An Economic Production Quantity Model with Imperfect Quality Raw Material and Backorders

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Abstract: In this paper the classical EPQ model is extended to account for the cost and quality of the raw material used in the production process and to incorporate the effects of shortages into the model. A production process that uses *n* different types of raw material is considered. The various types of raw material acquired in batches from the suppliers are assumed to contain a percentage of imperfect quality items of raw material. The proportion of imperfect quality raw material found in a batch is a random variable having a known probability distribution. A mathematical model describing the inventory/production situation is formulated and used to derive a system of equations whose solution is the optimal production and shortage quantities that minimizes the total cost. It is shown that the total cost function depends on the determination of the maximum of a set of *n* independent random variables obtained from the proportions of imperfect quality raw material. A process for obtaining the probability function of the maximum along with its expected value is described. Expressions for the probability density function and the expected value of the maximum are developed for the case when the random variables are uniformly distributed. A numerical example illustrating the determination of the optimal policy is presented.

1 INTRODUCTION

The classical economic production quantity (EPQ) model describes a situation where an item is produced to meet the demand. Let α denote the production rate, β the demand rate, C_0 the production setup cost, C the unit production cost, and h the holding cost per unit per unit time. The total inventory cost per unit time function is given by

$$TCU(Y) = C\beta + C_0\beta/Y + h\left(1-\frac{\beta}{\alpha}\right)Y/2, \quad (1)$$

where Y is the quantity ordered for production at the beginning of each production cycle. The optimal production quantity, or the economic production quantity, that minimizes the TCU function is

$$y^* = \sqrt{\frac{2C_0\beta}{h\left(1-\frac{\beta}{\alpha}\right)}}. \quad (2)$$

Note that the classical model does not take into account the cost or quality of the raw materials used in the production process and considers only the cost of the finished product. Also, the classical model assumes that shortages are not allowed.

The classical EPQ model is based on several assumptions that simplify the model. Numerous research studies have extended the classical EPQ model by relaxing some of its underlying assumptions so that the model becomes more realistic (Yassine, 2018; Khan & Jaber, 2011). Some of the factors introduced to relax the simplifying assumptions of the classical EPQ model include cost of raw material (Salameh & El-Kassar, 2007), quality of items produced (Salameh & Jaber, 2000; Khan et al., 2011), quality of the raw material used in the production process (Yassine, 2016; Yassine 2018), deterioration (Bandaly & Hassan, 2019), supply chain considerations (Khan et al., 2011; Khan & Jaber, 2011; Bandaly et al. 2014; Bandaly et al. 2016), and green and sustainable practices (Yassine, 2018).

Environmental concerns and resource limitations coupled with pressure from internal and external stakeholders have forced corporations to not only consider efficient and effective operations (El-Khalil & El-Kassar, 2016), but also to engage in responsible and environmentally friendly activities. Driven by ethical practices, engaging in responsible activities has been shown to enhance performance (El-Kassar & Singh, 2019; Singh et al., 2019, El-Khalil & El-Kassar, 2018), improve governance (ElGammal et al., 2018), and lead to employee and customer favorable outcomes (El-Kassar et al. 2017). In addition to the environmental and responsible practices, companies in general and manufacturers in particular are utilizing strategic resources, such as information and communication technologies and innovation, for enhancing their competitiveness level (Singh & El-Kassar, 2019; Yunis et al., 2017; Yunis et al., 2018). Recently, these factors have been incorporated into the classical EPQ model (Lamba et al., 2019; Yassine, 2018).

Salameh and Jaber (2000) introduced a new modeling approach to account for the quality of items produced or aquired. This approach triggered a new line of research (Khan et al., 2011; El-Kassar, 2009; Yassine et al. 2018). Incorporating the costs and quality of raw material used in the production process has been the focus of several studies (Salameh & El-Kassar, 2007; El-Kassar et al., 2012). Yassine (2018) considered an EPQ model that takes into account the quality of raw material; however, the model assumes that shortages are not allowed. Yassine and AlSagheer (2017) examined a production model with shortages and raw materials but did not account for the quality of the raw material.

The purpose of this paper is to extend the classical EPQ model to account for the cost and quality of the raw materials used in the production process and to incorporate the effects of shortages into the model. We consider the case that n different types of raw material are used in the production process in which each unit of the finished product requires one unit of each type of raw material. At beginning of each production/inventory cycle, the various types of raw material are acquired in batches from the suppliers. Each batch is assumed to contain a percentage of imperfect quality items of raw material. The proportion of imperfect quality raw material found in a batch is a random variable having a known probability.

The model also allows for shortages and backorders and accounts for two types of shortage cost, a constant administrative cost and a linear time dependent cost.

A mathematical model describing the problem at hand is formulated and used to derive a system of equations whose solution is the optimal policy. It is shown that the formulation of the mathematical model depends on the determination of the maximum of a set of n independent random variables obtained from the proportions of imperfect quality raw material. Thus, a process for obtaining the probability function of the maximum along with its expected value is described. Moreover, expressions for the probability density function and the expected value of the maximum are developed for the case when the random variables are uniformly distributed. The results are then applied to the EPQ model considered in this paper. A numerical example illustrating the determination of the optimal policy is presented.

The rest of this paper is organized in the following manner. In section 2, the mathematical model is formulated. The determination of the distribution and the expectation of the maximum of a set of independent random variables is discussed in section 3. In section 4, a case is presented to illustate the calculation of optimal solution. The paper concludes in section 5.

2 MATHEMATICAL MODEL

In this section, the mathematical model describing the problem at hand is formulated and used to derive a system of equations whose solution is the optimal policy.

2.1 Notation

The following notation is used throughout the rest of this paper:

Y	Order size of finished product
S	Size of planned shortage
M	Maximum inventory level
U_j	Order size of raw material of type j
A	Production rate
β	Demand rate
C_0	Production set up cost
C_p	Unit production cost
C_j	Ordering cost of raw material of type j
C_{rj}	Unit purchasing cost of raw material of type j
C_{dj}	Screening cost per unit of raw material of type j

C_b	Administrative cost per unit short of the finished product
C_s	Cost per unit short of the finished product per unit time
h_{rj}	Holding cost per unit of raw material of type j per unit time
h_P	Holding cost per unit of finished product per unit time
γ_j	Screening rate of raw material of type j
δ_j	Percentage of imperfect quality of raw material of type j
$g_j(\delta_j)$	Probability density function of δ_j
μ_j	Expected value of δ_j
S_{rj}	Salvage value per unit of imperfect quality raw material of type j
Tp	Length of the production period
T	Length of the inventory cycle
T_1	Time to fulfil the backorder of size S
T_2	Time to build the maximum inventory level
T_3	Time to deplete the maximum inventory
T_4	Time to build a backorder of size S

2.2 Problem Formulation

Let Y be the order size of the finished product, an unknown to be determined by minimizing the total cost per unit time function. At the start of each production cycle, the various types of raw material acquired from the suppliers are processed into a finished product at a production rate α. The batch of raw material of type j acquired from supplier j is screened for imperfect quality items at a rate γ_j. The screening period is U_j/γ_j, where U_j is the order size of raw material of type j. Suppose that is

$$U_j = Y/(1-\mu_j), \tag{3}$$

where μ_j is the expected value of δ_j, the proportion of imperfect quality raw material of type j. From Eq. (3), the amount of perfect quality raw material of type j is

$$(1-\delta_j)U_j = (1-\delta_j)Y/(1-\mu_j), \tag{4}$$

so that its expected value is

$$E[(1-\delta_j)U_j] = E[1-\delta_j]Y/(1-\mu_j) = Y. \tag{5}$$

On the other hand, the amount of imperfect quality raw material of type j is

$$\delta_j U_j = \delta_j Y/(1-\mu_j), \tag{6}$$

and its expected value is

$$E[\delta_j U_j] = E[\delta_j Y/(1-\mu_j)] = \mu_j Y/(1-\mu_j). \tag{7}$$

Since each unit of the finished product requires exactly one unit of perfect quality raw material of type j, U_j must be larger than the order size of the finished product Y. Note that the imperfect quality of raw material of type j is accounted for as follows:

$$U_j = \frac{Y}{1-\mu_j} = \frac{Y-Y\mu_j+Y\mu_j}{1-\mu_j} = Y + \frac{Y\mu_j}{1-\mu_j}. \tag{8}$$

The additional amount ordered is exactly the expected amount of imperfect quality of raw material of type j. However, the actual amount of perfect quality raw material may differ. Let Z_j denote this difference. From Eqs. (6) and (7),

$$Z_j = \frac{\delta_j Y}{1-\mu_j} - \frac{Y\mu_j}{1-\mu_j} = \frac{(\delta_j-\mu_j)Y}{1-\mu_j}. \tag{9}$$

This difference determines the number of finished items produced using the perfect quality raw material received during the current production cycle. Let W_c denote this number. Then,

$$\begin{aligned} W_c &= Y - \text{Max}\{Z_j : 1 \le j \le n\} \\ &= Y - Y\text{Max}\{(\delta_j - \mu_j)/(1-\mu_j) : 1 \le j \le n\}. \end{aligned} \tag{10}$$

Hence, the determination of the optimal production quantity depends on calculating the maximum of the n independent continuous random variables

$$X_j = \frac{\delta_j - \mu_j}{1-\mu_j}. \tag{11}$$

Note that each of these variables has a mean of 0.

Define the expected value of the maximum of $X_1, X_2, \ldots, X_n$ to be

$$\mu = E[\text{Max}\{X_j : 1 \le j \le n\}]. \tag{12}$$

The value of μ depends on the distribution of the variables X_j, $1 \le j \le n$. In section 3, we consider the case where the random variables $\delta_1, \delta_2, \ldots, \delta_n$ are uniformly distributed.

From Eq. (10), the expected number of finished items produced using the perfect quality raw materials received during the current cycle is

$$E[W_c] = Y(1-\mu). \tag{13}$$

Note that, from Eqs. (4) and (10), the number of unused good quality items of raw material of type *j* received during the current cycle is

$$e_j = \frac{(1-\delta_j)Y}{1-\mu_j} - W_c. \tag{14}$$

Using Eqs. (5) and (13), the expected number of unused good quality items of type *j* raw material is

$$E[e_j] = \frac{(1-\mu_j)Y}{1-\mu_j} - Y(1-\mu) = \mu Y. \tag{15}$$

The on-hand good quality raw materials are processed at a rate α until the end of the production period. The length of the production period is

$$T_P = W/\alpha, \tag{16}$$

where W is the total number of items produced during the current production cycle using both the perfect quality raw materials received at the beginning of the inventory cycle as well as the excess perfect quality raw materials kept in stock from previous cycles. Let W_p be the number of finished items produced using the excess perfect quality raw materials kept in stock from previous cycles. Hence, $W = W_c + W_p$.

Since each excess amount has the same expected value of $E[e_j] = \mu Y$, the expected number of finished product produced using the excess amount is also μY. Hence, the expected total number of finished product produced during a production cycle is exactly the order quantity Y. That is,

$$E[W] = E[W_c + W_p] = Y(1-\mu) + \mu Y = Y. \tag{17}$$

From Eqs. (16) and (17), the expected length of the production cycle is

$$E[T_p] = E[W/\alpha] = Y/\alpha. \tag{18}$$

During the production period, items of the finished product are produced at a rate α and used at a rate β to meet the demand. At the start of the production period and until time T_1, the excess amount of the finished product is used to fulfil the backorders at a rate of α–β. Hence,

$$T_1 = S/(\alpha-\beta). \tag{19}$$

After such time and until the end of the production period, the excess amount of the finished product is used to accumulate finished product inventory at a rate of α–β. This occurs during a time period of T_2, where $T_p = T_1 + T_2$. Hence,

$$T_2 = T_P - T_1 = W/\alpha - S/(\alpha-\beta). \tag{20}$$

At the end of this period, a maximum inventory level M is reached. Then,

$$M = T_2(\alpha-\beta) = W(1-\beta/\alpha) - S. \tag{21}$$

This maximum level will be used to meet the demand at a rate β until time T_3, when the inventory level of the finished product reaches zero. Hence,

$$T_3 = \frac{M}{\beta} = \frac{W(1-\beta/\alpha)-S}{\beta} = \frac{W}{\beta}\left(1-\frac{\beta}{\alpha}\right) - \frac{S}{\beta}. \tag{22}$$

Throughout the remainder of the inventory cycle, a planned shortage of size S is accumulated at a rate β during a time period of T_4, where

$$T_4 = S/\beta. \tag{23}$$

The finished product inventory level is shown in Fig. 1. Note that the length of the inventory cycle is $T = T_1 + T_2 + T_3 + T_4$. Eqs. (19), (20), (22) and (23) give that

$$T = W/\beta. \tag{24}$$

From Eqs. (13) and (24), the expected inventory length is

$$E[T] = Y/\beta. \tag{25}$$

2.3 The Cost Function

The optimal production quantity Y^* and the optimal shortage quantity S^* are determined by minimizing the total cost per unit time function given by

$$TCU(Y,S) = \frac{TC(Y,S)}{T}, \tag{26}$$

where $TC(Y, S)$ is the total cost per inventory cycle function. The $TC(Y, S)$ function comprises of the following cost components:

- Ordering, purchasing, screening and holding costs of raw material.
- Setup cost of production.
- Production and holding costs of finished product.
- Shortage and backorder costs.

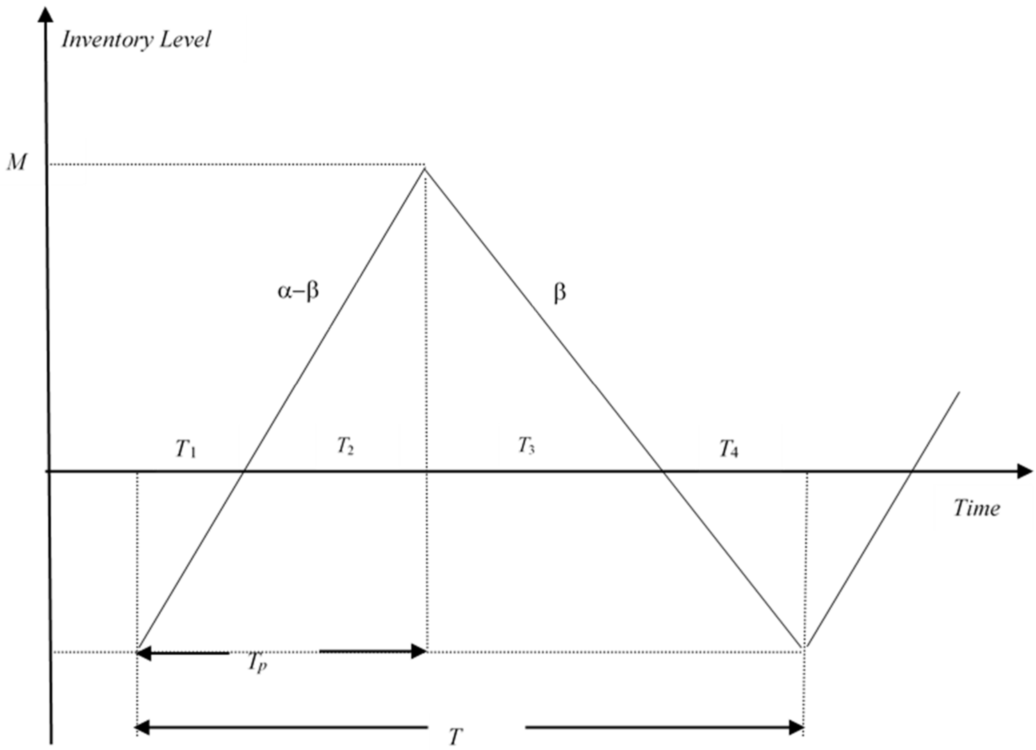

Figure 1: Finished Product Inventory level.

The ordering cost of raw materials of type j is C_j and the purchasing cost is $C_{rj}U_j$. The purchasing cost of raw material is reduced by an amount $S_{rj}\delta_j U_j$, which is the salvage value resulting from discarding the imperfect quality items at a discount price. Fig. 2 depicts the inventory level of raw material. Note that the drop in inventory level represents the selling of the $\delta_j U_j$ imperfect quality items of raw material.

The raw materials holding cost is the holding cost per unit of raw material per unit time, namely h_{rj}, multiplied by the average on hand inventory of raw material times the cycle length. That is, h_{rj} multiplied by the area under the curve in Fig. 2. Hence, the total holding cost of raw material per inventory cycle is

$$\text{Raw Material Holding Cost} = \sum_{j=1}^{n} hr_j \left(\frac{U_j(1-\delta_j)T_P}{2} + \frac{\delta_i U_j^2}{\gamma_j} + e_j\right). \tag{27}$$

The cost of producing the W units of the finished product is the sum of the setup C_0 and the variable production cost given by C_pW. The holding cost per unit of the finished product per unit time is h_p. Thus, the finished product holding cost is the average inventory of on hand finished product times the inventory cycle length times the holding cost per unit per unit time, which is h_p times the area in Fig. 1 under the curve and above the x-axis. Using Eqs. (20) to (22),

$$\text{Finished Product holding Cost} = \frac{h_P}{2}.M.(T_2 + T_3). \tag{28}$$

From Fig. 1 and using Eqs. (19) to (23), the shortage cost is

$$\begin{aligned}\text{Shortage Cost} &= C_bS + \tfrac{1}{2}C_sS(T_1 + T_4).\\ &= C_bS + \frac{1}{2}C_sS\left(\frac{S}{\alpha-\beta} + \frac{S}{\beta}\right)\\ &= C_bS + \frac{C_s}{2\beta(1-\beta/\alpha)}S^2.\end{aligned} \tag{29}$$

Hence, the total inventory cost per cycle is

$$TC(Y,S) = C_0 + \sum_{j=1}^{n} C_j + \sum_{j=1}^{n}\left(C_{rj} + C_{dj} - \delta_i S_{rj}\right)U_j + C_PW + C_bS + \tfrac{1}{2}C_sS(T_1 + T_4) + \sum_{j=1}^{n} h_{rj}\left(\frac{U_j(1-\delta_j)T_P}{2} + \frac{\delta_i U_j^2}{\gamma_j} + e_j\right) + \frac{h_P}{2}.M.(T_2 + T_3). \tag{30}$$

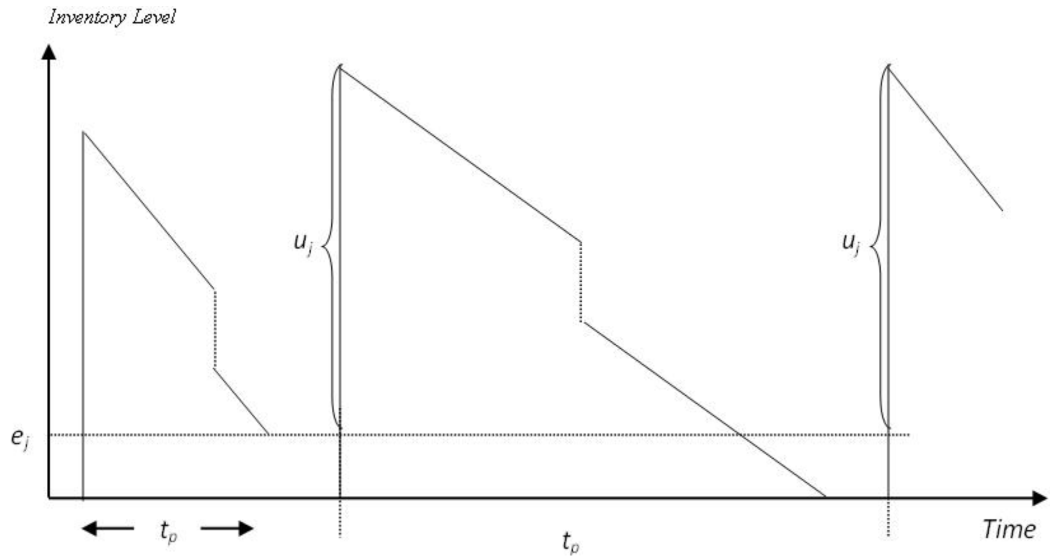

Figure 2: Inventory level of raw material of type *j* component.

The next step is to determine the expected total cost per inventory cycle. For this purpose, we note that in a typical inventory cycle, depicted in Fig. 1, the expected time required to build up the maximum inventory of finished items obtained using Eqs. (17) and (20) is

$$E[T_2] = Y/\alpha - S/(\alpha - \beta). \tag{31}$$

Similarly, Eqs. (17) and (21) give the expected maximum inventory of finished items as

$$E[M] = Y\left(1 - \frac{\beta}{\alpha}\right) - S. \tag{32}$$

Also, the expected time to deplete the maximum inventory is obtained from Eqs. (17) and (22) as

$$E[T_3] = \frac{Y}{\beta}\left(1 - \frac{\beta}{\alpha}\right) - \frac{S}{\beta}. \tag{33}$$

The area under the curve in Fig. 1 representing the on-hand inventory of the finished product is used to calculate the expected holding cost of the finished product. From Eqs. (28) and (31) to (33), we have

$$\text{Expected Finished Product Holding Cost} = \frac{h_P}{2}.\left(Y\left(1 - \frac{\beta}{\alpha}\right) - S\right) .\left(\frac{Y}{\alpha} - \frac{S}{\alpha - \beta} + \frac{Y}{\beta}\left(1 - \frac{\beta}{\alpha}\right) - \frac{S}{\beta}\right) = h_P \frac{(1 - \beta/\alpha)Y^2 - 2YS + \frac{S^2}{(1 - \beta/\alpha)}}{2\beta}. \tag{34}$$

Similarly, during a typical cycle, the expected area under the curve representing the on-hand inventory of the raw material of type *j* can used to calculate the expected total holding cost of the raw material. From Eqs. (15), (18) and (27), we have

$$\text{Expected Raw Material Holding Cost} = \sum_{j=1}^{n} hr_j \left(\frac{\frac{Y}{1-\mu_j}(1-\mu_j)Y}{2\alpha} + \frac{\mu_j\left(\frac{Y}{1-\mu_j}\right)^2}{\gamma_j} + \mu Y \right) = \sum_{j=1}^{n} hr_j \left(\frac{Y^2}{2\alpha} + \frac{\mu_j Y^2}{\gamma_j (1-\mu_j)^2} + \mu Y \right) \tag{35}$$

The expected total inventory cost per cycle $ETC(Y,S) = E[TC(Y, S)]$ obtained by taking the expected value of the various costs in Eq. (30) is

$$ETC(Y,S) = C_0 + \sum_{j=1}^{n} C_j + \sum_{j=1}^{n}\left(C_{rj} + C_{dj} - \mu_j S_{rj}\right)\frac{Y}{1-\mu_j} + C_P Y + C_b S + \frac{C_s}{2\beta(1-\beta/\alpha)}S^2 + \sum_{j=1}^{n} hr_j \left(\frac{Y^2}{2\alpha} + \frac{\mu_j Y^2}{\gamma_j(1-\mu_j)^2} + \mu Y\right) + \frac{h_P}{2\beta}\left(\left(1 - \frac{\beta}{\alpha}\right)Y^2 - 2YS + \frac{S^2}{(1-\beta/\alpha)}\right). \tag{36}$$

The expected total inventory cost per unit time, $ETCU(Y,S) = E[TCU(Y, S)] = E[TC(Y, S)/T]$, is approximated using the Renewal Reward Theorem as $ETCU(Y,S) = E[TC(Y, S)]/ E[T]$. Dividing Eq. (36) by the expected cycle length T given by Eq. (25). Hence,

$$ETCU(Y,S) = \frac{\beta}{Y}\Big(C_0 + \sum_{j=1}^{n} C_j + C_b S + \frac{C_s}{2\beta(1-\beta/\alpha)} S^2\Big) + \sum_{j=1}^{n}\big(C_{rj} + C_{dj} - \mu_i S_{rj}\big)\frac{\beta}{1-\mu_j} + C_P\beta + \beta\sum_{j=1}^{n} hr_j\left(\frac{Y}{2\alpha} + \frac{\mu_j Y}{\gamma_j(1-\mu_j)^2} + \mu\right) + \frac{h_P}{2}\left(\left(1-\frac{\beta}{\alpha}\right)Y - 2S + \frac{S^2}{(1-\beta/\alpha)Y}\right). \quad (37)$$

Note that the expected total cost per unit function depends on the determination of the expected value μ of the maximum of the random variables in $X_1, X_2, \ldots, X_n$. In section 3, the calculation of μ is described in the case where the random variables are uniformly distribution.

2.4 The Optimal Solution

To obtain the optimal production quantity Y^* and the optimal shortage size S^*, we find the first partial derivatives of $ETCU(Y, S)$ and set these derivatives equal to zero. Differentiating $ETCU(Y, S)$ with respect to S, we get

$$\frac{\partial ETCU(Y,S)}{\partial S} = \frac{\beta}{Y}\left(C_b + \frac{C_s}{\beta(1-\beta/\alpha)}S\right) + h_P\left(-1 + \frac{S}{(1-\beta/\alpha)Y}\right). \quad (38)$$

Setting the derivative in Eq. (38) equal to zero and rearranging, we get

$$h_P Y - \frac{(C_s + h_P)}{(1-\beta/\alpha)}S - C_b\beta = 0. \quad (39)$$

Now we differentiate $ETCU(Y, S)$ with respect to Y, we get

$$\frac{\partial ETC(Y,S)}{\partial Y} = -\frac{\beta}{Y^2}\left(C_0 + \sum_{j=1}^{n} C_j + C_b S + \frac{C_s}{2\beta\left(1-\frac{\beta}{\alpha}\right)}S^2\right) + \sum_{j=1}^{n}\beta hr_j\left(\frac{1}{2\alpha} + \frac{\mu_j}{\gamma_j(1-\mu_j)^2}\right) + \frac{h_P}{2}\left(\left(1-\frac{\beta}{\alpha}\right) - \frac{S^2}{\left(1-\frac{\beta}{\alpha}\right)Y^2}\right). \quad (40)$$

Setting the derivative in Eq. (40) to zero and rearranging, we have

$$-2\beta\big(C_0 + \sum_{j=1}^{n} C_j + C_b S\big) + Y^2\left(h_P\left(1-\frac{\beta}{\alpha}\right) + 2\beta\sum_{j=1}^{n} hr_j\left(\frac{1}{2\alpha} + \frac{\mu_j}{\gamma_j(1-\mu_j)^2}\right)\right) - \frac{h_p + C_s}{\left(1-\frac{\beta}{\alpha}\right)}S^2 = 0. \quad (41)$$

The solution of the system of equations (39) and (41) provide the optimal production quantity Y^* and optimal shortage quantity S^*. Note that the second partial derivatives obtained from (38) and (40) may be used to either demonstrate the uniqueness of the optimal solution or provide conditions that guarantee it. In the following section, we describe how the expected value of the maximum of a set of independent random variables can be calculated.

3 MAXIMUM OF A SET OF RANDOM VARIABLES

Functions of random variables have many applications in various fields, see (Yassine, 2018; Yassine and El-Rabih, 2019). The optimal solution derived in section 2 depends on the exepected value of the maximum of random variables each having a mean equal to 0. Hence, a process for obtaining the probability function of the maximum along with its expected value is needed. In the following, we describe such a process based argumentes similar to those Yassine (2018) used to determine the probability distribution and expected value of the minimum of uniformly distributed random variables each having a mean equal to 1.

Let $X_1, X_2, \ldots, X_n$ be n independent continuous random variables and let $g_j(X_j)$ denote the probability distribution of X_j. Since $X_1, X_2, \ldots, X_n$ are independent, the cumulative distribution of the random variable $Max(X_1, X_2, \ldots, X_n)$ is

$$\begin{aligned} H(t) &= P(Max(X_1, X_2, \ldots, X_n) \le t) \\ &= P(X_1 \le t)P(X_2 \le t) \ldots P(X_n \le t) \\ &= G_1(t).G_2(t) \ldots G_n(t), \end{aligned} \quad (42)$$

where $G_j(t)$ is the cumulative distribution of X_j.
In the case where each X_j is uniformly distributed over an interval $[-m_j, m_j]$ centered at zero, the probability distribution of X_j is

$$g_j(x_j) = \begin{cases} 0 & if \quad x_j < -m_j \\ \frac{1}{2m_j} & if \quad -m_j \le x_j \le m_j, \\ 0 & if \quad x_j > m_j \end{cases} \quad (43)$$

and its cumulative distribution, a continuous function, is

$$G_j(t) = \begin{cases} 0 & if \quad t \le -m_j \\ \dfrac{t + m_j}{2m_j} & if \quad -m_j \le t \le m_j. \\ 1 & if \quad t \ge m_j \end{cases} \quad (44)$$

Since each interval [$-m_j$, m_j] is centered at 0, we may assume, without loss of generality, that these intervals are nested so that

$$-m_n \le \ldots \le -m_2 \le -m_1 \le 0 \le m_1 \le m_2 \le \ldots \le m_n. \quad (45)$$

From Eqs. (42) and (44), the cumulative distribution of the maximum is

$$H(t) = \begin{cases} 0 & if \quad t \le -m_1 \\ \prod_{i=1}^{n} \frac{t+m_i}{2m_i} & if \quad -m_1 \le t \le m_1 \\ \prod_{i=j}^{n} \frac{t+m_i}{2m_i} & if \quad m_{j-1} \le t \le m_j \\ 1 & if \quad t \ge m_n \end{cases}. \quad (46)$$

The expected value μ of $Max(X_1, X_2,\ldots, X_n)$ is calculated using

$$\mu = \int_{-\infty}^{-\infty} th(t)dt, \quad (47)$$

where $h(t)$ is the derivative of $H(t)$.

In case when $n = 2$, the cumulative distribution in Eq. (46) reduces to

$$H(t) = \begin{cases} 0 & if \quad t \le -m_1 \\ \frac{(t+m_1)(t+m_2)}{4m_1m_2} & if \quad -m_1 \le t \le m_1 \\ \frac{t+m_2}{2m_2} & if \quad m_1 \le t \le m_2 \\ 1 & if \quad t \ge m_2 \end{cases}, \quad (48)$$

and the probability density function $h(t)$ of $Max(X_1, X_2,\ldots, X_n)$ is

$$h(t) = \begin{cases} 0 & if \quad t < -m_1 \\ \frac{2t+m_1+m_2}{4m_1m_2} & if \quad -m_1 \le t < m_1 \\ \frac{1}{2m_2} & if \quad m_1 \le t \le m_2 \\ 0 & if \quad t > m_2 \end{cases}. \quad (49)$$

Hence,

$$\mu = \int_{-m_1}^{m_1} \frac{t(2t + m_1 + m_2)}{4m_1m_2} dt \quad (50)$$

$$+ \int_{m_1}^{m_2} t/(2m_2)dt = \frac{m_1^2}{12m_2} + \frac{m_2}{4}.$$

When each δ_j is uniformly distributed over an interval [a_j, b_j], the random variable X_j is also uniformly distributed over an interval centred at 0, say [$-m_j$, m_j], where

$$m_j = \frac{b_j - \mu_j}{1-\mu_j} = \frac{b_j-(a_j+b_j)/2}{1-(a_j+b_j)/2} = \frac{b_j - a_j}{2-a_j-b_j}. \quad (51)$$

4 NUMERICAL EXAMPLE

Consider a production process where the demand rate for an item is 100 units per day and the production rate is 400 units per day. Assume that the percentage of imperfect raw material of type 1 used in production is uniformly distributed over [10%, 30%] so that the mean is 20%. Similarly, the percentage of imperfect raw material of type 2 is uniformly distributed over [10%, 40%] so that the mean is 25%. Screening for imperfect quality items of the raw material of type 1 is conducted at a rate of 1200 units per day and at a cost of \$0.20 per unit, and for type 2 at a rate of 800 units per day and at a cost of \$0.25 per unit. The ordering cost for the raw material of type 1 is \$2,000, of type 2 is \$3,000, and the production setup cost is \$4750. The holding cost of raw material of type 1 is \$0.2 per unit per day and \$0.3 per unit per day for raw material of type 2. The holding cost per unit of the finished product per day is \$0.92. The production cost is \$ 30 per unit. The purchasing cost of one item of raw material of type 1 is \$10 and \$20 for type 2. Planned shortages are permitted. The cost of having one finished short is \$2.6 per day and the administrative cost of a unit short is \$10. The production cost per unit is \$30. The Salvage value per unit of imperfect quality raw material of type 1 is \$5 and \$10 for type 2.

The parameters of the problem are $\alpha = 400$, $\beta = 100$, $C_0 = 4750$, $C_p = 30$, $C_1 = 2000$, $C_2 = 3000$, $C_{r1} = 10$, $C_{r2} = 20$, $C_{d1} = 0.2$, $C_{d2} = 0.25$, $C_b = 10$, $C_s = 2.6$, $h_{r1} = 0.2$, $h_{r2} = 0.3$, $h_P = 0.92$, $\gamma_1 = 1200$, $\gamma_2 = 800$, $S_{r1} = 5$, $S_{r2} = 20$, $\delta_1 \sim$ [10%, 30%], $a_1 = 0.10$, $b_1 = 0.30$, $g_1(\delta_1) = 1/(0.3–0.1) = 5$; $\mu_1 = (0.1+0.3)/2 = 0.2$; $\delta_2 \sim$ [10%, 40%]; $a_2 = 0.10$, $b_2 = 0.40$, $g_2(\delta_2) = 1/(0.4–0.1) = 3.33$; $\mu_2 = (0.1+0.4)/2 = 0.25$.

To determine the optimal production policy, first we need to determine the random variables X_1, X_2, and Max(X_1, X_2). Also, the expected value $\mu = E(\text{Max}(X_1, X_2))$ needs to be calculated. From Eq. (51), the value of m_1 is obtained as m_1 = (0.3–0.1)/(2–0.1–0.3) = 0.125. Hence, X_1 is uniformly distributed over [−0.125, 0.125]. Similarly, m_2 = (0.4–0.1)/(2–0.1–0.4) = 0.2 so that X_2 is uniformly distributed over [−0.2, 0.2]. The expected value of Max(X_1, X_2) can now be calculated using Eq. (50) as $\mu = \frac{0.125^2}{12(0.2)} + \frac{0.2}{4} = 0.05651$.

Solving the system in Eqs. (39) and (41) results in two solutions. The first has negative values for S and Y, which is rejected. The second solution gives the optimal production quantity $Y^* = 1600.09 \approx 1600$ and the optimal shortage quantity $S^* = 100.59 \approx 100$. Then, $ETCU(1600, 100) = 7801.03$. The order quantity of raw material of type 1 is $U_1 = Y/(1-\mu_1) = 1600/(1-0.8) = 2000$. Similarly, $U_2 = Y/(1-\mu_2) = 1600/(1-0.75) = 2133$. The expected number of finished items produced from the raw materials obtained during the current production cycle is $E[W_c] = Y(1-\mu) = 1510$. Also, the expected number of finished items produced from the excess perfect quality raw material kept in stock from previous periods is $E[W_p] = E[e_1] = E[e_2] = \mu Y = 90$.

The expected cycle length and production period are $E[T] = 1600/100 = 16$ and $E[T_p] = 1600/400 = 4$. The maximum inventory level of the finished product is $E[M] = 1600(1-100/400) - 100 = 1100$.

5 CONCLUSION

In this paper, an economic production model that accounts for the cost and quality of the raw materials was presented. Also, the effects of shortages were incorporated into the model. A mathematical model describing this production/inventory situation was formulated. It was shown that the optimal production and shortage quantities that minimize the total inventory cost per unit time function are the solution of a system of equations derived using the mathematical model. The total cost function was shown to depend on the maximum of a set of n independent random variables obtained from the proportion of imperfect quality raw material.

A process for obtaining the probability function of the maximum and its expected value was developed and described. Moreover, expressions for the probability density function and the expected value of the maximum when the random variables are uniformly distributed were obtained. The results were applied to the EPQ model considered in this paper. A numerical example illustrating the determination of the optimal policy was presented.

This study has some limitations. Due to the restriction on the length of the paper, uniqueness of the optimal solution was not demonstrated nor sensitivity analysis was performed. Also, the model considered the producer as the decision maker and ignored the other supply chain members. These limitations can be tackled in future research.

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