Interpreting Xor Intuitionistic Fuzzy Connectives from Quantum Fuzzy Computing

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Abstract: Computer systems based on intuitionistic fuzzy logic are capable of generating a reliable output even when handling inaccurate input data by applying a rule based system, even with rules that are generated with imprecision. The main contribution of this paper is to show that quantum computing can be used to extend the class of intuitionistic fuzzy sets with respect to representing intuitionistic fuzzy Xor operators. This paper describes a multi-dimensional quantum register using aggregations operators such as t-(co)norms based on quantum gates allowing the modeling and interpretation of intuitionistic fuzzy Xor operations.

1 INTRODUCTION

Fuzzy logic (FL) and its extensions as the Atanassov's fuzzy intuitionistic logic (A-IFL)(Atanassov and Gargov, 1998; Atanassov, 2017) together with quantum computing (QC) are relevant research areas consolidating the analysis and the search for new solutions for difficult problems faster than the classical logical approach by extending results from conventional computing.

Similarities between these areas in the representation and modelling of uncertainty have been explored (Pykacz, 1993; Kreinovich et al., 2009; Mannucci, 2006). The uncertainty of human being's reasoning can be modelled in A-IFL as a mathematical model inheriting the indeterminacy from membership and non-membership degrees which are not necessarily complementary. It provides techniques helping physicists and mathematicians to transform their uncertainty ideas into new computational programs (Kosheleva et al., 2015).

The uncertainty of the real world is concerned with fundamental concepts of QC by making use of properties of quantum mechanics as the superposition which suggest an improvement in the efficiency regarding complex tasks. In addition, simulations using classical computers improve the development and validation of basic quantum algorithms, anticipating the knowledge related to their behaviours when executed in a quantum computer.

In spite of quantum computers being restricted to a few research centers and laboratories, the studies from quantum information and quantum computation are a reality nowadays.

In this context, new methods dealing with fuzzy approaches to quantum computational logics have been proposed (Chiara et al., 2018). Moreover, it contributes to increase the interest in quantum algorithm applications representing fuzzy and intuitionistic fuzzy systems.

1.1 Synergy between FL and QC

Potentialities from quantum parallelism and superposition of quantum states are explored by new methodologies. See, e.g. providing a model for humanoid behaviours based on FL (Raghuvanshi and Perkowski, 2010) and using a density operator and logical connectives defined by quantum gates (Bertini and Leporini, 2017). In other areas, QC is used to improve computations as Econometric Modeling (Sriboonchitta et al., 2019) and to integrate concepts from quantum genetic algorithm and fuzzy neural network (Li et al., 2010) as a model employed to control an inverted pendulum system.

From a logical perspective, many-valued fuzzy approach is a consolidated research area (Hájek, 1998) and recent efforts integrating the quantum computational logic are considered (Chiara et al., 2018). See, e.g. (Freytes et al., 2010) making use of quantum states to represent information processing in QC.

In (Rigatos and Tzafestas, 2002), a parallel control fuzzy algorithm is proposed showing that both QC and fuzzy learning algorithms are relevant research

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areas which can profit from each other.

In this work, the modelling and interpretation of A-IFL via QC provides the description of intuitionistic fuzzy connectives by using quantum registers and quantum transformations from the traditional model of quantum circuits.

The information regarding each intuitionistic fuzzy set is represented by pairs of quantum registers, guaranteeing the inherent unitarity of quantum states and quantum transformations well as the flexibility of the complementarity relation of membership and non-membership functions characterizing intuitionistic fuzzy sets.

Some results of such research area are mainly related to interpretation of fuzzy connectives via QC, as negation, conjunction, disjunction, implications and xor operators (Avila et al., 2015) and to intuitionistic fuzzy connectives (Reiser et al., 2016).

This work extends that approach by the interpretation of two classes of intuitionistic fuzzy Xor connectives via QC, the $\otimes_I$ and $\oplus_I$ intuitionistic fuzzy Xor operators, which can be expressed as composition of conjunctions (S_I), disjunctions (T_I) and the complementary operators (N_I). But different from other representable intuitionistic fuzzy connectives, an intuitionistic fuzzy Xor connective cannot be expressed as an N-dual pair of Xor and EXor fuzzy connectives.

So, quantum states and quantum operators provide interpretation for intuitinistic fuzzy values and connectives, respectively. In particular, Xor and EXor can be applied to explore regular applications based on symmetric functions by making use of reversible logic (He et al., 2017) and can be also extended to QC (Perkowski et al., 2001).

1.2 Paper Outline

The remainder of this paper is organized as follows. Section 2 presents the foundations on A-IFL. Section 3 brings the main concepts of QC. In Section 4, the study and modeling of intuitionistic fuzzy Xors using QC is described. An interpretation of classical A-IFS (Atanassov, 2016; Mannucci, 2006) from quantum states is also considered, presenting the operations on A-IFS modelled from quantum transformations, relating the Atanassov's intuitionistic fuzzy approach(Atanassov, 1986) to two classes of Xor operators and also considering representable Xor connectives obtained by composition of standard negation together with the Product and Algebraic Sum. Finally, conclusions and further work are discussed in Section 5.

2 FUZZY LOGIC APPROACH

The A-IFL is a type-2 fuzzy logic conceived as a generalization of FL overcoming the limitations related to fuzzy sets for dealing with problems where the rules applied to the system could not be defined with precision, mainly related to non-membership degree which cannot be defined as a complement of its membership degree.

An element $x \in \mathcal{X}$ belongs to the subset A such that $0 \leq \mu_A(x) + \nu_A \leq 1$, meaning that a non-membership degree $\nu_A(x)$ is not necessary the complement of its membership degree $\mu_A(x)$. Thus, A is given as:

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in \mathcal{X}, 0 \leq \mu_A(x) + \nu_A \leq 1\} \quad (1)$$

Taking $\mu_A(x) = x_1, \nu_A(x) = x_2 \in [0,1]$ for an element $x \in \chi$, the set of all Atanassov's intuitionistic fuzzy values is given as $\tilde{U} = \{(x_1, x_2) : x_1 + x_2 \leq 1\}$. The least and greatest element on $\tilde{U}$ are given as $\tilde{0} = (0,1)$ and $\tilde{1} = (1,0)$, respetively.

Considering the research of A-IFL, it makes possible to extend the usual logic connectives, as follows:

1. *Conjunction*, usually modelled by a *triangular norm* (t-norm) operator (Klement et al., 2013), which is an additive aggregation used in the framework of intersection in A-IFL;
2. *Disjunction* which is frequently modelled by a *triangular conorm* (Klement et al., 2013) (t-conorm) representing the t-norm dual operation;
3. *Complement* as a negation, a non-increasing function, defined in the Atanassov's seminal work (Atanassov, 2006), reverting the extremes of the unit interval $[0,1]$.
4. *Xor operators*, considering in this work two representable classes $\oplus_{I_P}$ and $\otimes_{I_P}$ (Bedregal et al., 2013), both defined by composition of the above connectives verifying the symmetry, associativity, neutral element and boundary conditions.

Now, the connectives used to make the correlation between QC and A-IFL will be described in terms of representability based on fuzzy negations and aggregations.The standard intuitionistic fuzzy negation (Atanassov, 2006) is expressed as follows:

$$N_{I_S}(\tilde{x}) = (x_2, x_1), \forall \tilde{x} = (x_1, x_2) \in \tilde{U}. \quad (2)$$

Meanwhile, for all $\tilde{x} = (x_1, x_2), \tilde{y} = (y_1, y_2) \in \tilde{U}$, the intersection and union can be defined, respectively, in terms of a t-norm T and a t-conorm S, given as:

$$T_I(\tilde{x}, \tilde{y}) = T_I((x_1, x_2), (y_1, y_2)) = (T(x_1, y_1), S(x_2, y_2));$$
$$S_I(\tilde{x}, \tilde{y}) = S_I((x_1, x_2), (y_1, y_2)) = (S(x_1, y_1), T(x_2, y_2)).$$

We consider the Product t-norm T_{I_P} and Algebraic Sum S_{I_P}, respectively described as follows:

$$T_{I_P}(\tilde{x}, \tilde{y}) = (T_P(x_1, y_1), S_P(x_2, y_2)) = (x_1 y_1, x_2 + y_2 + x_2 y_2);$$
$$S_{I_P}(\tilde{x}, \tilde{y}) = (S_P(x_1, y_1), T_P(x_2, y_2)) = (x_1 + y_1 + x_1 y_1, x_2 y_2)).$$

The classical xor expressions considered for this work are described as follows:

$$A \oplus B \equiv (\neg A \wedge B) \vee (A \wedge \neg B);$$
$$A \otimes B \equiv (A \vee B) \wedge (\neg A \vee \neg B).$$

Definition 2.1. *The Atanassov's intuitionistic fuzzy Xor operator is a function $E_I : \tilde{U}^2 \to \tilde{U}$ verifying:*

E1: $E_I(\tilde{0},\tilde{0}) = E_I(\tilde{1},\tilde{1}) = \tilde{0}$ *and* $E_I(\tilde{0},\tilde{1}) = E_I(\tilde{1},\tilde{0}) = \tilde{1}$*;*
E2: $E_I(\tilde{x},\tilde{y}) = E_I(\tilde{y},\tilde{x})$*;*
E3: $\tilde{y}_1 \leq \tilde{y}_2 \Rightarrow E_I(\tilde{0},\tilde{y}_1) \leq E_I(\tilde{0},\tilde{y}_2)$*;*
E4: $\tilde{y}_1 \leq \tilde{y}_2 \Rightarrow E_I(\tilde{1},\tilde{y}_1) \geq E_I(\tilde{1},\tilde{y}_2)$*.*

Proposition 2.1. *The functions $\oplus_I, \otimes_I : \tilde{U}^2 \to \tilde{U}$ given as follows:*

$$\oplus_I(\tilde{x},\tilde{y}) = S_I(T_I(N_{SI}(\tilde{x}),\tilde{y}), T_I(\tilde{x},N_{SI}(\tilde{y}))) \quad (3)$$
$$\otimes_I(\tilde{x},\tilde{y}) = T_I(S_I(\tilde{x},\tilde{y}), S_I(N_{SI}(\tilde{x}),N_{SI}(\tilde{y}))) \quad (4)$$

are intuitionistic fuzzy Xor operators.

Proof. Consider $\oplus_I$ operator. It holds that:

E_I1: The following boundary conditions are verified in the endpoints of unit interval:

$$\oplus_I(\tilde{0},\tilde{0}) = S_I(T_I(N_{SI}(\tilde{0}),\tilde{0}), T_I(\tilde{0},N_{SI}(\tilde{0}))) = S_I(\tilde{0},\tilde{0}) = \tilde{0};$$
$$\oplus_I(\tilde{1},\tilde{1}) = S_I(T_I(N_{SI}(\tilde{1}),\tilde{1}), T_I(\tilde{1},N_{SI}(\tilde{1}))) = S_I(\tilde{0},\tilde{0}) = \tilde{0};$$
$$\oplus_I(\tilde{0},\tilde{1}) = S_I(T_I(N_{SI}(\tilde{0}),\tilde{1}), T_I(\tilde{0},N_{SI}(\tilde{1}))) = S_I(\tilde{1},\tilde{0}) = \tilde{1};$$
$$\oplus_I(\tilde{1},\tilde{0}) = S_I(T_I(N_{SI}(\tilde{1}),\tilde{0}), T_I(\tilde{1},N_{SI}(\tilde{0}))) = S_I(\tilde{0},\tilde{1}) = \tilde{1};$$

$E_I2 : \forall \tilde{x},\tilde{y} \in \tilde{U}$ the following is verified

$$E_I(\tilde{x},\tilde{y}) = S_I(T_I(N_{SI}(\tilde{x}),\tilde{y}), T_I(\tilde{x},N_{SI}(\tilde{y})))$$
$$= S_I(T_I(\tilde{y},N_{SI}(\tilde{x})), T_I(N_{SI}(\tilde{y}),\tilde{x})) = E_I(\tilde{y},\tilde{x});$$

E_I3 : If $\tilde{y}_1 \leq \tilde{y}_2$ then the following is verified

$$\oplus_I(\tilde{0},\tilde{y}_1) = S_I(T_I(N_{SI}(\tilde{0}),\tilde{y}_1), T_I(\tilde{0},N_{SI}(\tilde{y}_1)))$$
$$\leq S_I(T_I(N_{SI}(\tilde{0}),\tilde{y}_2), T_I(\tilde{0},N_{SI}(\tilde{y}_2))) = E_I(\tilde{0},\tilde{y}_2);$$

E_I4 : If $\tilde{y}_1 \leq \tilde{y}_2$ then the following is verified

$$\oplus_I(\tilde{1},\tilde{y}_1) = S_I(T_I(N_{SI}(\tilde{1}),\tilde{y}_1), T_I(\tilde{1},N_{SI}(\tilde{y}_1)))$$
$$\geq S_I(T_I(N_{SI}(\tilde{1}),\tilde{y}_2), T_I(\tilde{1},N_{SI}(\tilde{y}_2))) = E_I(\tilde{1},\tilde{y}_2);$$

Therefore, Proposition 2.1 is verified. □

Based on those expressions, one can use the operators T_I, S_I, N_{I_S} to construct the Atanassov's intuitionistic fuzzy xor $(\oplus_I, \otimes_I)$ respectively given as follows:

$$\begin{aligned}
&\oplus_I(\tilde{x},\tilde{y}) = \\
&= S_I(T_I(N_{SI}(\tilde{x}),\tilde{y}), T_I(\tilde{x},N_{SI}(\tilde{y}))) \\
&= S_I(T_I((x_2,x_1),(y_1,y_2)), T_I((x_1,x_2),(y_2,y_1)))) \\
&= S_I((T(x_2,y_1),S(x_1,y_2)),(T(x_1,y_2),S(x_2,y_1)))) \\
&= (S(T(x_2,y_1),T(x_1,y_2)), T(S(x_1,y_2),S(x_2,y_1))) \quad (5)
\end{aligned}$$

$$\begin{aligned}
&\otimes_I(\tilde{x},\tilde{y}) = \\
&= T_I(S_I(\tilde{x},\tilde{y}), S_I(N_{SI}(\tilde{x}),N_{SI}(\tilde{y}))) \\
&= T_I(S_I((x_1,x_2),(y_1,y_2)), S_I((x_2,x_1),(y_2,y_1))) \\
&= T_I((S(x_1,y_1),T(x_2,y_2)),(S(x_2,y_2),T(x_1,y_1))) \\
&= (T(S(x_1,y_1),S(x_2,y_2)), S(T(x_2,y_2),T(x_1,y_1))) \quad (6)
\end{aligned}$$

By considering T_{I_P} and S_{I_P} in Eqs. (5) and (6), the xor $(\oplus_I, \otimes_I)$ operators can be respectively expressed as follows:

$$\begin{aligned}
\oplus_I(\tilde{x},\tilde{y}) &= ((x_1y_2 + x_2y_1 - x_1x_2y_1y_2), \\
&= (x_1x_2 + x_1y_1 + x_2y_2 + y_1y_2 - x_1x_2y_1 - \\
&\quad x_1x_2y_2 - x_1y_1y_2 - x_2y_1y_2 + x_1x_2y_1y_2)). \quad (7)
\end{aligned}$$

$$\begin{aligned}
\otimes_I(\tilde{x},\tilde{y}) &= ((x_1x_2 + x_1y_2 + x_2y_1 + y_1y_2 - x_1x_2y_1 \\
&= -x_1x_2y_2 - x_1y_1y_2 - x_2y_1y_2 + x_1x_2y_1y_2), \\
&= (x_1y_1 + x_2y_2 - x_1x_2y_1y_2)). \quad (8)
\end{aligned}$$

According to (Mannucci, 2006), fuzzy sets can be obtained by quantum superposition of classical fuzzy states associated with a quantum register.

3 QUANTUM COMPUTING

In *QC*, the *qubit* is the basic information unit, being the simplest quantum system, defined by a unitary and bi-dimensional state vector.

Qubits are generally described, in Dirac's notation (Nielsen and Chuang, 2003), by the following expression

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

when the coefficients α and β are complex numbers for the amplitudes of the corresponding states in the computational basis (state space), respecting the condition $|\alpha|^2 + |\beta|^2 = 1$, which guarantees the unitary of the state vectors of the quantum system, represented by $(\alpha,\beta)^t$ (Kaye et al., 2007).

The state space of a quantum system with multiple *qubits* is obtained by the tensor product of the space states of its subsystems. Considering a quantum system with two *qubits*, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$, the state space comprehends the tensor product given by

$$|\Pi\rangle = |\psi\rangle \otimes |\varphi\rangle = \alpha\cdot\gamma|00\rangle + \alpha\cdot\delta|01\rangle + \beta\cdot\gamma|10\rangle + \beta\cdot\delta|11\rangle.$$

The state transition of a quantum systems is performed by controlled and unitary transformations associated with orthogonal matrices of order 2^N, with N being the number of *qubits* within the system, preserving norms, and thus, probability amplitudes (Imre and Balázs, 2005). For instance, the *NOT* operator (*Pauli-X* transformation) and its application over 1-dimensional and 2-dimensional quantum systems are presented in the following.

$$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}; \quad (9)$$

$$X^{\otimes 2}|\Pi\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha\cdot\gamma \\ \alpha\cdot\delta \\ \beta\cdot\gamma \\ \beta\cdot\delta \end{pmatrix} = \begin{pmatrix} \alpha\cdot\gamma \\ \alpha\cdot\delta \\ \beta\cdot\delta \\ \beta\cdot\gamma \end{pmatrix} \quad (10)$$

Furthermore, the action of a Toffoli QT is also shown in next Eq. (11), describing a controlled operation for a 3-dimensional quantum system

$$T|\chi\rangle = T(\psi \otimes \varphi \otimes \sigma).$$

In this case, the *NOT* operator is applied to the third *qubit* $|\sigma\rangle$ when the current states of the first two *qubits* $|\psi\rangle$ and $|\varphi\rangle$ are both $|1\rangle$:

Similarly to QTs of multiple *qubits* which were obtained by the tensor product performed over unitary transformations, Eq.(11) presents the matrix structure of such QT, when $|\mathcal{X}\rangle$ is the initial state:

$$\mathrm{T}|\mathcal{X}\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \\ \theta \\ \upsilon \\ \sigma \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \\ \theta \\ \sigma \\ \upsilon \end{pmatrix} \quad (11)$$

In order to obtain information from a quantum system, it is necessary to apply measurement operators, defined by a set of linear operators M_m, called projections. The index m refers to the possible measurement results. If the state of a 1-dimensional quantum system is $|\psi\rangle$ immediately before the measurement, the probability of an outcome occurrence is given by $p(|\psi\rangle) = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$. When measuring a *qubit* $|\psi\rangle$ with $\alpha, \beta \neq 0$, the probability of observing $|0\rangle$ and $|1\rangle$ are, respectively, given by the following expressions:

$$p(0) = \langle\phi|M_0^\dagger M_0|\phi\rangle = \langle\phi|M_0|\phi\rangle = |\alpha|^2;$$
$$p(1) = \langle\phi|M_1^\dagger M_1|\phi\rangle = \langle\phi|M_1|\phi\rangle = |\beta|^2.$$

After the measuring process, the quantum state $|\psi\rangle$ has $|\alpha|^2$ as the probability to be in the state $|0\rangle$ and $|\beta|^2$ as the probability to be in the state $|1\rangle$.

In multidimensional systems, the operators M_m^n and $p_N(m)$ denote the m-projection and corresponding probability measure, both performed on the n-qubit.

4 INTERPRETING XOR OPERATORS BASED ON QC

The description of intuitionistic fuzzy sets from the QC viewpoint extends the work in (Mannucci, 2006) by modeling an element $\tilde{x} = (x_1, x_2)$ by a pair of one-dimensional qubit quantum states $(|x_1\rangle, |x_2\rangle)$ where:

$$|x_1\rangle = \sqrt{1-x_1}|0\rangle + \sqrt{x_1}|1\rangle; \quad (12)$$
$$|x_2\rangle = \sqrt{1-x_2}|0\rangle + \sqrt{x_2}|1\rangle. \quad (13)$$

By modeling fuzzy operators in QC(Avila et al., 2015), the Product t-norm T_P and Algebriac Sum t-conorm can be represented through the *Toffoli* gate (T) and the standard negation through the *Pauli-X* gate (N).

So the first step to generate the quantum representation for both Xor operators $\oplus_{I_P}$ and $\otimes_{I_P}$ is to apply *De Morgan's law* related to t-(co)norms $T(S)$ and fuzzy negation N as present in Eqs. (5) and (6), resulting on the following expressions for their membership and non-membership degrees when considering $(\tilde{x}, \tilde{y})$ as input:

$$\mu_{\oplus_I} = N(T(N(T(x_2,y_1)), N(T(x_1,y_2)))) \quad (14)$$
$$\nu_{\oplus_I} = T(N(T(N(x_1),N(y_2))), N(T(N(x_2),N(y_1)))) \quad (15)$$

$$\mu_{\otimes_I} = T(N(T(N(x_1),N(y_1))), N(T(N(x_2),N(y_2)))) \quad (16)$$
$$\nu_{\otimes_I} = N(T(N(T(x_2,y_2)), N(T(x_1,y_1)))) \quad (17)$$

Taking $\tilde{x} = (x_1, x_2)$, $\tilde{y} = (y_1, y_2)$, the initial quantum stated $|\phi\rangle$ is the 10-dimensional quantum register:

$$|\phi\rangle = |x_1\rangle \otimes |x_2\rangle \otimes |y_1\rangle \otimes |y_2\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle.$$

Then, the quantum representation of the Xor operator $\oplus_{I_P}$ can be obtained by translating Eqs. (14) and (15), respectively resulting in the following QT compositions:

$$\mu_{\oplus_{\overline{I}}} = M_1^9 \circ N_{5,6,9} \circ \mathrm{T}_9^{5,6} \circ N_{5,6} \circ \mathrm{T}_6^{1,4} \circ \mathrm{T}_5^{2,3} \quad (18)$$
$$\nu_{\oplus_{\overline{I}}} = M_1^{10} \circ \mathrm{T}_{10}^{7,8} \circ N_{2,3,8} \circ \mathrm{T}_8^{2,3} \circ N_{2,3} \circ N_{1,4,7} \circ \mathrm{T}_7^{1,4} \circ N_{1,4} \quad (19)$$

Analogously, considering the initial quantum state $|\phi\rangle$, the quantum representation of the Xor operator $\otimes_{I_P}$ can be obtained by translating Eqs. (16) and (17), respectively resulting in QT compositions given as follows:

$$\mu_{\otimes_{\overline{I}}} = M_1^9 \circ \mathrm{T}_9^{5,6} \circ N_{2,4,6} \circ \mathrm{T}_6^{2,4} \circ N_{2,4} \circ N_{1,3,5} \circ \mathrm{T}_5^{1,3} \circ N_{1,3} \quad (20)$$
$$\nu_{\otimes_{\overline{I}}} = M_1^{10} \circ N_{10,7,8} \circ \mathrm{T}_{10}^{7,8} \circ N_{7,8} \circ \mathrm{T}_8^{1,3} \circ \mathrm{T}_7^{2,4} \quad (21)$$

For both quantum representations of membership functions, it is used 10 qubits: 2 pairs for the inputs $(\tilde{x}, \tilde{y})$, 4 ancillaries qubits to store intermediate results and 1 pair for the final result. The membership degree obtained is stored on qubit 9 and the non-membership degree on qubit 10.

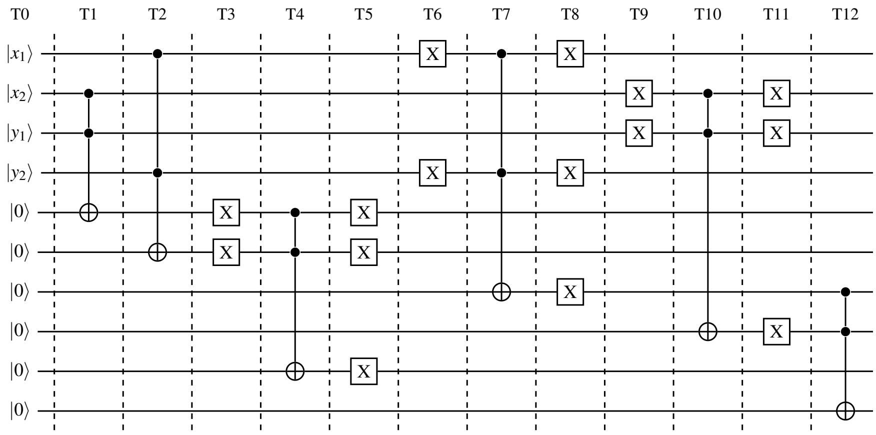

Figure 1: Quantum circuit modelling $\oplus_{I_P}(\tilde{x},\tilde{y})$.

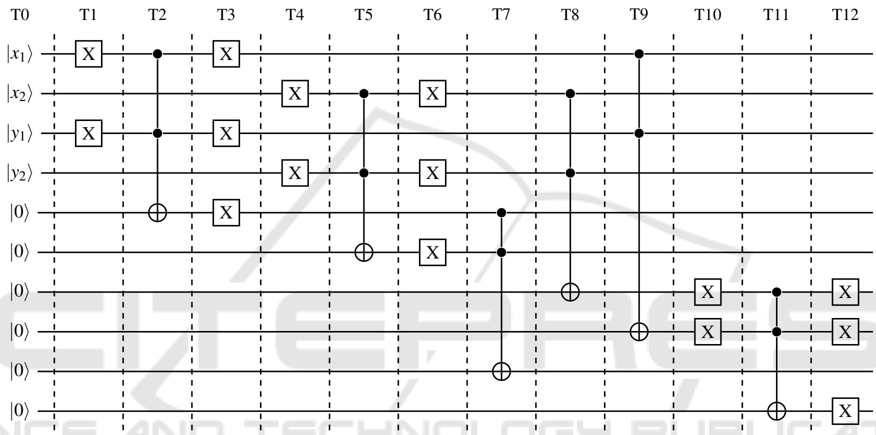

Figure 2: Quantum circuit modelling $\otimes_{I_P}(\tilde{x},\tilde{y})$.

4.1 Modelling $\oplus_{I_P}$ Quantum Operator

See in Fig. 1 the quantum circuit for the $\oplus_{I_P}$ Xor operator, resulting from the composition of Eq.(18) (from T1 to T5) and Eq. (19) (from T6 to T12) which is anticipating the measure operations.

Columns in Table 1 show the non-void amplitude evolution for the most relevant points of this quantum circuit, with T0 denoting the initial quantum state and T12 is the final quantum state resulting on $\oplus_{I_P}$ Xor operator obtained from the application of all composition quantum operators. And, in such columns, the changed qubits are highlighted.

After performing the circuit in Fig. 1, the measure operator M_1^9 is applied, that is, on the 9^{th} qubit and related to $|1\rangle$, it has probability

$$p_{9_{\oplus_I}}(1) = x_1y_2 + x_2y_1 - x_1x_2y_1y_2,$$

corresponding to the membership degree of $\oplus_{I_P}$ Xor operator, obtained by the $\mu_{\oplus_I}$ expression. Therefore,

$$p_{9_{\oplus_I}}(1) = \mu_{\oplus_{I_P}(\tilde{x},\tilde{y})}.$$

Analogous, the resulting measure operator M_0^9 has the probability expressed as follows

$$p_{9_{\oplus_I}}(1)(0) = 1 - x_1y_2 - x_2y_1 + x_1x_2y_1y_2,$$

corresponding to the complement of the membership degree of $\oplus_{I_P}(\tilde{x},\tilde{y})$, meaning that

$$p_{9_{\oplus_I}}(1)(0) = 1 - \mu_{\oplus_I}(\tilde{x},\tilde{y}).$$

Thus, the 9^{th} qubit is given by the following expression:

$$|\mu_{\oplus_I}\rangle = \sqrt{1-\mu_{\oplus_I}(\tilde{x},\tilde{y})}|0\rangle + \sqrt{\mu_{\oplus_I}(\tilde{x},\tilde{y})}|1\rangle.$$

In addition, by applying the measure operator M_1^{10}, that is, on the 10^{th} qubit and related to $|1\rangle$, the result quantum state has probability

$$p_{10_{\oplus_I}}(1) = x_1x_2 + x_1y_1 + x_2y_2 + y_1y_2 - x_1x_2y_1 - x_1x_2y_2 - x_1y_1y_2 - x_2y_1y_2 + x_1x_2y_1y_2,$$

corresponding to the non-membership degree of $\oplus_{I_P}(\tilde{x},\tilde{y})$. Then, it means that

$$p_{10_{\oplus_I}}(1) = \nu_{\oplus_I}(\tilde{x},\tilde{y}).$$

Table 1: Evolution of superposition quantum registers in modelling quantum circuit: $\oplus_{I_P}(\tilde{x},\tilde{y})$.

non-void amplitudes	$T0$	$T1$	$T2$	$T5$	$T8$	$T11$	$T12$
$(1-x_1)(1-x_2)(1-y_1)(1-y_2)$	0000000000	0000000000	0000000000	0000000000	0000000000	0000000000	0000000000
$(1-x_1)(1-x_2)(1-y_1)y_2$	0001000000	0001000000	0001000000	0001000000	0001001000	0001001000	0001001000
$(1-x_1)(1-x_2)y_1(1-y_2)$	0010000000	0010000000	0010000000	0010000000	0010000000	0010000100	0010000100
$(1-x_1)(1-x_2)y_1y_2$	0011000000	0011000000	0011000000	0011000000	0011001000	0011001100	0011001101
$(1-x_1)x_2(1-y_1)(1-y_2)$	0100000000	0100000000	0100000000	0100000000	0100000000	0100000100	0100000100
$(1-x_1)x_2(1-y_1)y_2$	0101000000	0101000000	0101000000	0101000000	0101001000	0101001100	0101001101
$(1-x_1)x_2y_1(1-y_2)$	0110000000	0110100000	0110100000	0110100010	0110100010	0110100110	0110100110
$(1-x_1)x_2y_1y_2$	0111000000	0111100000	0111100000	0111100010	0111101010	0111101110	0111101111
$x_1(1-x_2)(1-y_1)(1-y_2)$	1000000000	1000000000	1000000000	1000000000	1000001000	1000001000	1000001000
$x_1(1-x_2)(1-y_1)y_2$	1001000000	1001000000	1001010000	1001010010	1001011010	1001011010	1001011010
$x_1(1-x_2)y_1(1-y_2)$	1010000000	1010000000	1010000000	1010000000	1010001000	1010001100	1010001101
$x_1(1-x_2)y_1y_2$	1011000000	1011000000	1011010000	1011010010	1011011010	1011011110	1011011111
$x_1x_2(1-y_1)(1-y_2)$	1100000000	1100000000	1100000000	1100000000	1100001000	1100001100	1100001101
$x_1x_2(1-y_1)y_2$	1101000000	1101000000	1101010000	1101010010	1101011010	1101011110	1101011111
$x_1x_2y_1(1-y_2)$	1110000000	1110100000	1110100000	1110100010	1110101010	1110101110	1110101111
$x_1x_2y_1y_2$	1111000000	1111100000	1111110000	1111110010	1111111010	1111111110	1111111111

In analogous way, the resulting measure operator M_0^{10} has the probability

$$p_{10_{\oplus_I}}(0) = 1 - x_1x_2 - x_1y_1 - x_2y_2 - y_1y_2 + x_1x_2y_1 + x_1x_2y_2 + x_1y_1y_2 + x_2y_1y_2 - x_1x_2y_1y_2,$$

corresponding to the complement of the non-membership degree of $\oplus_{I_P}(\tilde{x},\tilde{y})$, that is,

$$p_{10_{\oplus_I}}(0) = 1 - \nu_{\oplus_I}(\tilde{x},\tilde{y}).$$

Thus, the 10^{th} qubit is given as follows:

$$|\nu_{\oplus_I}\rangle = \sqrt{1-\nu_{\oplus_I}(\tilde{x},\tilde{y})}|0\rangle + \sqrt{\nu_{\oplus_I}(\tilde{x},\tilde{y})}|1\rangle.$$

Concluding, the interpretation of intuitionistic fuzzy values related to the $\oplus_{I_P}$ Xor connective is provided by the pair $(|\mu_{\oplus_I}\rangle, |\nu_{\oplus_I}\rangle)$ of quantum registers.

Remark 4.1. *As a relevance, the entanglement of such qubits makes the fuzzy quantum circuits different from circuits modeling other logical approaches.*

For instance, by taking the results from $\oplus_{I_P}$ Xor operator one can easily observe that when the measurement applied to the 10^{th} qubit is related to $|1\rangle$, then it returns 7^{th} and 8^{th} qubits also related to $|1\rangle$, meaning that these three qubits collapse to value 1.

This is a phenomenon that does not occur in standard fuzzy set theory: the result of measurement (observation) affects the state of other arguments.

So, these values can be used in next calculations performed by other functions but involving such qubit systems, independently of the reuse of circuit inputs, which are also restored from 1^{rs} to 4^{th} qubits.

4.2 Modelling $\otimes_{I_P}$ Quantum Operator

Fig. 2 shows the $\otimes_{I_P}$ quantum circuit related to composition of Eq.(20) (from T1 to T7) and Eq.(21) (from T8 to T12) without the measure operations.

Analogously, based on the entanglement of such qubits, one can easily observe that when the measurement 10^{th} qubit is related to $|0\rangle$ simultaneously, the 5^{th} and 6^{th} qubits are also related to $|0\rangle$, meaning that these three qubits collapse to value 0.

Moreover, see columns in Table 2 presenting the evolution of the non-void amplitudes for the most relevant points of this circuit, with T0 being the initial quantum state and T12 is the final quantum state, ie, the quantum state resulting after the application of all the quantum operators related to the $\otimes_{I_P}(\tilde{x},\tilde{y})$.

After executing this circuit, applying the measure operator M_1^9, that is, on the 9^{th} qubit and related to $|1\rangle$, it has the following distribution of probability:

$$p_{9_{\otimes_{I_P}}}(1) = x_1x_2 + x_1y_2 + x_2y_1 + y_1y_2 - x_1x_2y_1 - x_1x_2y_2 - x_1y_1y_2 - x_2y_1y_2 + x_1x_2y_1y_2,$$

corresponding to the membership degree of $\otimes_{I_P}(\tilde{x},\tilde{y})$, then we obtain that

$$p_{9_{\otimes_I}}(1) = \mu_{\otimes_{I_P}(\tilde{x},\tilde{y})}. \quad (22)$$

Analogously, the measure M_0^9 has the following probability

$$p_{9_{\otimes_I}}(0) = 1 - x_1x_2 - x_1y_2 - x_2y_1 - y_1y_2 + x_1x_2y_1 + x_1x_2y_2 + x_1y_1y_2 + x_2y_1y_2 - x_1x_2y_1y_2,$$

which means that the complement of the membership degree of $\otimes_{I_P}$ Xor operator is given as follows

$$p_{9_{\otimes_I}}(0) = 1 - \mu_{\otimes_{I_P}}(\tilde{x},\tilde{y}). \quad (23)$$

Thus, By Eqs. (22) and (23) we have that the 9^{th} qubit can be given as follows:

$$|\mu_{\otimes_I}\rangle = \sqrt{1-\mu_{\otimes_I}(\tilde{x},\tilde{y})}|0\rangle + \sqrt{\mu_{\otimes_I}(\tilde{x},\tilde{y})}|1\rangle.$$

Table 2: Evolution of superposition quantum registers in modelling quantum circuit: $\otimes_{I_P}(\tilde{x},\tilde{y})$.

non-void amplitudes	$T0$	$T3$	$T6$	$T7$	$T8$	$T11$	$T12$
$(1-x_1)(1-x_2)(1-y_1)(1-y_2)$	0000000000	0000000000	0000000000	0000000000	0000000000	0000000000	0000000000
$(1-x_1)(1-x_2)(1-y_1)(1-y_2)$	0000000000	0000000000	0000000000	0000000000	0000000000	0000000000	0000000000
$(1-x_1)(1-x_2)(1-y_1)y_2$	0001000000	0001000000	0001010000	0001010000	0001010000	0001010000	0001010000
$(1-x_1)(1-x_2)y_1(1-y_2)$	0010000000	0010100000	0010100000	0010100000	0010100000	0010100000	0010100000
$(1-x_1)(1-x_2)y_1y_2$	0011000000	0011100000	0011110000	0011110010	0011110010	0011110010	0011110010
$(1-x_1)x_2(1-y_1)(1-y_2)$	0100000000	0100000000	0100010000	0100010000	0100010000	0100010000	0100010000
$(1-x_1)x_2(1-y_1)y_2$	0101000000	0101000000	0101010000	0101010000	0101011000	0101011000	0101011001
$(1-x_1)x_2y_1(1-y_2)$	0110000000	0110100000	0110110000	0110110010	0110110010	0110110010	0110110010
$(1-x_1)x_2y_1y_2$	0111000000	0111100000	0111110000	0111110010	0111111010	0111111010	0111111011
$x_1(1-x_2)(1-y_1)(1-y_2)$	1000000000	1000100000	1000100000	1000100000	1000100000	1000100000	1000100000
$x_1(1-x_2)(1-y_1)y_2$	1001000000	1001100000	1001110000	1001110010	1001110010	1001110010	1001110010
$x_1(1-x_2)y_1(1-y_2)$	1010000000	1010100000	1010100000	1010100000	1010100000	1010100100	1010100101
$x_1(1-x_2)y_1y_2$	1011000000	1011100000	1011110000	1011110010	1011110010	1011110110	1011110111
$x_1x_2(1-y_1)(1-y_2)$	1100000000	1100100000	1100110000	1100110010	1100110010	1100110010	1100110010
$x_1x_2(1-y_1)y_2$	1101000000	1101100000	1101110000	1101110010	1101111010	1101111010	1101111011
$x_1x_2y_1(1-y_2)$	1110000000	1110100000	1110110000	1110110010	1110110010	1110110110	1110110111
$x_1x_2y_1y_2$	1111000000	1111100000	1111110000	1111110010	1111111010	1111111110	1111111111

Moreover, applying the measure operator M_1^{10}, that is, measurement on the 10^{th} qubit and related to $|1\rangle$, the resulting probability is given as follows:

$$p_{10_{\otimes_{I_P}}}(1) = x_1y_1 + x_2y_2 - x_1x_2y_1y_2,$$

which corresponds to the non-membership degree of $\otimes_I(\tilde{x},\tilde{y})$, therefore, we obtain that

$$p_{10_{\otimes_I}}(1) = \nu_{\otimes_I}(\tilde{x},\tilde{y}). \quad (24)$$

Analogous, the measure M_0^{10} has the probability

$$p_{10_{\otimes_I}}(0) = 1 - x_1y_1 - x_2y_2 + x_1x_2y_1y_2,$$

corresponding to the complement of the non-membership degree of $\otimes_{I_P}(\tilde{x},\tilde{y})$, that is,

$$p_{10_{\otimes_I}}(0) = 1 - \nu_{\otimes_I}(\tilde{x},\tilde{y}). \quad (25)$$

And finally, by Eqs. (24) and (25) we have that the 10^{th} qubit is given by:

$$|\nu_{\otimes_I}\rangle = \sqrt{1-\nu_{\otimes_I}(\tilde{x},\tilde{y})}|0\rangle + \sqrt{\nu_{\otimes_I}(\tilde{x},\tilde{y})}|1\rangle.$$

Concluding, the interpretation of intuitionistic fuzzy values related to the $\otimes_{I_P}$ Xor connective is provided by the pair $(|\mu_{\otimes_I}\rangle, |\nu_{\otimes_I}\rangle)$ of quantum registers.

5 CONCLUSIONS

This paper describes the interpretation of two classes of Xor operations on A-IFS through concepts of QC. It was modelled using a quantum register using operations over fuzzy sets described by QTs.

Therefore, the presented approach to interpretation of intuitionistic fuzzy valued from quantum registers and quantum states shows another basic construction in the specification of fuzzy expert systems from QC, in order to obtain new information technologies based on intuitionistic fuzzy approach.

Computer systems based on A-IFL and performed over quantum computers may be able to generate an output considering the manipulation of inaccurate data and also dealing with imprecision in the model of rule-based system, by taking advantage of properties as quantum parallelism.

Further work aims at consolidation of this specification including not only other fuzzy connectives but also constructors (e.i. automorphisms and reductions) and the corresponding extension of (de)fuzzyfication methodology from formal structures provided by QC.

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REFERENCES

Atanassov, K. and Gargov, G. (1998). Elements of intuitionistic fuzzy logic. *Fuzzy Sets and Systs.*, 9(1):39–52.

Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20:87–96.

Atanassov, K. T. (2006). On intuitionistic fuzzy negations. In *Computational Intelligence, Theory and Applications*, pages 159–167. Springer.

Atanassov, K. T. (2016). Mathematics of intuitionistic fuzzy sets. In *Fuzzy Logic in Its 50th Year - New Developments, Directions and Challenges*, pages 61–86.

Atanassov, K. T. (2017). *Intuitionistic Fuzzy Logics*, volume 351 of *Studies in Fuzziness and Soft Computing*. Springer.

Avila, A., Schmalfuss, M., Reiser, R., and Kreinovich, V. (2015). Fuzzy xor classes from quantum computing. In *International Conference on Artificial Intelligence and Soft Computing*, pages 305–317. Springer.

Bedregal, B., Reiser, R., and Dimuro, G. (2013). Revising XOR–implications: Classes of fuzzy (Co)implications based on fuzzy XOR (XNOR) connectives. *International Journal of Uncertainty, Fuzziness and Knowlegde– Based Systems*, 14(6):1–29.

Bertini, C. and Leporini, R. (2017). A fuzzy approach to quantum logical computation. *Fuzzy Sets and Systems*, 317:44–60.

Chiara, M. L. D., Giuntini, R., Sergioli, G., and Leporini, R. (2018). A many-valued approach to quantum computational logics. *Fuzzy Sets and Systems*, 335:94–111.

Freytes, H., Giuntini, R., Sergioli, G., and Aricò, A. (2010). Representing fuzzy structures in quantum computation with mixed states. In *40th IEEE International Symposium on Multiple-Valued Logic, ISMVL 2010, Barcelona, Spain, 26-28 May 2010*, pages 162–166.

Hájek, P. (1998). *Metamathematics of fuzzy logic*, volume 4. Springer Science & Business Media.

He, X., Li, Y., Qin, K., and Meng, D. (2017). On the characterizations of fuzzy xnor connectives. *Journal of Intelligent and Fuzzy Systems*, 32(3):2733–2745.

Imre, S. and Balázs, F. (2005). *Quantum Computing and Communications An Engineering Approach*. Johm Wiley & Sons, Ltd.

Kaye, P., Laflamme, R., and Mosca, M. (2007). *An Introduction to Quantum Computing*. Oxford University Press.

Klement, E. P., Mesiar, R., and Pap, E. (2013). *Triangular norms*, volume 8. Springer Science & Business Media.

Kosheleva, O., Reiser, R., and Kreinovich, V. (2015). Formalizing the informal, precisiating the imprecise: How fuzzy logic can help mathematicians and physicists by formalizing their intuitive ideas. In Trillas, E., Seising, R., and Kacprycz, J., editors, *Fuzzy Logic: Towards the Future*, volume 325 of *LNCS*, pages 301–321. Springer International Publishing, Netherlands, 2015.

Kreinovich, V., Kohout, L., and Kim, E. (2009). Similarity between quantum logic and fuzzy logic.

Li, P., Song, K., and Yang, E. (2010). Quantum genetic algorithm and its application to designing fuzzy neural controller. In *Sixth International Conference on Natural Computation, ICNC 2010, Yantai, Shandong, China, 10-12 August 2010*, pages 2994–2998.

Mannucci, M. (2006). Quantum fuzzy sets: Blending fuzzy set theory and quantum computation. *CoRR*, abs/cs/0604064.

Nielsen, M. and Chuang, I. (2003). *Quantum Computation and Quantum Information*. Cambridge University Publisher, Cambridge.

Perkowski, M. A., Chrzanowska-Jeske, M., Mishchenko, A., Song, X., Al-Rabadi, A., Massey, B., Kerntopf, P., Buller, A., Józwiak, L., and Coppola, A. J. (2001). Regular realization of symmetric functions using reversible logic. In *Euromicro Symposium on Digital Systems Design 2001 (Euro-DSD 2001), 4-6 September 2001, Warsaw, Poland*, pages 245–253.

Pykacz, J. (1993). Fuzzy quantum logic i. *International Journal of Theoretical Physics*, 32:1691–1708.

Raghuvanshi, A. and Perkowski, M. A. (2010). Fuzzy quantum circuits to model emotional behaviors of humanoid robots. In *Proceedings of the IEEE Congress on Evolutionary Computation, CEC 2010, Barcelona, Spain, 18-23 July 2010*, pages 1–8.

Reiser, R., Lemke, A., de Avila, A. B., Vieira, J., Pilla, M. L., and Bois, A. R. D. (2016). Interpretations on quantum fuzzy computing: Intuitionistic fuzzy operations $\times$ quantum operators. *Electr. Notes Theor. Comput. Sci*, 324:135–150.

Rigatos, G. G. and Tzafestas, S. G. (2002). Parallelization of a fuzzy control algorithm using quantum computation. *IEEE Trans. Fuzzy Systems*, 10(4):451–460.

Sriboonchitta, S., Nguyen, H. T., Kosheleva, O., Kreinovich, V., and Nguyen, T. N. (2019). Quantum approach explains the need for expert knowledge: On the example of econometrics. In *Structural Changes and their Econometric Modeling*, pages 191–199.