Subjective Markov Process with Fuzzy Aggregations

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Keywords: Fuzzy Logic, Markov Process, Decision-making, Subjective Reasoning.

Abstract: Dynamical models of autonomous systems usually follow general assumption about rationality of the systems and their judgements. In particular, the systems acting under uncertainty are defined using probabilistic methods with the reasoning based on minimization or maximization of the expected payoffs or rewards. However, in the systems that deal with rare events or interact with human usually demonstrating irrational behaviour correctness of the use of probability measures and of the utility functions is problematic. In order to solve this problem, in the paper we suggest a Markov-like process that is based on a certain type of possibility measures and uninorm and absorbing norm aggregators. Together these values and operators form an algebraic structure that, on one hand, extends Boolean algebra and, on the other hand, operates on the unit interval as arithmetic system. We demonstrate the basic properties of the suggested subjective Markov process that go in parallel to the properties of usual Markov process, and stress formal differences between two models. The actions of the suggested process are illustrated by the simple model of search that clarifies the differences between Markov and subjective Markov processes and corresponding decision-making.

1 INTRODUCTION

Usually the models of autonomous systems acting under uncertainties are based on the Markov processes that define the evolution of the probabilities of the system states. The decision-making in such systems deals with the choice of the system's activities in each state.

In spite of a wide variety of methods and algorithms used in such models, the starting point of these probabilistic techniques is an assumption about rationality of the systems and their judgements (Luce and Raiffa, 1964; Raiffa, 1968). Consequently, the reasoning in such systems is based on minimization or maximization of the expected payoffs or rewards (White, 1993).

However, in the models those consider the systems with rare events or deal with the systems interacting with humans, who usually demonstrate irrational behaviour (Kahneman and Tversky, 1979), application of probabilistic measures and minimization/maximization criterions are rather problematic. Subjective factors in such models usually are considered on the base of certain utility functions that represent preferences of the observer (Friedman and Savage, 1948), and by extending usual Markov processes (MP) up to partially observable or hidden Markov processes (HMP) (Monahan, 1982; Rabiner, 1989) or hierarchical HMP (HHMP) (Fine, Singer and Tishbi, 1998). Such techniques allow effective modelling of many types of particular systems, but as direct successors of the Markov decision processes (MDP) (White, 1993), these techniques are also based on probabilistic methods with no concern to the existence or correctness of the required probabilistic measures.

In order to resolve these problems, in the paper we suggest subjective MP (μP) that goes in parallel to usual MP, but instead of probabilities is defined on a certain type of possibility measures (Dubois and Prade, 1988) that represent beliefs of the observer. Together with the uninom (Yager and Rybalov, 1996) and the absorbing norm (Batyrshin, Kaynak and Rudas, 2002), these measures form an algebraic structure that, on one hand, extends Boolean algebra and, on the other hand, operates on the unit interval as usual arithmetic system (Kagan, Rybalov, Siegelmann and Yager, 2013). In addition to clear definition of the process, such property allows consideration of the μP in comparison with the MP and stressing their similarities and differences.

Kagan, E., Rybalov, A. and Yager, R.
Subjective Markov Process with Fuzzy Aggregations.
DOI: 10.5220/0008915203860394
In Proceedings of the 12th International Conference on Agents and Artificial Intelligence (ICAART 2020) - Volume 2, pages 386-394
ISBN: 978-989-758-395-7; ISSN: 2184-433X

In particular, in the paper we present a basic classification of the µP states that goes in parallel to the classification of the MP states and demonstrates formal differences between two models. The actions of the µP are illustrated by the simple model of search (Pollock, 1970; Kagan and Ben-Gal, 2013) that clarifies the differences between µP and MP and corresponding decision-making.

2 UNINORM AND ABSORBING NORM

Let us briefly recall the definitions of the uninorm (Yager and Rybalov, 1996) and absorbing norm (Batyrshin, Kaynak and Rudas, 2002).

Consider the truth values that, in contrast to the Boolean logic, are drawn from the interval $[0, 1]$. In the theory of fuzzy sets and in fuzzy logic (Bellman and Giertz, 1973; Zade, 1965), such truth values are associated with the "grades of membership" $\mu_A(d)$ of the points d of some domain D to the set $A \subset D$. The function $\mu_A: D \to [0, 1]$ is, respectively, called the membership function.

For the non-binary truth values are defined the multivalued "and" $\curlywedge$, "or" $\curlyvee$ and "not" $\sim$ operators, such that for any $x, y \in [0, 1]$ also $(x \curlywedge y) \in [0, 1]$, $(x \curlyvee y) \in [0, 1]$ and $(\sim x) \in [0, 1]$. These operators go in parallel to the Boolean "and" $\wedge$, "or" $\vee$ and "not" $\neg$ operators and coincide with them such that for binary truth values $x, y \in \{0, 1\}$ it holds true that $x \curlywedge y = x \wedge y$, $x \curlyvee y = x \vee y$ and $\sim x = \neg x$. In the most applications, they are also associated with statistical triangular norms (Klement, Mesiar and Pap, 2000) (conjunction $\curlywedge$ with t-norm and disjunction $\curlyvee$ with t-conorm) or are defined arithmetically (Dubois and Prade, 1985).

Later (Yager and Rybalov, 1996), conjunction $\curlywedge$ and disjunction $\curlyvee$ operators were united into a single uninorm aggregator $\oplus_\theta: [0, 1] \times [0, 1] \to [0, 1]$ with neutral element $\theta \in [0, 1]$ such that for $\theta = 1$ it is $x \oplus_1 y = x \curlywedge y$ and for $\theta = 0$ it is $x \oplus_0 y = x \curlyvee y$. In parallel, there was introduced (Batyrshin, Kaynak and Rudas, 2002) an absorbing norm aggregator $\otimes_\vartheta: [0, 1] \times [0, 1] \to [0, 1]$ with absorbing element $\vartheta \in [0, 1]$; this aggregator extends the Boolean not xor operator.

Both uninorm $\oplus_\theta$ and absorbing norm $\otimes_\vartheta$ are the functions that specify aggregation of the variables $x, y \in [0, 1]$ resulting in $(x \oplus_\theta y) \in [0, 1]$ and $(x \otimes_\vartheta y) \in [0, 1]$, and for all $x, y, z \in [0, 1]$ they meet the commutative and associative properties:

$$x \oplus_\theta y = y \oplus_\theta x, \tag{1a}$$

$$x \otimes_\vartheta y = y \otimes_\vartheta x, \tag{1b}$$

$$(x \oplus_\theta y) \oplus_\theta z = x \oplus_\theta (y \oplus_\theta z) \text{ and} \tag{2a}$$

$$(x \otimes_\vartheta y) \otimes_\vartheta z = x \otimes_\vartheta (y \otimes_\vartheta z); \tag{2b}$$

for the uninorm it also holds true that

$$x \le y \text{ implies } x \oplus_\theta z \le y \oplus_\theta z. \tag{3}$$

Neutral θ and absorbing ϑ elements play a role of zero for their operators that is

$$\theta \oplus_\theta x = x \text{ and} \tag{4a}$$

$$\vartheta \otimes_\vartheta x = \vartheta. \tag{4b}$$

Interpretation of the aggregators $\oplus_\theta$ and $\otimes_\vartheta$ is the following (Rybalov and Kagan, 2017). The uninorm is an operator such that its truth value is defined by the extent (called also true by extent), to which both its arguments are true, and the absorbing norm is an extension of the not xor comparison and specifies the grade of similarity between its arguments. In the other situations, these aggregators can be considered as parameterized logical operators and applied for design of logical schemes (Rybalov, Kagan and Yager, 2012), or even as a tool for modelling operations in quantum information theory (Rybalov, Kagan, Rapoport and Ben-Gal, 2014).

Together with formal properties of the aggregators $\oplus_\theta$ and $\otimes_\vartheta$, it was also proven (Fodor, Yager and Rybalov, 1997; Fodor, Rudas and Bede, 2004) that for any $x, y \in [0, 1]$ there exist functions u and v called generator functions such that

$$x \oplus_\theta y = u^{-1}\big(u(x) + u(y)\big), \tag{5a}$$

$$x \otimes_\vartheta y = v^{-1}\big(v(x) \cdot v(y)\big). \tag{5b}$$

while the inverse functions u^{-1} and v^{-1} considered on the open interval $(0, 1)$ are probability distributions (Kagan, Rybalov, Siegelmann and Yager, 2013). Such equivalence between inverse generator functions and probability distributions demonstrates deep relation between probabilistic and fuzzy logics (Kagan, Rybalov, Siegelmann and Yager, 2013; Kagan, Rybalov and Yager, 2014) that, however, requires additional considerations.

The truth values from the interval $[0, 1]$ together with the uninorm $\oplus_\theta$ and absorbing norm $\otimes_\vartheta$ aggregators form an algebraic structure. In the next section we define this structure.

3 ALGEBRAIC STRUTURE WITH UNINORM AND ABSORBING NORM

Let $\oplus_\theta$ be a uninorm with the neutral element θ and $\otimes_\vartheta$ be an absorbing norm with the absorbing element ϑ. As operators on the interval $[0,1]$, uninorm $\oplus_\theta$ defines monoid $\mathcal{M}_\oplus = \langle [0,1],\ \oplus_\theta, \theta \rangle$ with a unit θ, and absorbing norm $\otimes_\vartheta$ defines monoid $\mathcal{M}_\otimes = \langle [0,1],\ \otimes_\vartheta, \vartheta \rangle$ with a unit ϑ.

The algebraic structure $\mathcal{A} = \langle [0,1], \oplus_\theta, \otimes_\vartheta \rangle$ on the interval $[0,1]$ with uninorm $\oplus_\theta$ and absorbing norm $\otimes_\vartheta$ aggregators is defined as a triple that joins monoids $\mathcal{M}_\oplus$ and $\mathcal{M}_\otimes$. It is clear that this structure extends Boolean algebra $B = \langle \{0,1\},\ \wedge,\ \vee \rangle$ defined for the operators $\wedge$ and $\vee$, and to its multivalued version $\mathcal{B} = \langle [0,1],\ \curlywedge,\ \curlyvee \rangle$ defined for the t-norm $\curlywedge$ and t-conorm $\curlyvee$ and acts both as a multivalued logical system and as an arithmetic system on the interval $[0,1]$.

The basic properties of the structure $\mathcal{A}$ are the following.

– if $u(x) = v(x)$ for all $x \in [0,1]$, then $\theta = \vartheta$;

– the value $\lambda = v^{-1}(1)$ is an identity element of the absorbing norm that is $\lambda \otimes_\vartheta x = x$; below this value will be denoted by $\mathbb{I}_\otimes$.

Moreover 0 (Fodor, Rudas and Bede, 2004):

– if $\theta = \vartheta$, then absorbing norm is distributive with respect to the uninorm, that is

$$(x \oplus_\theta y) \otimes_\vartheta z = (x \otimes_\vartheta z) \oplus_\theta (y \otimes_\vartheta z); \tag{6}$$

– for any $x \in [0,1]$ there exists an opposite element $\ominus_\theta x = u^{-1}(-u(x)) \in [0,1]$ such that

$$x \oplus_\theta (\ominus_\theta x) = x \ominus_\theta x = \theta; \tag{7}$$

– for any $x \in [0,1]$, $x \neq \vartheta$, there exists an inverse element $\lambda \oslash_\vartheta x = v^{-1}(1/u(x)) \in (0,1)$ such that

$$x \otimes_\vartheta (\lambda \oslash_\vartheta x) = x \oslash_\vartheta x = \lambda. \tag{8}$$

In addition, notice that (Kagan, Rybalov, Siegelmann and Yager, 2013)

– if $u(x) = v(x)$, $x \in [0,1]$ and so $\theta = \vartheta$, then the structure $\mathcal{A}$ is a commutative ring isomorphic to the ring of real numbers; otherwise, the structure $\mathcal{A}$ is a non-distributive algebra such that

$$(x \oplus_\theta y) \otimes_\vartheta z \neq (x \otimes_\vartheta z) \oplus_\theta (y \otimes_\vartheta z). \tag{9}$$

In the other words, the structure $\mathcal{A} = \langle [0,1], \oplus_\theta, \otimes_\vartheta \rangle$ defines formal algebra on the interval [0,1], where the aggregator $\oplus_\theta$ is considered as an operation of summation and the aggregator $\otimes_\vartheta$ – as an operation of multiplication. In addition to these operations, there are obviously defined the operators of subtraction $\ominus_\theta$ and of division $\oslash_\vartheta$ such that for any $x, y \in [0,1]$ are

$$x \ominus \theta y = u - 1ux - uy, \tag{10}$$

and

$$x \oslash \vartheta y = v - 1vx/vy, \tag{11}$$

In the further considerations, we will need the following properties of the uninorn and absorbing norm:

uninorm $\oplus_\theta$:

– if $x, y > \theta$ then $x \oplus_\theta y > \theta$;

– if $x, y < \theta$ then $x \oplus_\theta y < \theta$;

– if $x > \theta$ and $y < \theta$ and $|x - \theta| > |y - \theta|$ then $x \oplus_\theta y > \theta$;

absorbing norm $\otimes_\vartheta$:

– if $x, y > \vartheta$ or $x, y < \vartheta$ then $x \otimes_\vartheta y > \vartheta$;

– if $x > \vartheta$ and $y < \vartheta$ then $x \otimes_\vartheta y < \vartheta$.

In order to prove these properties it is enough to exhibit a continuous monotonically increasing function $\tau_v : [0,1] \to [-1,1]$ with parameter v (that stands for θ in the uninorm $\oplus_\theta$ and for ϑ in the absorbing norm $\otimes_\vartheta$) such that $\tau_v(0) = -1$, $\tau_v(v) = 0$ and $\tau_v(1) = 1$. The simplest example of the required function τ_v is a partially linear function

$$\tau_v(x) = \begin{cases} \frac{1}{v}x - 1, & x < v, \\ \left(1 - \frac{v}{v-1}\right)x + \frac{v}{v-1}, & x > v, \end{cases} \tag{12}$$

and such that $\tau_v(x) = -1$ when $v = 0$ and $\tau_v(x) = 1$ when $v = 1$.

Then aggregation using uninorm and absorbing norm are equivalent to the normalized summation and multiplication in the interval $[-1,1]$, for which the required properties hold.

Notice that since the properties of aggregation using uninorm and absorbing norm are equivalent to the summation and multiplication in the interval $[-1,1]$, operations using these norms can be considered as a multivalued extension of the operations of the three-valued logic.

Finally, in the further considerations we need the following values (Kagan, Rybalov and Ziv, 2016; Kagan, Rybalov and Yager, 2018)

$$\mathbb{O}_\oplus = u^{-1}(-1), \quad \mathbb{I}_\oplus = u^{-1}(1), \tag{13a}$$

$$\mathbb{O}_\otimes = v^{-1}(-1), \quad \mathbb{I}_\otimes = v^{-1}(1). \tag{13b}$$

The values $\mathbb{O}_{\oplus}$ and $\mathbb{O}_{\otimes}$ are called subjective false and the values $\mathbb{I}_{\oplus}$ and $\mathbb{I}_{\otimes}$ are called subjective true (both with respect to $\oplus_{\theta}$ and $\otimes_{\vartheta}$). These values are certainly differ from neutral element $\theta = u^{-1}(0)$ and absorbing element $\vartheta = v^{-1}(0)$. Moreover, from the properties of the generator functions for both aggregators it immediately follows that

$$0 < \mathbb{O}_{\oplus} < \theta < \mathbb{I}_{\oplus} < 1, \tag{14a}$$

$$0 < \mathbb{O}_{\otimes} < \vartheta < \mathbb{I}_{\otimes} < 1. \tag{14b}$$

where 0 and 1 represent Boolean false and true values that are limiting values for subjective false and subjective true, respectively.

Using the presented properties of algebra $\mathcal{A}$ in the next section we define the Markov-like process called subjective Markov process.

4 Markov AND SUBJECTIVE Markov PROCESSES

We define subjective Markov process (μP) as a Markov process (MP) in algebra $\mathcal{A}$. In order to demonstrate similarities and differences between μP and MP, we start with recalling the definition of MP and then define the μP in parallel to MP. Since we are interested in decision-making, we will consider only the discrete time processes with finite number of states that are the Markov chains.

4.1 Markov Process

Let $S = \{s_1, s_2, \dots, s_n\}$ be a finite set of some abstract states, and consider a system that in time t can be in one of the states from this set such that an exact state $s(t) \in S$ is unknown. In order to handle this uncertainty, assume that for the unknown system state $s(t)$ at time t and each state s_i from the indicated set S of abstract states there is defined the probability

$$p_i(t) = Pr\{s(t) = s_i\} \tag{15}$$

that the state $s(t)$ is equal to the state $s_i \in S$, $i = 1,2, \dots, n$.

The dynamics of the system is defined using conditional probabilities ρ_{jk}, $j, k = 1,2, \dots, n$, such that for each pair $(s_j, s_k) \in S \times S$ of abstract states probability ρ_{jk} represents the chance of transition from the state s_j to the state s_k. In the other words, if it is known that at time t the system is in the state $s(t) = s_j$, then the probability that at the next time $t + 1$ it will be in the state $s(t + 1) = s_k$ is

$$\rho_{jk} = Pr\{s(t + 1) = s_k | s(t) = s_j\}. \tag{16}$$

Starting from the initial state probabilities $p_i(0)$, $i = 1,2, \dots, n$, defined at time $t = 0$, evolution of the system is formally defined by the product of the probabilities vector $p(t) = (p_1(t), p_2(t), \dots, p_n(t))$ and the transition matrix $\rho = \|\rho_{jk}\|_{n \times n}$

$$p(t + 1) = p(t) \cdot \rho = p(0) \cdot \rho^{t+1}. \tag{17}$$

The resulting state probabilities form a basis for making decision about the action that should be conducted at time $t + 1$ and about possible rewards at this time. Repetition of such multiplication allows prediction of the system's state for some future time and correction of the decisions according to the predicted future rewards.

However, as indicated above, definition of the state probabilities is problematic; even at the initial time it requires consideration of internal and external parameters of the system that usually are not available. Correct definition of the transition probabilities, in its turn, requires deep analysis of the system and its behaviour. But if such analysis was already conducted, then the behaviour of the system is known and its probabilistic modelling becomes meaningless. Additional problem rises because of the assumption about rationality of the system's behaviour since usually the real-world systems interacting with humans follow irrational judgements based on the subjective factors.

In order to resolve these problems, we suggest to use a μP that is a Markov process in algebra $\mathcal{A}$.

4.2 Subjective Markov Process

As above, let $S = \{s_1, s_2, \dots, s_n\}$ be a finite set of abstract states, and assume that at time t the system is in the state $s(t) \in S$ that is unknown to the observer. However, for each abstract state $s_i \in S$, $i = 1,2, \dots, n$, the observer can ask the question: "Is it true that at time t the system is in the state s_i?" and can conclude that the truth level of the answer: "At time t the system is in the state s_i" is $\mu(s_i, t) = \mu_i(t) \in [0, 1]$. For convenience, we consider this truth level as an observer's belief that state $s(t)$ is s_i and denote it as

$$\mu_i(t) = Bel\{s(t) = s_i\}. \tag{18}$$

Boundary value $\mu_i(t) = 0$ means that the statement "at time t the system is in the state s_i" is false and boundary value $\mu_i(t) = 1$ means that this statement is true. In the other words, belief $\mu_i(t) = 1$ is interpreted as an exact knowledge about the occurrence of the event and belief $\mu_i(t) = 0$ is interpreted as an exact knowledge about nonoccurrence of the event. The intermediate truth values represent the grades of the observer's belief that at time t the system is in the state s_i, and belief $\mu_i(t) = 0.5$ means an absence of any knowledge whether the event occurred or not.

Denote by $\omega_{jk} \in [0, 1]$, $j, k = 1,2, \ldots, n$, the truth value, which for each pair $(s_j, s_k) \in S \times S$ of abstract states represents the belief that from the state s_j the system transits to the state s_k. In the other words, if the observer asks the question: "Is it true that the system transits from the state s_j to the state s_k?", then ω_{jk} is the truth level of the answer: "The system transits from the state s_j to the state s_k". In the other interpretation the value ω_{jk} can be considered as a possibility of transition from the state s_j to the state s_k. Such interpretation allows application of ω_{jk} in the analysis of *coincidentia oppositorium* (Rybalov and Kagan, 2017; Rybalov and Kagan, 2018) and consider it as a truth value of the statement that the system is both in the state s_j and in the state s_k (that happens when the system is transiting from s_j to s_k: at some moment it is both in s_j and in s_k, or neither in s_j nor in s_k).

Dynamics of the system is defined in the algebra $\mathcal{A}$ as follows. Let $\mu(t) = (\mu_1(t), \mu_2(t), \ldots, \mu_n(t))$ be a vector of the states' $s_i \in S$ truth values, $i = 1,2, \ldots, n$, and by $\omega = \|\omega_{jk}\|_{n\times n}$ a matrix of transition possibilities ω_{jk}, $j, k = 1,2, \ldots, n$. Then, in parallel to MP, the update of the state truth values is specified as

$$\mu(t+1) = \mu(t) \otimes_\vartheta \omega. \tag{19}$$

where the product of vector $\mu(t)$ and matrix ω in the algebra $\mathcal{A}$ is defined by application of the aggregators $\oplus_\theta$ and $\otimes_\vartheta$ following usual "the row to the column" rule: for each $i = 1,2, \ldots, n$

$$\begin{aligned}\mu_i(t+1) = (\mu_1(t) \otimes_\vartheta \omega_{1i})& \\ \oplus_\theta (\mu_2(t) \otimes_\vartheta \omega_{2i}) \oplus_\theta \ldots& \\ \oplus_\theta (\mu_n(t) \otimes_\vartheta \omega_{ni}).&\end{aligned} \tag{20}$$

This process considers behaviour of the system from the observer's point of view, and by changing the values of neutral θ and absorbing ϑ elements the observer's beliefs and preferences can be tuned. For decision-making, the values $\mu_i(t)$, $i = 1,2, \ldots, n$, can be used either directly (such as in the example in section 4) or can be transformed into the states probabilities using the means of possibility theory (Dubois and Prade, 1985) or of probabilistic logic (Nilsson, 1986; Kagan, Rybalov and Yager, 2014).

4.3 Basic Types of the States in Subjective Markov Process

Using the properties of algebra $\mathcal{A}$ we can consider the basic types of the μP states. In parallel to usual MP, the states of μP are classified according to the corresponding beliefs and transition possibilities that allow prediction of possible states and beliefs of the observer.

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of states, and denote by $s(t) \in S$ the state of the system at time t. Then, in parallel to the probabilities that characterize MP, for μP we introduce the following beliefs and possibilities:

– the first passage belief

$$\beta_{ij}^{(l)} = Bel\left\{\begin{matrix} s(t+l) = s_j, \\ s(t+m) \neq s_j, 0 < m < l \\ \mid s(t) = s_i \end{matrix}\right\} \tag{21}$$

is a belief that if at time t the system is in the state s_i, then at first time it will be in the state s_j in l steps, $i, j = 1,2, \ldots, n$.

– the l-step transition belief

$$\psi_{jk}^{(l)} = Bel\{s(t+l) = s_k \mid s(t) = s_j\} \tag{22}$$

is a belief that if at time t the system is in the state s_j, then it will reach the state s_k, $j, k = 1,2, \ldots, n$, in exactly l steps.

It is clear that by definition, 1-step transition belief is equivalent to the transition possibility that is

$$\begin{aligned}\psi_{jk}^{(1)} &= \omega_{jk} \\ &= Bel\{s(t+1) = s_k \mid s(t) = s_j\}.\end{aligned} \tag{23}$$

Following usual notation, denote by

$$\beta_{ij} = \oplus_{\theta}{}_{l=1}^{\infty} \beta_{ij}^{(l)} \tag{24}$$

the belief that starting from the state s_i in some time the system will reach the state s_j. Then, we say that the state s_i

– is believed to be persistent (or recurrent) if $\mathbb{I}_\oplus \leq \beta_{ii} \leq 1$,

– is believed to be transient if $\mathbb{O}_\oplus < \beta_{ii} < \mathbb{I}_\oplus$, and

– is believed to be separate (non-persistent and non-transient or non-recurrent and non-transient) if $0 \leq \beta_{ii} \leq \mathbb{O}_\oplus$.

It means that the state is persistent if the observer highly believes that the system will sooner or later return to this state, is transient if the observer's belief about return to this state is low, and is separate if the observer highly beliefs that the system will never return to this state. In addition, notice that, in contrast to MP, in μP belief β_{ii} can change its value such that persistent state will become separate state and backwards. Such state is called oscillating state and represents the observer's hesitations regarding this state. Finally, the state s_i

– is believed to be periodic (with period T) if these exists an integer number T such that if $l = kT$, $k = 1, 2, 3, \ldots$, then $\psi_{ii}^{(l)} \neq \vartheta$, otherwise $\psi_{ii}^{(l)} = \vartheta$.

Formula "the state is believed to be…" is used for avoiding unambiguousness and stresses that the values $\beta_{ij}^{(l)}$, β_{ij} and $\psi_{jk}^{(l)}$ are beliefs and makes sense only in the context of observer's knowledge; also it allows distinguishing the values and states used in μP from the probabilities and corresponding states used in MP.

Relation between the first passage belief $\beta_{ij}^{(l)}$ and the l-steps transition belief $\psi_{jk}^{(l)}$ is similar to the relation between first passage and l steps transition probabilities in MP and is defined as follows:

$$\psi_{ij}^{(l)} = \oplus_{\theta}{}_{r=1}^{l} \left(\beta_{ij}^{(r)} \otimes_{\vartheta} \psi_{jj}^{(l-r)}\right), \tag{25}$$

where $\beta_{ij}^{(0)} = \mathbb{O}_{\otimes}$ (it is believed that in zero steps the system will not move from the state i to the state j, $j \neq i$), $\psi_{jj}^{(0)} = \mathbb{I}_{\otimes}$ (it is believed that during time unit the system will stay in its current state), $\psi_{ij}^{(0)} = \mathbb{O}_{\otimes}$ (it is believed that in zero steps the system will not move from the state i to the state j, $j \neq i$), and $\beta_{ij}^{(1)} = \omega_{ij}$ (belief that starting from state i the system will reach state j at first time in one step is equivalent to the possibility of transition from state i to state j), $i, j, k = 1, 2, \ldots, n$.

Relation between first passage belief $\beta_{ij}^{(l)}$ and l-steps transition belief $\psi_{jk}^{(l)}$ for persistent, transient and separate states in μP is the following. The state s_i

– is believed to be persistent (or recurrent) if and only if there are no any hesitations about the possibility of return to the state, that is

$$\oplus_{\theta}{}_{l=0}^{\infty} \psi_{ii}^{(l)} = 1; \tag{26}$$

– is believed to be transient if and only if there exists some possibility of return but this possibility is not exact, that is

$$0 < \left(\oplus_{\theta}{}_{l=0}^{\infty} \psi_{ii}^{(l)}\right) < 1; \tag{27}$$

– is believed to be separate (non-persistent and non-transient or non-recurrent and non-transient) if and only if it is exactly known that there is no any possibility to return to this state, that is

$$\oplus_{\theta}{}_{l=0}^{\infty} \psi_{ii}^{(l)} = 0. \tag{28}$$

The proofs of these propositions are based on direct application of the monotonicity of the uninorn and of the convergence of its results to 0 or 1 for the terms less than or greater than θ, respectively. The other way to prove these propositions is based on the application of the function τ_{θ} and its reverse that allows consideration of the propositions in the interval $[-1,1]$ with usual arithmetic operations (together with normalization of sum).

The formulated properties of the states in μP go in parallel to the properties of the states proven for MP (Feller, 1970). However, it is seen that both the meaning and formal characteristics of these states are different.

In order to stress this difference and to illustrate the actions of μP in the next section we consider the simple model of search (Pollock, 1970; Kagan and Ben-Gal, 2013) using both models.

5 SIMPLE MODEL OF SEARCH WITH MP AND μP

We clarify the actions of μP and the difference between MP and μP by running example of classical Pollock model of search. In this model, the target moves between two boxes and the observer should catch the target by checking one of the boxes: if the target is in the chosen box, then the search terminates and if not, then the search continues (Pollock, 1970; Kagan and Ben-Gal, 2013).

Below we do not address the optimization issues and do not compare MP and μP from this point of view; our goal is only to demonstrate that μP provides additional information about the considered system and can lead to decisions that differ from the decisions led by MP.

Assume that the set $S = \{s_1, s_2\}$ includes only two states that are associated with the boxes. At each time $t = 0, 1, 2, \ldots$, the target can be in one of the boxes s_1 and s_2 with the probabilities $(i = 1, 2)$

$$\begin{aligned} p_i(t) &= Pr\{s(t) = s_i\}, \\ p_1(t) &= 1 - p_2(t), \end{aligned} \tag{29}$$

these probabilities are called location probabilities.

The chances of movements between the boxes s_1 and s_2 are defined by the transition probabilities $(j, k = 1, 2)$

$$\rho_{jk} = Pr\{s(t+1) = s_k | s(t) = s_j\},$$
$$\rho_{12} + \rho_{11} = 1, \rho_{21} + \rho_{22} = 1, \quad (30)$$

where the probabilities ρ_{11} and ρ_{22} represent the chances that the target stays in its current box 1 or 2, respectively.

Dynamics of the target is governed by MP as follows

$$\big(p_1(t+1), p_2(t+1)\big) = \big(p_1(t), p_2(t)\big) \cdot \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}. \quad (31)$$

On the base of probabilities $p_1(t+1)$ and $p_2(t+1)$ the searcher decides which box should be checked at time $t+1$: if the target is found, then the search is terminated, and if it is not found, then the location probabilities are updated (for the checked box it is set to zero and to the other box – to one) and the search continues. The one-step decision is clear and prescribes to choose the box with maximal location probability; however, since the unsuccessful decision leads to the probabilities update, the long-term decision-making is rather nontrivial problem.

Now let us describe the process using the suggested μP. For the same set $S = \{s_1, s_2\}$ of states, let

$$\mu_i(t) = Bel\{s(t) = s_i\}, \text{i} = 1, 2, \quad (32)$$

be truth values that represent the beliefs of the searcher that the target is located in the boxes s_1 and s_2; for briefness we call these values the location beliefs. The beliefs that the target can move from one box to another are represented by the transition possibilities $(j, k = 1, 2)$

$$\omega_{jk} = Bel\{s(t+1) = s_k | s(t) = s_j\}, \quad (33)$$

where the values ω_{11} and ω_{22} represent the beliefs that the target stays in its current box 1 or 2, respectively.

Then, the system is described directly from the searcher's point of view as a search process that is governed by μP such that

$$\big(\mu_1(t+1), \mu_2(t+1)\big) = \big(\mu_1(t), \mu_2(t)\big) \otimes_\vartheta \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}. \quad (34)$$

Similar to MP, in the obtained μP the searcher decides which box should be checked at time $t+1$ following location beliefs $\mu_1(t+1)$ and $\mu_2(t+1)$. As indicated above, it can be done either directly or by the means of possibility theory or of the probabilistic logic.

However, the values of the beliefs obtained in the μP, in general, differ from the values of the location probabilities and lead to the decision that differs from the decision made in the MP. The long-term decision-making also differs; since the beliefs represent the observer's subjective point of view, they do not updated and the next step beliefs are calculated using the current beliefs with no concern to the observation result.

In order to illustrate the difference between MP and μP, we implement the distributive version of algebra $\mathcal{A}$ with the aggregators $\oplus_\theta$ and $\otimes_\vartheta$ defined by equations (5) with equivalent generator functions $u = v = w$, where w is the inverse of Cauchy distribution

$$w(x) = m + \alpha \tan\left[\pi\left(x - \tfrac{1}{2}\right)\right]. \quad (35)$$

Then

$$w^{-1}(\xi) = \tfrac{1}{2} + \tfrac{1}{\pi}\arctan\left(\tfrac{\xi - m}{\alpha}\right), \quad (36)$$

where $\xi, m \in (-\infty, \infty)$ and $\alpha > 0$. From the requirement $w(\theta) = w(\vartheta) = 0$ it also follows that

$$\theta = \vartheta = \tfrac{1}{2} - \tfrac{1}{\pi}\arctan\left(\tfrac{m}{\alpha}\right), \quad (37)$$

In addition, we assume that parameters of the distribution are $m = 0$ and $\alpha = 1$; thus $\theta = \vartheta = \frac{1}{2}$.

Consider the first step of the process. For convenience, we assume that the initial values of the location probabilities and location beliefs are equal and are

$$\mu(0) = p(0) = (0.8, 0.2), \quad (38)$$

Direct calculations result in the following. Let transition matrices (transition beliefs and transition probabilities) be

$$\omega = \rho = \begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}. \quad (39)$$

Then

$$p(1) = (0.44, 0.56),$$
$$\mu(1) = (0.27, 0.73). \quad (40)$$

It is seen that the probability $p_1(1) = 0.44$ that the target will be at the first box is smaller than the probability $p_2(1) = 0.56$ that it will be in the second box and the same is true for the beliefs that are $\mu_1(1) = 0.27$ and $\mu_2(1) = 0.73$. Then, in both cases the searcher should check box 2. However, the difference between the probabilities $p_1(1)$ and $p_2(1)$ is essentially smaller than the difference between the beliefs $\mu_1(1)$ and $\mu_2(1)$ that is the belief that the target is in the box 2 is greater than the probability that it is there.

The further iterations of the processes demonstrate that in the MP the relation $\mathrm{p}_1(\mathrm{t}) < \mathrm{p}_2(\mathrm{t})$ remains until reaching the steady state $\mathrm{p}_1(\mathrm{t}) = \mathrm{p}_2(\mathrm{t}) = 0.5$. In the μP, in contrast, each iteration changes the relation between $\mu_1(t)$ and $\mu_2(t)$ such that $\mu_1(0) < \mu_2(0)$, $\mu_1(1) > \mu_2(1)$, $\mu_1(2) < \mu_2(2)$, $\mu_1(3) > \mu_2(3)$, … up to reaching the steady state $\mu_1(t) = \mu_2(t) = 0.5$.

Thus, in the MP the searcher should all times check box 2, while in the μP the searcher should change the checked box at each step.

Now assume that transition matrices (transition beliefs and transition probabilities) are

$$\omega = \rho = \begin{pmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \end{pmatrix}. \quad (41)$$

Then the picture essentially changes and

$$p(1) = (0.38, 0.62), \\ \mu(1) = (0.66, 0.34). \quad (42)$$

Here location probability $p_1(1) = 0.38$ for the first box is again smaller than the location probability $p_2(1) = 0.62$ for the second box, but the belief $\mu_1(1) = 0.66$ that the target will be in the first box is greater that the belief $\mu_2(1) = 0.34$ that it will be in the second box. Consequently, in the first case the searcher should check box 2, but in the second case – box 1.

In the further iterations both relations $p_1(t) < p_2(t)$ and $\mu_1(t) > \mu_2(t)$ remain until reaching the steady states $p_1(t) = \frac{1}{3}$, $p_2(t) = \frac{2}{3}$ and $\mu_1(t) = \mu_2(t) = 0.5$. This state in the MP prescribes to continue checking box 2 and in the μP it prescribes to choose the box by random.

In addition notice that in both cases of transition matrices in the MP the steady state is reached faster than in the μP, thus the μP provides more information for making decision about the box for check.

It is clear that the considered model is the simplest one and is used only as an example. However, even such simple model stresses the difference between the MP and μP and demonstrates that the decisions made in μP can differ from the decisions made in MP.

More complex processes and decisions are obtained by the use of non-distributive version of the algebra $\mathcal{A}$, where in the aggregators $\oplus_\theta$ and $\otimes_\vartheta$ the elements θ and ϑ differ or even differ generation functions u and v; but these issues we remain for further research.

6 CONCLUSIONS

The suggested subjective Markov process (μP) goes in parallel to the usual Markov process (MP), but, in contrast to MP, it acts in the recently constructed algebra $\mathcal{A}$ that implements uninorm and absorbing norm aggregators and combines logical and arithmetical operations.

The values, with which μP deals, are considered as observer's beliefs about the system's states and can be associated with the grades of membership or with possibilities of the system to be in certain states. Such definition allows to use the suggested μP instead or in parallel to the MP for analysis of the systems that include rare events or follow subjective irrational decisions.

For the suggested process, we considered the basic types of the states with respect to the transition beliefs that specify the possibilities of transitions among the states. The essential role in this consideration play recently introduced concepts of subjective false and subjective true that allow precise and meaningful classification of the states.

The difference between the suggested μP and usual MP is illustrated by running example of the Pollock model of search. It was shown that even in such simple model (with maximization of the probabilities of finding the target) μP provides additional information and leads to the decisions that can differ from the decisions prescribed by MP.

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