The Maximum Feasible Scenario Approach for the Capacitated Vehicle Routing Problem with Uncertain Demands

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Abstract: This study deals with the Capacitated Vehicle Routing Problem where customer demands are uncertain with unknown probability distribution. We follow the robust optimization methodology to formulate and solve the Robust Vehicle Routing Problem with Demand Uncertainty. Since the robust solution is a route plan, which optimizes the worst case that could arise, our focus is concentrated on determining the worst-case demands to solve the robust optimization model. The computational experiments examined two proposed strategies to indicate their performance in terms of the extra cost and unmet demands.

1 INTRODUCTION

The Capacitated Vehicle Routing Problem (CVRP) is one of the combinatorial optimization problems which aims to find a set of minimum total cost routes for a fleet of capacitated vehicles based at one depot, to serve a set of customers under the following constraints:

(1) each route begins and ends at the depot,

(2) each customer is visited exactly once,

(3) the total demand of each route does not exceed the capacity of the vehicle (Laporte, 2007).

The first mathematical formulation and algorithm for the solution of the CVRP was proposed by Dantzig and Ramser (1959) and five years later, Clarke and Wright (1964) proposed the first heuristic for this problem. Till to date many solution methods for the CVRP have been published. General surveys can be found in (Toth and Vigo, 2014) and (Laporte, 2009). The CVRP belongs into the category of NP hard problems that can be exactly solved only for small instances of the problem. Therefore, researchers have concentrated on developing heuristic algorithms to solve this problem, for example (Laporte et al., 2014), (Gendreau et al., 2010).

Contrary to the deterministic CVRP, which assumes that the problem parameters (e.g. the customer demands, travelling costs, service times, etc.) are deterministic and known, the robust CVRP (RVRP) considers the parameters affected by uncertainty with unknown probability distribution. In robust optimization methodology introduced by Ben-Tal and Nemirovski (1998), uncertainty is modeled as a bounded set U which contains all possible continuous or discrete values referred as scenarios. The objective of the RVRP is to obtain a robust solution, that is feasible for all scenarios in U.

There are some works in the literature dealing with the RVRP with different (combined or separated) uncertain parameters. For example, Sungur et al. (2008), Moghaddam et al. (2012), Gounaris et al. (2013), Gounaris et al. (2016) and Pessoa et al. (2018) study the RVRP with uncertain demands. Toklu et al. (2013), Han et al. (2013) and Solano-Charris et al. (2014) apply robust optimization for the RVRP with uncertain travel costs. Lee et al. (2012) and Sun et al. (2015) consider RVRP in which travel times and also demands are uncertain.

This paper deals with the RVRP in which the customer demands are uncertain and the distribution of them is unknown. Our research is inspired by a work of Sungur et al. (2008), who proposed the first solution procedure for the RVRP with demand uncertainty. The authors used a robust version of the CVRP formulation with Miller-Tucker-Zemlin constraints based on specific uncertainty to determine vehicle routes that minimize transportation costs while satisfying all possible demand realizations. Their model may be perceived as a worst-case instance of the deterministic CVRP in which the nominal demand parameter is replaced by a modified one from the uncertainty set, thus solving the RVRP is no more difficult than solving a single deterministic CVRP. But, depending on the nature of the scenarios, some RVRPs may become infeasible problems. Therefore, we are interested in extending of this method to infeasible instances.

The structure of this paper is organized as follows. Section 2 is devoted to introduce the RVRP formulation whith uncertain demands which is derived from our two-index CVRP formulation (Borčinová, 2017). Next, our robust solution approach is presented in Section 3. In Section 4 the computational results are reported and compared with adopted method of Sungur et al. Finally in Section 5, conclusions are drawn.

2 THE RVRP WITH UNCERTAIN DEMANDS FORMULATION

The RVRP with demand uncertainty can be defined by a complete directed graph $G = (V, H, c)$ with $V = \{0, 1, 2, \ldots, n\}$ as the set of nodes and $H = \{(i, j) : i, j \in V, i \neq j\}$ as the set of arcs, where node 0 represents the depot for a fleet of p vehicles with the same capacity Q and remaining n nodes represent geographically dispersed customers.

The positive travel cost c_{ij} is associated with each arc $(i, j) \in H$. The cost matrix is symmetric, i.e. $c_{ij} = c_{ji}$ for all $i, j \in V, i \neq j$ and satisfies the triangular inequality, $c_{ij} + c_{jl} \geq c_{il}$ for all $i, j, l \in V$.

The uncertain demands associated with each customer $j \in V - \{0\}$ are modeled as discrete scenarios. The scenarios are constructed as deviations around an expected demand d_j^0, i.e. demand d_j^k of customer j in scenario k is positive value, $d_j^0 - \varepsilon d_j^0 \leq d_j^k \leq d_j^0 + \varepsilon d_j^0$, where ε is a non negative constant, which determines the maximum perturbation percentage.

Two-index decision variables x_{ij} are used as binary variables equal to 1 if arc (i, j) belongs to the optimal solution and 0 otherwise. For all pairs of nodes $i, j \in V - \{0\}, i \neq j$ we calculate the savings s_{ij} for joining the cycles $0 \to i \to 0$ and $0 \to j \to 0$ using arc (i, j) as in Clarke and Wright's saving method (1964), i.e. $s_{ij} = c_{i0} + c_{0j} - c_{ij}$. Then, instead of minimizing the total cost, we will maximize the total saving (Borčinová, 2017). To ensure the continuity of the route and to eliminate sub-tours, we define an auxiliary continuous variable y_j, which shows (in the case of collecting of the goods) the vehicle load after visiting customer j. To simplify mathematical modelling, we replace each feasible route $0 \to v_1 \to v_2 \to \cdots \to v_t \to 0$ by a path from node 0 to node v_t, i.e. $0 \to v_1 \to v_2 \to \cdots \to v_t$.

The RVRP with uncertain demands seeks the optimal solution that satisfies all demand realizations. Let U_d denotes a set which contains all scenario vectors, $U_d = \{d^k, k = 1, \ldots, m\}$, where m is a given number of scenarios. Then, according to Bertsimas and Sim (2003), the robust optimization model of the problem can be stated as

RVRP 1:

$$\max \quad \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} s_{ij} x_{ij}, \tag{1}$$

subject to

$$\sum_{j=1}^{n} x_{0j} = p, \tag{2}$$

$$\sum_{i=1}^{n} x_{i0} = 0, \tag{3}$$

$$\sum_{\substack{i=0 \\ i \neq j}}^{n} x_{ij} = 1, \quad \forall j \in V - \{0\}, \tag{4}$$

$$\sum_{\substack{j=1 \\ i \neq j}}^{n} x_{ij} \leq 1, \quad \forall i \in V - \{0\}, \tag{5}$$

$$y_i + d_j^k x_{ij} - Q(1 - x_{ij}) \leq y_j, \quad \forall d^k \in U_d, i, j \in V - \{0\}, i \neq j, \tag{6}$$

$$d_j^k \leq y_j \leq Q, \quad \forall d^k \in U_d, j \in V - \{0\}, \tag{7}$$

$$x_{ij} \in \{0, 1\}, \quad \forall i, j \in V, i \neq j. \tag{8}$$

In this formulation, the objective function (1) maximizes the total travel saving. The constraints (2), (3), (4) and (5) are the indegree and outdegree constraints for depot and customers. Constraints (6) are the route continuity and sub-tour elimination constraints and the constraints given in (7) are capacity bounding constraints which restrict the upper and lower bounds of y_j. Finally, (8) are the obligatory constraints.

The general approach of robust optimization is to optimize the worst-case value over all data uncertainty (Sungur et al., 2008). Let d_j^w denotes the demand of customer j in the worst scenario. Then, we replace the constraints (6) and (7) with the constraints

$$y_i + d_j^w x_{ij} - Q(1 - x_{ij}) \leq y_j, \quad \forall i, j \in V - \{0\}, i \neq j, \tag{9}$$

$$d_j^w \leq y_j \leq Q, \quad \forall j \in V - \{0\}. \tag{10}$$

Thus, similar to Sungur et al.(2008), we can solve the RVRP as an instance of the CVRP. We refer to the resulting model as *RVRP 2*.

3 DETERMINING THE WORST-CASE SCENARIO

The key step in this approach is to identify the worst-case scenario and subsequently substitute its values

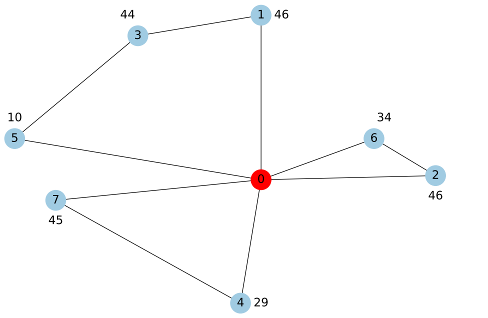

Figure 1: CVRP: Optimal deterministic solution.

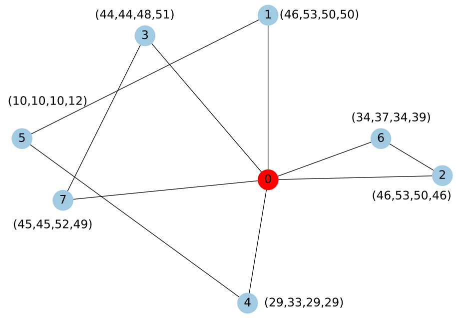

Figure 2: RVRP: Robust solution.

d_j^w into the constraints (9) and (10). We suggest following two strategies for determining the worst-case scenario: the maximum demand scenario and the maximum feasible demand scenario.

3.1 The Maximum Demand Scenario

In the first strategy (*Strategy 1*) we consider that all customer demands take their maximum values in each scenario (Ordóñez, 2010), i.e. the worst-case scenario values are $d_j^w = d_j^{max} = \max\{\max_k d_j^k, d_j^0\}$ for $j \in V - \{0\}$, $k \in \{1,\ldots,m\}$, where m is a number of scenarios. This strategy gives a solution, which is immune to demand variation under each scenario. However, the CVRP with the maximum demand scenario may be infeasible if vehicles have not sufficient capacity. Then a robust solution can not be found.

3.2 The Maximum Feasible Demand Scenario

The main idea of this strategy (*Strategy 2*) is to serve as many demands as possible, i.e. we find such a scenario with values $d_j^w \in \{d_j^k, k = 0,\ldots,m\}$ to maximize the sum of satisfied demands with respect the fleet size and vehicle capacities.

We assume the decission variables t_{ijk}, which indicate whether vehicle i serves demand d_j^k or not. The mathematical model that determines feasible scenario with maximum sum of satisfied demands can be described as follows:

$$\max \quad \sum_{i=1}^{p}\sum_{j=1}^{n}\sum_{k=0}^{m} t_{ijk}\, d_j^k, \tag{11}$$

subject to

$$\sum_{j=1}^{n}\sum_{k=0}^{m} t_{ijk} \geq 1, \quad \forall i \in \{1,\ldots,p\}, \tag{12}$$

$$\sum_{i=1}^{p}\sum_{k=0}^{m} t_{ijk} = 1, \quad \forall j \in \{1,\ldots,n\}, \tag{13}$$

$$\sum_{j=1}^{n}\sum_{k=0}^{m} t_{ijk}\, d_j^k \leq Q, \quad \forall i \in \{1,\ldots,p\}, \tag{14}$$

$$t_{ijk} \in \{0,1\},$$
$$\forall i \in \{1,\ldots,p\}, j \in \{1,\ldots,n\}, k \in \{0,\ldots,m\}. \tag{15}$$

The objective function (11) maximizes the sum of the demand values selected from a set of scenarios to be served. The constraints (12) express that each vehicle must serve at least one customer. The constraints (13) impose that every customer is visited once by one vehicle and exactly one its demand value is selected to be served. The constraints (14) ensure that the total load does not exceed the capacity of any vehicle. Finally, the constraints (15) define the binary decission variables. Obviously, it is not guarantee that all the demand of each customer will be met under this strategy.

4 COMPUTATIONAL EXPERIMENTS

In order to assess the efficiency and quality of the proposed robust solution approach, we design computational experiments and analyze the robust solutions obtained with alternative strategies in terms of their demand and cost performances. The mathematical models were coded in Python 3.7 (2019) and solved by the solver Gurobi 8.1 (2019). A computer with an Intel Xeon 32 cores, 2.4 GHz processor and 256 GB of RAM was used to perform the computational experiments.

4.1 Test Instances

We use standard instances publicly available at `www.coin-or.org/SYMPHONY/branchandcut/VRP/data/` for our computational experiments. Because the instances are originally designed for deterministic CVRP, it was necessary to modify them to include demand uncertainty. The customer demands specified in the benchmark were taken to be their nominal values d_j^0. For each deterministic CVRP benchmark, we construct four classes of uncertainty sets of 5 scenarios within the allowed perturbation percentage $\varepsilon \in \{0.05, 0.1, 0.15, 0.2\}$. All instances were solved with a runtime limit of one hour.

4.2 Performance Measures

To evaluate the performance of the proposed strategies, we use the performance measures presented in (Sungur, 2008) including the relative extra cost and unmet demands.

Let z_r and z_d be the cost of robust and deterministic solutions respectively. The ratio $\zeta = \frac{z_r - z_d}{z_d}$ quantifies the relative extra cost of the robust solution versus the deterministic optimal one. It is clear, that smaller ζ means the better cost performance.

Let g_d and g_r represent the maximum unmet demand that can occur when using the deterministic and robust solution respectively and D is the total nominal demand, i.e. $D = \sum_{i=1}^{n} d_i^0$. The unmet demand is the sum of demands in each route that exceeds the vehicle capacity. The ratio $\gamma = \frac{g_d - g_r}{D}$, reflects the relative decrease of unmet demand in the robust solution compare to deterministic optimal one when it faces the maximum demand scenario. Obviously, the larger γ indicate the better demand performance. We notice that every solution found by *Strategy 1* has $g_r = 0$.

Example:

Figure 1 shows a CVRP optimal solution with $n = 7$ customers and $p = 3$ vehicles of the capacity $Q = 100$. Customer demands $d = (46,46,44,29,10,34,45)$ are displayed next to the nodes. The cost of this solution $z_d = 227$.

Figure 2 illustrates a robust solution of the same problem with $m = 4$ discrete scenarios of uncertain demands, which were generated randomly by pertubation $\varepsilon = 0.20$:

$$d^1 = (46,46,44,29,10,34,45),$$
$$d^2 = (53,53,44,33,10,37,45),$$
$$d^3 = (50,50,48,29,10,34,52),$$
$$d^4 = (50,46,51,29,12,39,49).$$

The route demands in particular scenarios are:

- Route 1, 5, 4 : 85, 96, 89, 91
- Route 3, 7 : 89, 89, 100, 100
- Route 2, 6 : 80, 90, 84, 85

It is evident, that the total demand of any route in each scenario does not exceed vehicle capacity. The cost of robust solution $z_r = 291$, i.e. the relative extra cost $\zeta = 0.282$.
Since *Strategy 1* failed in solving this problem, depicted robust solution was found by *Strategy 2*, whereby the maximum feasible scenario is $d^w = (53,53,51,33,12,39,49)$. To evaluate the demand performance, we calculate the total demands in each route for a case if all customer demands take their maximum value $d^{max} = (53,53,51,33,12,39,52)$. The summations of demands in the optimal solution routes are 116, 85 and 92, i.e. there is $g_d = 16$ unsatisfied demands. In the robust solution, the summations of demands in the routes are 98, 103 and 92, therefore $g_r = 3$ and the relative decrease of unmet demand $\gamma = 0.051$.

4.3 Numerical Results

The results of both proposed strategies are summarized in the Table 1.

In this table, the first column represents the name of instances. The name of each instance allows determine its characteristics, since it has a format X-nA-kB-eC, where A is the number of nodes, B represents the number of vehicles and C indicates perturbation percentage. For example an instance P-n16-k8-e5 has 16 nodes, 8 vehicles and was derived from the instance P-n16-k8 by demand generation with $\varepsilon = 0.05$.

The columns *Cost performance* and *Demand performance* show the two performance measures ζ and γ respectively, as explained before. An indicator "*in*" denotes an infeasible instance.

As we can observe from Table 1 the strategy, which optimizes the maximum demand scenario (*Strategy 1*) results to infeasible solutions in some cases, while strategy based on the maximum feasible demand scenario approach (*Strategy 2*) has an appropriate solution in all cases. For all other cases both of strategies have achieved the same results, because the maximum feasible demand scenario is equal to the maximum demand scenario. It means, that *Strategy 2* is an extension of *Strategy 1* for problems with infeasible maximum demand scenario.

Table 1: The comparison of two strategies.

Instance	*Strategy 1* Cost perform.	*Strategy 1* Demand perform.	*Strategy 2* Cost perform.	*Strategy 2* Demand perform.
P-n16-k8-e5	0.000	0.000	0.000	0.000
P-n16-k8-e10	0.000	0.004	0.000	0.000
P-n16-k8-e15	0.027	0.008	0.027	0.008
P-n16-k8-e20	*in*	*in*	0.027	0.004
P-n20-k2-e5	0.005	0.006	0.005	0.006
P-n20-k2-e10	0.005	0.010	0.005	0.010
P-n20-k2-e15	*in*	*in*	0.023	0.003
P-n20-k2-e20	*in*	*in*	0.009	0.000
P-n23-k8-e5	0.091	0.006	0.091	0.006
P-n23-k8-e10	0.091	0.006	0.091	0.006
P-n23-k8-e15	*in*	*in*	0.174	0.000
P-n23-k8-e20	*in*	*in*	0.102	0.000
E-n13-k4-e5	0.121	0.021	0.121	0.021
E-n13-k4-e10	0.121	0.053	0.121	0.053
E-n13-k4-e15	0.121	0.082	0.121	0.082
E-n13-k4-e20	0.166	0.117	0.166	0.117
E-n22-k4-e5	0.043	0.008	0.043	0.008
E-n22-k4-e10	0.043	0.008	0.043	0.008
E-n22-k4-e15	0.043	0.007	0.043	0.007
E-n22-k4-e20	0.043	0.006	0.043	0.006
E-n23-k3-e5	0.002	0.005	0.002	0.005
E-n23-k3-e10	0.002	0.026	0.002	0.026
E-n23-k3-e15	*in*	*in*	0.000	0.000
E-n23-k3-e20	*in*	*in*	0.000	0.000
A-n34-k5-e5	0.000	0.000	0.000	0.000
A-n34-k5-e10	0.000	0.000	0.000	0.000
A-n34-k5-e15	0.042	0.004	0.042	0.004
A-n34-k5-e20	*in*	*in*	0.042	0.004
A-n44-k6-e5	0.000	0.000	0.000	0.000
A-n44-k6-e10	0.002	0.004	0.002	0.004
A-n44-k6-e15	0.049	0.019	0.049	0.019
A-n44-k6-e20	*in*	*in*	0.044	0.009
A-n53-k7-e5	0.032	0.000	0.032	0.000
A-n53-k7-e10	0.028	0.008	0.028	0.008
A-n53-k7-e15	*in*	*in*	0.049	0.035
A-n53-k7-e20	*in*	*in*	0.046	0.026

5 CONCLUSIONS

In this paper the robust optimization was used to solve the Capacitated Vehicle Routing Problem with demand uncertainty. The main contribution of our work is to design a new strategy to cope with uncertain demands. At first, we present the mathematical model *RVRP 2*, derived from our two-index CVRP formulation, which optimizes the worst-case scenario. Then we are concerned with determining the worst-case scenario. In *Strategy 1*, the worst-case scenario takes the maximum demand values, which sometimes leads to an infeasible problem. Therefore we propose strategy (*Strategy 2*) based on an idea to satisfy as many demands as possible with respect the fleet size and vehicle capacities. Hence, in contrast to *Strategy 1*, it always designes an appropriate solution, even though it is not strictly robust.

A possible future research is to extend this study to solving large-scale problem instances.

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