = \det\begin{pmatrix} a-\delta & 0 \\ 0 & b-\delta \end{pmatrix} = 0$. Hence, it holds that $\delta^2 - (a+b)\delta + ab$ =0, which implies that the eigenvalues are given by $\delta_{1,2} = a, b$. The diagonal matrix D is given by

$$D = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \quad (7.3)$$

For $\delta_{1,2} = a, b$, the corresponding eigen vectors are $\binom{1}{0}, \binom{0}{1}$. Hence,

$$T = T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (7.4)$$

Hence, the exponential matrix of the FIM of the stable M/G/1 queue manifold is given by

$$e^A = Te^DT^{-1} = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix} \quad (7.5)$$

The result obtained in (7.5) shows that the exponential of the FIM of the stable M/G/1 queue manifold is a solution of a differential equation of the form

$$\frac{dx}{dt} = Ax \quad (7.6)$$

8 CONCLUSIONS AND FUTURE WORK

The stable M/G/1 QM is characterized from the viewpoint of IG, KD and J-D were determined. Moreover, the matrix exponential of information of the M/G/1 QM is devised. This paper opens a new ground for research linking queueing theory with many other mathematical disciplines such as information theory, differential geometry and matrix theory. Specifically, adding information geometric links with queueing theory enables the study of the dynamics of a queueing system from the Riemannian Geometric point of view (c.f., Amari, 1985) and (c.f., Dodson, 1999) and in turn, enabling the analysis of a queueing system based on the Theory of Relativity (c.f., Norton, 2020).

This paper has introduced for the first time the FIM and its inverse and has also obtained the Fisher information metric for the M/G/1 QM.

Moreover, a novel expression for α (or $\nabla^{(\alpha)}$)-*connection* of the stable M/G/1 queue manifold was devised. In addition, the KD and the J-D of the stable M/G/1 queue manifold were devised. In this context, it was shown that the stable M/G/1 queue manifold can be described as compressible or non-solenoidal, in which case any closed surface has no net flux across it The latter, was justified by the Divergence Theorem of (c.f., MIT Open Course Ware, 2010), which states that the flux of a vector field across a closed boundary curve C is equal to the integral of the divergence over its interior. It is implied that, when the J-D is zero, any closed surface has no net flux across the M/G/1 QM. Moreover, it was revealed that the stable M/G/1 QM has a zero 0-Gaussian curvature and a non-zero Ricci Curvature Tensor. Finally, it was proven that the exponential of the FIM of the stable M/G/1 queue manifold is a solution of a differential equation of the form $\frac{dx}{dt}$= Ax. Specifically, the main original contributions of this paper are summarised below.

- The proposed new approach to visualize queueing systems via computational information geometry;
- The establishment of new links between queueing theory and other mathematical disciplines such as information geometry, matrix theory Riemannian geometry and the theory of Relativity.
- Providing a novel link between Ricci Curvature (RCT) and the stability analysis of the stable M/G/1 QM.
- Having introduced several information geometric concepts, we have managed for first time to capture the M/G/1 queue as a manifold and analysed the M/G/1 QM by using information geometric methods. Consequently, classical Queueing Theory can be extended to become richer because of the application of IG.

An exponential family or mixture family of probability distributions has a natural hierarchical structure. Orthogonal decomposition of such a system based (c.f. Amari, 2001) on information geometry. A typical example is the decomposition of stochastic dependency among a number of random variables. In general, they have a complex structure of dependencies. The orthogonal decomposition is given in a wide class of hierarchical structures including both exponential and mixture families. As an example, we decompose the dependency in a higher order Markov chain into a sum of those in various lower order Markov chains.

Single-server, such as M/G/1 system is simple and can be utilized as preliminary models (c.f., Hamasha et al, 2016). Modelling of the systems state using Markov chain approach and queuing models provides a more rigid approach to better understand the dynamics of the service delivery system, which proposes a conceptual model using of Markov chain approach combined with M/G/1 queuing model to optimize general service delivery systems.

Based on the above discussion, clearly the lost link is now uncovered by our novel approach as it reveals the significant impact of IG on Queueing Theory.

The stability problem (Rachev, 1989) in queueing theory is concerned with the continuity of the mapping *F* from the set *U* of the input flows into the set V of the output flows. First, using the theory of probability metrics we estimate the modulus of *F*-continuity providing that *U* and *V* have structures of metric spaces. Then we evaluate the error terms in the approximation of the input flows by simpler ones assuming that we have observed some functionals of the empirical input flows distributions. This shows the strength of our novel approach as it derives for the first time ever the exact stability and instability phases of the underlying M/G/1 queueing system.

The beauty of our novel approach that revolutionizes Queueing Theory, is looking at a queue as a manifold, in which case, α is considered as the parameter of curvature as well as being the connection parameter of the underlying stable M/G/1 QM.

In other words, under a metric connection (c.f., Jefferson, 2018), parallel transport of two vectors preserves the inner product, hence their significance in Riemannian geometry. Any connection which is both metric and symmetric is Riemannian, of which there are generically an infinite number. However, the natural metrics on statistical manifolds are generically non-metric! Indeed, since only the special case $\alpha = 0$ defines a Riemannian connection $\nabla^{(0)}$ with respect to the Fisher metric (though observe that $\nabla^{(\alpha)}$ is symmetric for any value of α). While this may seem strange from a physics perspective, where preserving the inner product is of prime importance, there's nothing mathematically pathological about it. Indeed, the more relevant condition, that every point on the manifold have an interpretation as a probability distribution. In general, (c.f., Lee, 1950), exponentiating a matrix corresponds to exponentiating each of its Jordan blocks. In fact, this interpretation also holds for any analytic function f applied to a matrix and not just e^X. Also, it may be useful to think of the matrix exponential as the "Solution to the System of Ordinary Differential Equations (ODEs)".

Based on the contributions of this paper, there are several future research directions towards the new applications of information geometric queueing theory includes developing further advances on many existing queueing manifolds, such as the G/G/1 queue (c.f., Dodson, 2005 and Kouvatsos 1988) manifold and employing information geometrics on various statistical manifolds.

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