Some Geometric Objects Related to a Family of the Ballistic Trajectories in a Viscous Medium

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Abstract: Computer geometric modeling is important pre-processing steps in the object's mathematical representation using curves that may be constructed using analytic functions, a set of points, or other curves and surfaces. The paper describes some remarkable curves related to a family of the ballistic trajectories in a viscous medium with a linear resistance. The envelope of the family of trajectories, the trajectory of the farthest flight and the curve of maximum flight altitudes are presented in parametric form. A geometric interpretation of the entire set of ballistic trajectories in the form of some surface (the Galileo's dome) is also presented.

1 INTRODUCTION

Some classical problems of applied mathematics and mechanics seem inexhaustible. Each appeal to them reveals some new facets, highlighting the existence of hidden connections between various areas of mathematics. Galileo's problem about the motion of a body thrown at some angle to the horizon was the first solved problem of dynamics. It was solved by Galileo long before the appearance of the Newtonian mechanics. The solution is given in his last book "Discorsi e Dimostrazioni Matematiche Intorno a Due Nuove Scienze", published in Leiden in 1638. This book was translated from Italian and Latin into English by Henry Crew and Alfonso de Salvio in 1914. Now this translation is available in the Online Library of Liberty (Galilei, 1914).

"Fourth Day: The motion of projectiles" is the chapter title of (Galilei, 1914) treating the problem in the delightful and convincing language of geometry. This language of the era, perhaps, will seem somewhat heavy to the modern reader. But the epoch had no other language. Neither Newton's laws of mechanics nor differential equations existed.

This problem is a traditional and simple task, with which the study of mechanics and physics often begins. The design of the geometric modeling is widely used in Computational Fluid Dynamics (CFD) simulations. Simple and efficient geometric modeling can improve the efficiency of flow field simulations for various applications. Some of the applications described in (Bertin, 2017; Zhou et al., 2017; Ma et al., 2019).

We will consider in this paper some new geometric objects related to this problem.

In the paper (Seidametova and Temnenko, 2020) we considered the simplest Galilean version of this ballistic problem, assuming that only gravity acts on the flying object. In this paper we examined the ballistic problem in a viscous environment. We will assume that, in addition to gravity, a viscous resistance force $\vec{\mathbb{F}}_R$ acts on the flying object, which is linearly dependent on the speed of movement $\vec{v}$:

$$\vec{\mathbb{F}}_R = -b\vec{v}. \quad (1)$$

The constant b characterizes the resistance of the medium. For a physical object at low Reynolds numbers, the value b is determined by the well-known G. G. Stokes formula (Landau and Lifshitz, 1987):

$$b = 6\pi a \rho_m \nu_m, \quad (2)$$

where a is a sphere radius, ρ_m is a density of the medium, ν_m is a kinematic viscosity of the medium.

We take the value of the initial speed of the thrown body v_0 as a velocity unit, the acceleration of gravity g as an acceleration unit. With this choice, the unit of time is $\frac{v_0}{g}$, and the unit of length is $\frac{v_0^2}{g}$. Let t be the time, x the horizontal coordinate, y the vertical

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coordinate (we assume that $y \geq 0$), α is the angle that the initial speed vector makes up with the horizontal line ($0 \leq \alpha \leq \pi/2$).

2 FORMULATION OF THE PROBLEM

Newton's equations of motion are:

$$\begin{aligned} \dot{v}_x &= -\beta v_x, \\ \dot{v}_y &= -1 - \beta v_y. \end{aligned} \tag{3}$$

$$\begin{aligned} \dot{x} &= v_x, \\ \dot{y} &= v_y. \end{aligned} \tag{4}$$

Here the dot above the letter denotes the time derivative, v_x, v_y are the Cartesian components of the velocity $\vec{v}$; β is the dimensionless parameter characterizing the resistance of the medium:

$$\beta = \frac{bv_0}{mg}. \tag{5}$$

where m is the mass of a flying object.

If we assume that the flying object is a homogeneous sphere of radius a and density ρ_b, then, taking into account the Stokes formula (2), the coefficients of viscous resistance β can be given the following form

$$\beta = \left(\frac{9}{2}\frac{\rho_m}{\rho_b}\right) \cdot \frac{\nu_m v_0}{ga^2}. \tag{6}$$

In order for the equations of motion (3) to adequately describe the trajectory, two conditions must be met:

1. The size of the flying body should be much smaller than the characteristic dimensions of the flight path:
$$\frac{v_0^2}{ag} \gg 1. \tag{7}$$
2. The Reynold's number should be small enough
$$Re = \frac{v_0 \cdot a}{\nu_m} \ll 1. \tag{8}$$

Inequalities (7) and (8) limit the initial velocity from above and below:

$$\sqrt{ag} \ll v_0 \ll \frac{\nu_m}{a}. \tag{9}$$

For these constraints to be compatible, the object must be small enough:

$$a \ll \left(\nu_m^2/g\right)^{1/3}. \tag{10}$$

To prevent inequality (10) from being too burdensome, experiments with a flying object should be carried out in a medium with a high viscosity, for example, in glycerin.

The equations of motion (3) and (4) are supplemented by the initial conditions at $t = 0$:

$$\begin{aligned} v_x(t=0) &= \cos\alpha, \\ v_y(t=0) &= \sin\alpha, \end{aligned} \tag{11}$$

and

$$\begin{aligned} x(t=0) &= 0, \\ y(t=0) &= 0. \end{aligned} \tag{12}$$

In the equations of motion (11) α is the departure angle (the angle that makes the body's velocity vector with the axis x at the initial moment). The angle α obeys the condition:

$$0 < \alpha \leq \frac{\pi}{2}. \tag{13}$$

The formulated problem contains one physical parameter β and one geometric parameter α. Changes of α in region (13) at fixed β generates a family of ballistic trajectories. We investigate in this paper how resistance β affects the appearance of a family of trajectories. We considered the trajectories at $y \geq 0$, from the moment of departure of the object to its fall.

Of particular interest are three curves generated by the family of trajectories: the envelope of the family of trajectories, the trajectory of the farthest flight, and the locus of the points of maximum flight altitude when the departure angle changes. In (Seidametova and Temnenko, 2020) a new composite remarkable curve was constructed from these three curves, which we called Galileo's poleaxe. We will look at how the parameter β affects these wonderful curves.

3 TRAJECTORIES OF MOVEMENT

The solutions of the differential equations of motion (3), (4) with the initial conditions (11), (12) have the following form:

$$\begin{aligned} v_x &= \cos\alpha \cdot e^{-\beta t}, \\ v_y &= \frac{1}{\beta}\left((1+\beta\sin\alpha)\, e^{-\beta t} - 1\right). \end{aligned} \tag{14}$$

$$\begin{aligned} x &= \frac{\cos\alpha}{\beta}\left(1 - e^{-\beta t}\right), \\ y &= (1/\beta^2)\left((1+\beta\sin\alpha)\left(1 - e^{-\beta t}\right) - \beta t\right). \end{aligned} \tag{15}$$

Eliminating time t from (15), we can obtain an explicit equation for the family of ballistic trajectories in a medium with linear viscous resistance:

$$y = \frac{1}{\beta^2}\left((1+\beta\sin\alpha)\frac{\beta x}{\cos\alpha} + \ln\left(1 - \frac{\beta x}{\cos\alpha}\right)\right). \tag{16}$$

4 THE LOCUS OF THE MAXIMUM LIFTING HEIGHTS OF THE TRAJECTORIES

At the point of maximum rise of the flying body, the following condition is met:

$$v_y = 0. \tag{17}$$

Substituting into (17) the expression for v_y from (14), we find the flight time t_m to this point:

$$t_m = \frac{1}{\beta}\ln(1+\beta\sin\alpha). \tag{18}$$

Substituting the value t_m into the equations of motion (18), we obtain the equations for the geometric maximum rise of the trajectory:

$$\begin{aligned} x &= \frac{1}{2}\cdot\frac{\sin 2\alpha}{1+\beta\sin\alpha}, \\ y &= \frac{1}{\beta^2}(\beta\sin\alpha - \ln(1+\beta\sin\alpha)). \end{aligned} \tag{19}$$

Relations (19) in a parametric form define the curve of maximum heights. Figure 1 shows curves (18) at some values β.

For $\beta \to 0$ equation (19) yields the equations of the maximum height curve in the absence of medium resistance:

$$x = \frac{1}{2}\sin 2\alpha,$$
$$y = \frac{1}{2}\sin^2\alpha.$$

These equations were given in the paper (Seidametova and Temnenko, 2020). These equations describe the semi-ellipse:

$$\left(\frac{x}{1/2}\right)^2 + \left(\frac{y-1/4}{1/4}\right)^2 = 1.$$
$$(x \geq 0; y \geq 0).$$

5 THE ENVELOPE FOR A BALLISTIC TRAJECTORY FAMILY

The envelope of the family of ballistic trajectories (16) satisfies the equations of motion (15) and the condition for the vanishing of the Jacobian $\frac{D(x,y)}{D(t,\alpha)}$:

$$\frac{D(x,y)}{D(t,\alpha)} = \begin{vmatrix} \dot{x} & \dot{y} \\ \frac{\partial x}{\partial\alpha} & \frac{\partial y}{\partial\alpha} \end{vmatrix} = 0 \tag{20}$$

Relation (20) can be given the form:

$$v_x\frac{\partial y}{\partial\alpha} - v_y\frac{\partial x}{\partial\alpha} = 0. \tag{21}$$

Calculating the derivatives by (15) $\frac{\partial x}{\partial\alpha}$ and $\frac{\partial y}{\partial\alpha}$ and substituting this into (21), we obtain a relation connecting the departure angle α and the time t at which the trajectory touches the envelope:

$$e^{-\beta t} = \frac{\sin\alpha}{\beta+\sin\alpha}. \tag{22}$$

Substitute (22) into equation (15) generates the envelope equation in parametric form:

$$\begin{aligned} x &= \frac{\cos\alpha}{\beta+\sin\alpha}, \\ y &= \frac{1}{\beta^2}\left(\frac{\beta(1+\beta\sin\alpha)}{\beta+\sin\alpha} + \ln\left(\frac{\sin\alpha}{\beta+\sin\alpha}\right)\right). \end{aligned} \tag{23}$$

Since we considered only trajectories with $y \geq 0$, equations (23) describe the section of the envelope with $y \geq 0$ for values α of the parameter satisfying the inequalities:

$$\alpha_m \leq \alpha \leq \frac{\pi}{2}. \tag{24}$$

where α_m is the departure angle corresponding to the trajectory of the maximum flight range. Figure 2 shows the envelope of ballistic trajectories at some values of β.

6 FLIGHT DISTANCE AND THE FOLIUM OF GALILEO

The flight range l is the value of the horizontal coordinate x when the vertical coordinate y vanishes. Denote t_f the flight time of the object before falling. We also introduce the notation:

$$\tau = \beta t_f. \tag{25}$$

Assuming in (15) $y = 0$ we establish a relationship between the departure angle α and the total flight time t_f:

$$\sin\alpha = \frac{1}{\beta}\frac{\tau-(1-e^{-\tau})}{1-e^{-\tau}}. \tag{26}$$

Assuming in (15) $t = t_f$ and substituting t_f into the expression for the coordinate x, we find the flight range l:

$$l = \frac{1}{\beta}\sqrt{1-\left(\frac{1}{\beta}\cdot\frac{\tau-(1-e^{-\tau})}{1-e^{-\tau}}\right)^2}\cdot\left(1-e^{-\tau}\right). \tag{27}$$

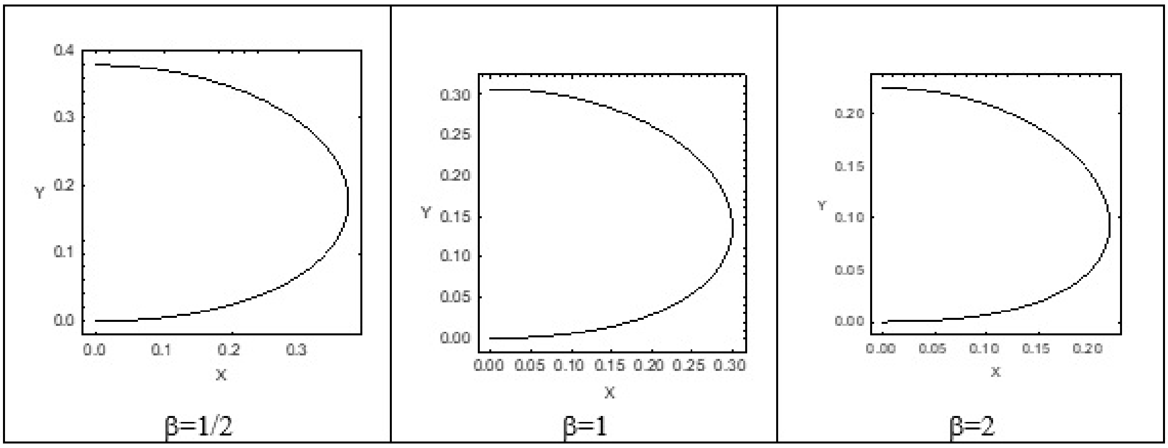

Figure 1: Curve of maximum heights of ballistic trajectories at a given β.

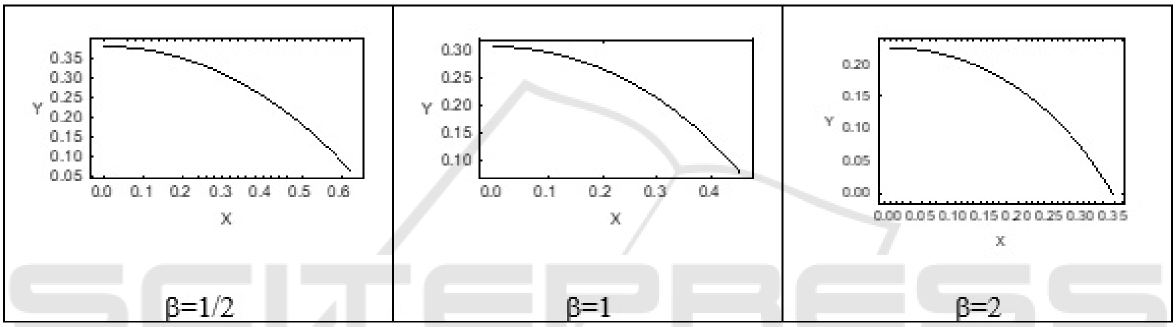

Figure 2: The envelope of the family of ballistic trajectories for some β.

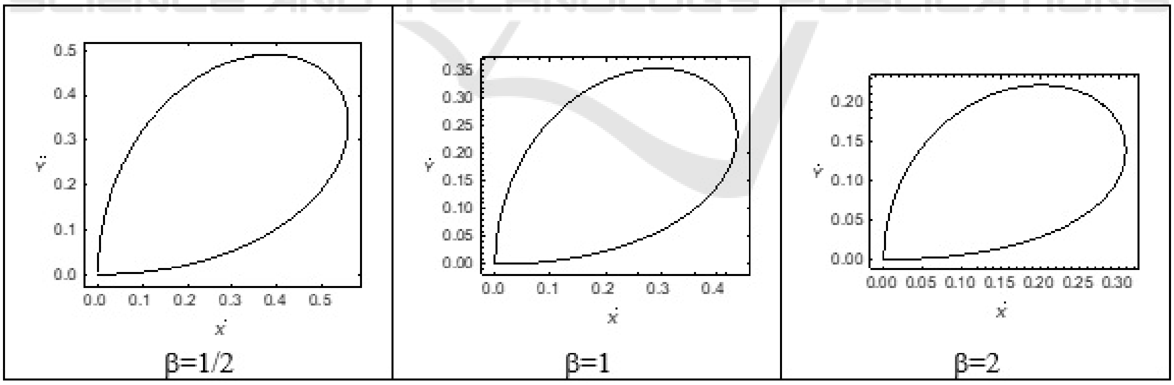

Figure 3: The folium of Galileo at some values of the dimensionless parameter of viscous resistance β.

Equations (27) together with the relation arising from (26):

$$\alpha = \arcsin\left(\frac{1}{\beta}\cdot\frac{\tau-(1-e^{-\tau})}{1-e^{-\tau}}\right), \qquad (28)$$

define in a parametric form the dependence of the flight range l on the departure angle α. The parameter of this curve is the value τ.

Figure 3 shows the dependence $l = l(\alpha)$ at some β. As suggested in (Seidametova and Temnenko, 2020), this dependence is constructed in the form of a polar diagram, which we called "The folium of Galileo". The flight range l is interpreted as a radial coordinate in polar coordinates, and the angle α is interpreted as an azimuthal angle in polar coordinates.

When constructing figure 3, it should be noted that

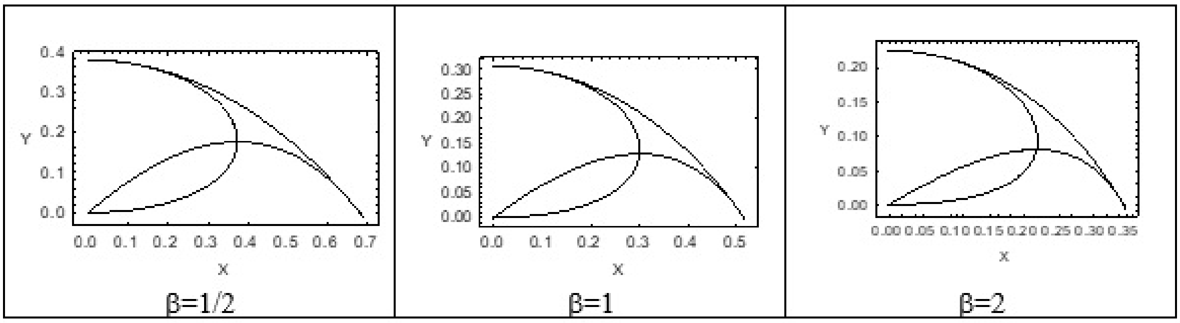

Figure 4: Galileo's Poleaxe for a ballistic problem with viscous resistance at some values of the resistance parameter β.

the parameter τ is bounded from above:

$$\tau \leq \tau_*, \tag{29}$$

where τ_* is the solution to the equation:

$$F(\tau) = \frac{\tau}{1 - e^{-\tau}} = 1 + \beta. \tag{30}$$

determined by (30) for a given β, we build the folium of Galileo (27), (28) on the interval of change τ:

$$0 \leq \tau \leq \tau_*. \tag{31}$$

In the figure 3 $\tilde{x}$ and $\tilde{y}$ some conditional cartesian coordinates

$$\tilde{x} = l(\alpha) \cdot \cos\alpha,$$
$$\tilde{y} = l(\alpha) \cdot \sin\alpha.$$

7 GALILEO'S POLEAXE

Knowing the envelope of the family of ballistic curves (23) and the curve of maximum altitudes (19), as well as adding to these curves the trajectory of the farthest flight, we can build a composite curve – Galileo's Poleaxe (figure 4).

When constructing the trajectory of the farthest flight, it is necessary using curve from figure 3, to set the angle α_{max} corresponding to the farthest flight and substitute this value of the angle α into equation (15).

8 GALILEO'S DOME

If in the equation of a one-parameter family of the ballistic trajectories (16) we reinterpret the triple (x, y, α) as a triplet of cylindrical coordinates (ρ, z, φ): $x \equiv \rho$; $y \equiv z$; $\alpha \equiv \varphi$, then the equation of the *family of curves* (16) turns into the equation of *one surface* given explicitly in cylindrical coordinates $z = z(\rho, \varphi)$:

$$z = \frac{1}{\beta^2}\left((1 + \beta\sin\varphi)\frac{\beta\rho}{\cos\varphi} + \ln\left(1 - \frac{\beta\rho}{\cos\varphi}\right)\right). \tag{32}$$

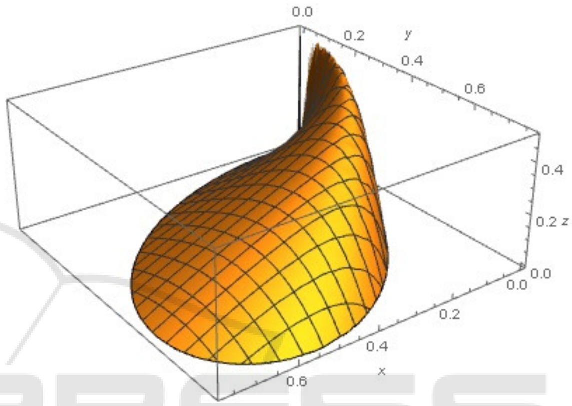

Figure 5: The Galileo's dome for $\beta = 0$.

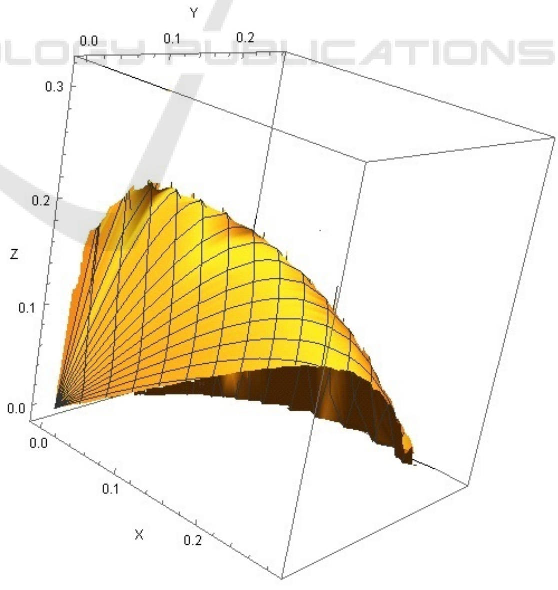

Figure 6: The Galileo's dome for $\beta = 2$.

It is assumed here that the polar coordinates (ρ, φ) are given in some auxiliary plane $(\widetilde{x}, \widetilde{y})$:

$$\widetilde{x} = \rho\cos\varphi; \ \widetilde{y} = \rho\sin\varphi.$$

Equation (32) describes (for $z \geq 0$ and

$0 \leq \varphi \leq \frac{\pi}{2}$) a certain surface (figure 5), which we call "Galileo's dome". Galileo's dome provides a visual representation of the entire set of ballistic trajectories as some whole geometric object (figure 6).

9 CONCLUSIONS

The paper presents a solution to the problem of a family of ballistic trajectories in a medium with linear viscous resistance. The equations of the envelope of the family of trajectories and the equation of the curve of the highest elevation of the trajectory are presented in a parametric form. The polar diagram of the flight range is presented in parametric form. The paper also presents a geometric interpretation of the entire set of ballistic trajectories in the form of the some surface – the Galileo's dome.

REFERENCES

Bertin, D. (2017). 3-d ballistic transport of ellipsoidal volcanic projectiles considering horizontal wind field and variable shape-dependent drag coefficients. *Journal of Geophysical Research: Solid Earth*, 122(2):1126–1151.

Galilei, G. (1914). *Dialogues concerning two new sciences*. Macmillan, New York. https://oll.libertyfund.org/title/galilei-dialogues-concerning-two-new-sciences.

Landau, L. D. and Lifshitz, E. M. (1987). *Fluid Mechanics*, volume 6 of *Course of Theoretical Physics*. Pergamon Press, 2nd edition. https://users-phys.au.dk/srf/hydro/Landau+Lifschitz.pdf.

Ma, L., Li, T., Liu, Z., Shao, J., Wu, H., Price, W. G., Temarel, P., and Hudson, D. A. (2019). Geometric modeling and mesh generation by radial basis functions and their application to ship flow simulations. In *The 29th International Ocean and Polar Engineering Conference*. International Society of Offshore and Polar Engineers. http://legacy.isope.org/publications/proceedings/ISOPE/ISOPE%202019/data/polopoly_fs/1.4537852.1558362433!/fileserver/file/975372/filename/19DCW-07Ma.pdf.

Seidametova, Z. and Temnenko, V. (2020). Some Geometric Objects Related to a Classical Problem of Galileo. *The College Mathematics Journal*, 51(1):57–65.

Zhou, S., Li, T., Jin, Q., Price, W. G., Temarel, P., and Hudson, D. A. (2017). Geometric modeling over a background cartesian grid using radial basis functions. In *The 27th International Ocean and Polar Engineering Conference*. International Society of Offshore and Polar Engineers. https://onepetro.org/ISOPEIOPEC/proceedings-abstract/ISOPE17/All-ISOPE17/ISOPE-I-17-039/17166.