Fuzzy–rough Fuzzification in General FL Classifiers

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Abstract: In this paper, a three-dimensional version of fuzzy-rough fuzzification is examined for classification tasks. Similar approach based on interval fuzzy-rough fuzzification has been demonstrated to classify with three decision labels of confidence, one of which were uncertain. The method proposed here relies on the use of fuzzification of inputs with a triangular membership function describing the nature of imprecision in data. As a result, we implement in fuzzy classifiers three dimensional membership functions using the calculus of general type-2 fuzzy sets. The approach is justified when more confidence labels are expected from the decision system, especially when the classifier is embedded in a recurrent hierarchical decision system working on easily available economic, extended, and advanced expensive real data.

1 INTRODUCTION

An extended concept of rough sets applied to fuzzy sets has been introduced in the form of an approximation of fuzzy sets by so-called fuzzy granules (Nakamura, 1988). In contrast to rough-fuzzy sets, fuzzy-rough sets is based on extended equivalence relations that only correspond to Zadeh's similarity relations, i.e. a fuzzy relation R on $\mathbb{X}$ should be reflexive $\mu_R(x,x) = 1\ \forall x \in \mathbb{X}$, symmetric $\mu_R(x,y) = \mu_R(y,x)\ \forall x,y \in \mathbb{X}$, and transitive $\mu_R(x,z) \geq \sup_y \min(\mu_R(x,y), \mu_R(y,z))\ \ \forall x,y,z \in \mathbb{X}$. This relation, as a typical fuzzy set, can be decomposed into α-cuts in order to construct the fuzzy-rough set as an α-composition of upper and lower rough approximations of A

$$\mu_{\overline{R}_\alpha(A)}(x) = \sup\{\mu_A(y) \,|\, \mu_R(x,y) \geq \alpha\}\ , \quad (1)$$

$$\mu_{\underline{R}_\alpha(A)}(x) = \inf\{\mu_A(y) \,|\, \mu_R(x,y) \geq \alpha\}\ . \quad (2)$$

The fuzzy-rough set relies on a single fuzzy relation R. A family of fuzzy equivalence relations R_i establishes a fuzzy partition on $\mathbb{X}$ by fuzzy sets F_i which may be complete in order to cover the whole domain, $\inf_x \max_i \mu_{F_i}(x) > 0$. Therefore, a fuzzy-rough set A is a family of lower and upper rough approximations of A calculated for each i-th partition set, i.e.,

$$\overline{\Phi}_{i,\alpha}(A) = \sup\{\mu_A(x) \,|\, x \in [F_i]_\alpha\}\ , \quad (3)$$

$$\underline{\Phi}_{i,\alpha}(A) = \inf\{\mu_A(x) \,|\, x \in [F_i]_\alpha\}\ . \quad (4)$$

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Exemplary fuzzy-rough sets under triangular fuzzy partition settings are illustrated in Figures 1.

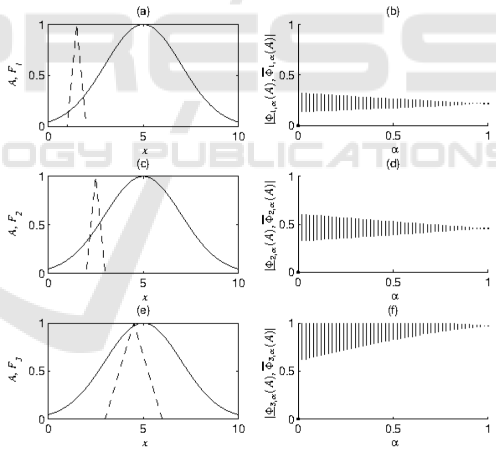

Figure 1: Construction of fuzzy-rough approximations using a triangular fuzzy set: (a,c,e) A — fuzzy set (*solid lines*), F_i — triangular fuzzy partition sets (*dashed lines*), (b,d,f) $\left[\underline{\Phi}_{i,\alpha}(A), \overline{\Phi}_{i,\alpha}(A)\right]$ — α-cuts of the fuzzy-rough set, $i = 1,2,3$.

The rough set theory is helpful when there are not enough attributes to fully describe an object, i.e., we have limited ability to classify particular objects since some other objects are indiscernible to the considered one. Considering fuzziness as a weaker form of indiscernibility, we are able to process uncertainty for ill-defined attributes such as measurement imprecision, vague estimations, three-point approximations, etc. The problem focuses on the use of an apriori

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knowledge about the imprecision of input data in order to determine an adequate shape of fuzzification. However, usually knowledge about the nature of such imprecisions is limited, then a three-point approximation can be considered as a satisfactory complexity of estimation. Triangular probability distributions are known to be successfully employed in financial analysis, management and business decision making. Triangular approximations become natural, when only its smallest, largest and the most likely values are known in the context of expected data inaccuracies. Such formulation lead the reasoning process to operate on three-dimensional membership functions. In (Starczewski, 2010), we have noticed that the composition of α-cuts $\bigcup_{\alpha\in(0,1]}\left[\underline{\Phi}_{i,\alpha}(A),\overline{\Phi}_{i,\alpha}(A)\right]$ formally represents a fuzzy grade of type-2. Systems constructed on linear type-2 fuzzy sets are a point of interest (Han et al., 2016; Najariyan et al., 2017).

The type-2 fuzzy set is understood as a set $\tilde{A}$ being a vague collection of elements characterized by membership function $\mu_{\tilde{A}}\colon \mathbb{X}\to\mathcal{F}([0,1])$, where $\mathcal{F}([0,1])$ is a set of all classical fuzzy sets in the unit interval $[0,1]$. Each $x\in\mathbb{X}$ is associated with a secondary membership function $f_x\in\mathcal{F}([0,1])$ i.e. a mapping $f_x\colon [0,1]\to[0,1]$. The fuzzy membership grade $\mu_{\tilde{A}}(x)$ is often refered as a fuzzy truth value, since its domain is the truth interval $[0,1]$. Regarding bounded secondary membership functions, the upper and lower bounds of $f_x>0$ with respect to $\mathbb{X}$ will be referred as upper an lower membership functions, respectively. Considering only secondary membership functions as fuzzy truth numbers, the function of x returning unique argument values of secondary functions for which $f_x=1$ will be called a principal membership function. With this formulation, we are able to construct a fuzzy-rough classifier which outputs a four-valued confidence label associated with the category label, extending our previous works (Starczewski, 2013; Nowicki and Starczewski, 2017; Nowicki, 2019).

2 TRIANGULAR FUZZIFICATION USING FUZZY-ROUGH APPROXIMATION

Using the definition of fuzzy-rough sets directly, fuzzy partitions F_i reflect the uncertainty of input data. This forms an automatic approach to perform non-singleton fuzzification. Namely, an input vector x should be mapped to the fuzzy-rough partition, i.e. a membership function, in this case, triangular with a peak value at x'.

In order to introduce a common notation, we may consider non-singleton fuzzification as a generalized membership function $\mu_F(x,x')=\mu_{F_i}(x)$ with implicit parameters of left and right deviations of triangles in our case. In the construction of triangular fuzzy-rough sets we may use our analytical results (Starczewski, 2010; Starczewski, 2013) expressing the secondary membership function of the antecedent $A_{k,n}$ as:

$$f_n\left(u,x_n'\right) = \max\left(\mu_{F_n}\left(\underline{\mu}_{A_{k,n}}^{-1}(u),x_n'\right),\mu_{F_n}\left(\overline{\mu}_{A_{k,n}}^{-1}(u),x_n'\right)\right), \quad (5)$$

where k indicates a rule and n is an index for inputs. In cases of symmetric and monotonic on slopes continuous membership functions, the secondary membership function can be expressed by cases, i.e.

$$f_n\left(u,x_n'\right)=\begin{cases}\mu_{F_n}\left(\overline{\mu}_{A_{k,n}}^{-1}(u),x_n'\right) & \text{if } m_{k,n}\le x_n' \\ \mu_{F_n}\left(\underline{\mu}_{A_{k,n}}^{-1}(u),x_n'\right) & \text{otherwise}\end{cases}. \quad (6)$$

2.1 Triangular Fuzzification of Triangular MFs

In the case of triangular fuzzification, $A_{k,n}$ can be generally asymmetric, i.e. $\mu_{A_{k,n}}(x_n)=\Big/\frac{x_n-m_{k,n}+\delta_{k,n}}{\delta_{k,n}},\frac{m_{k,n}-x_n+\gamma_{k,n}}{\gamma_{k,n}}\Big/$, while F_n are usually assumed to be symmetric triangular fuzzy numbers and $\mu_{F_n}(x_n)=\Big/\frac{x_n-x_n'+\Delta_n}{\Delta_n},\frac{x_n'-x_n+\Delta_n}{\Delta_n}\Big/$, respectively, where δ_n, γ_n, $\Delta_{k,n}$ denote spread values of triangular membership functions and a boundary operator is introduced as $/z/=\max(0,\min(1,z))$. Due to piecewise linear shapes of both functions, the result can be easily calculated as a composition of two linear transformations. Therefore, the secondary membership function of the fuzzy-valued fuzzy set induced by the fuzzy-rough approximation can be evaluated as follows:

$$f_{k,n}(u)=\frac{\max\begin{pmatrix}/\Delta_n-\left|\delta_{k,n}u+m_{k,n}-x_n'-\delta_{k,n}\right|/, \\ /\Delta_n-\left|m_{k,n}-x_n'+\gamma_{k,n}-\gamma_{k,n}u\right|/\end{pmatrix}}{\Delta_n}. \quad (7)$$

Apparently, the obtained expression represents a triangular function for applicable inputs $x_n'\in[m_{k,n}-\delta_{k,n},m_{k,n}+\delta_{k,n}]$. Now we are able to return to the primary domain, hence, the principal membership function is described by

$$\widehat{\mu}_{A_{k,n}}\left(x_n'\right)=\mu_{A_{k,n}}\left(x_n'\right), \quad (8)$$

The upper membership function has a trivial kernel $[m_{k,n}-\Delta_n, m_{k,n}+\Delta_n]$ and is characterized by the trapezoidal membership function alike

$$\overline{\mu}_{A_{k,n}}(x'_n) = \Bigg/ \begin{matrix} \frac{x'_n - m_{k,n}+\Delta_n+\delta_{k,n}}{\delta_{k,n}}, \\ \frac{m_{k,n}+\Delta_n - x'_n+\gamma_{k,n}}{\gamma_{k,n}} \end{matrix} \Bigg/, \qquad (9)$$

since $f_{k,n}(u,x') = 0$ for all $u \in [0,1]$ whenever $m_{k,n} - \delta_{k,n} \geq x'_n + \Delta_n$ or $m_{k,n} + \gamma_{k,n} \leq x'_n - \Delta_n$. The lower membership is a subnormal triangular function with its support $[m_{k,n} - \delta_{k,n} + \Delta_n, m_{k,n} + \gamma_{k,n} - \Delta_n]$ and its peak value can be calculated as follows. Since we search for bounds of u, i.e. limits for $f_{k,n}(u) > 0$, by omitting the boundary operator, we can calculate $f_{k,n}(u) = 0$ for the two slopes instead. Consequently, we obtain

$$u_1 = \frac{x'_n - m_{k,n} - \Delta_n + \delta_{k,n}}{\delta_{k,n}}, \qquad (10)$$

$$u_2 = \frac{m_{k,n} - x'_n - \Delta_n + \gamma_{k,n}}{\gamma_{k,n}} \qquad (11)$$

Obviously, the lower membership function needs to be aggregated in the following way:

$$\underline{\mu}_{A_{k,n}}(x'_n) = /\min(u_1, u_2)/. \qquad (12)$$

The center point of the triangle is calculated at interscetion of slope lines, i.e.

$$\frac{c_{k,n} - m_{k,n} - \Delta_n + \delta_{k,n}}{\delta_{k,n}} = \frac{m_{k,n} - c_{k,n} - \Delta_n + \gamma_{k,n}}{\gamma_{k,n}}$$

$$c_{k,n} = m_{k,n} - \frac{\delta_{k,n} - \gamma_{k,n}}{\delta_{k,n} + \gamma_{k,n}} \Delta_n. \qquad (13)$$

and the corresponding peak value is equal to

$$h_{k,n} = \frac{c_{k,n} - m_{k,n} - \Delta_n + \delta_{k,n}}{\delta_{k,n}} \qquad (14)$$

$$= 1 - \frac{2\Delta_n}{\delta_{k,n} + \gamma_{k,n}}. \qquad (15)$$

The construction of secondary memberships functions is demonstrated in Figure 2 for three exemplary x' values. To construct continuous type-2 fuzzified antecedent sets, we need to vary $\mu_F(x,x')$ in the whole spectrum of x' values. Unfavorably, for $x'_n \notin [m_{k,n} - \delta_{k,n}, m_{k,n} + \delta_{k,n}]$, the intersection between the fuzzy partition set and the antecedent fuzzy set is not sufficient, hence, secondary memberships the results are no longer triangular. In the sequel, however, using axiomatic operations on type-2 fuzzy sets, we will lose this non-triangularity, as we used triangular approximations for the clipped secondary membership functions.

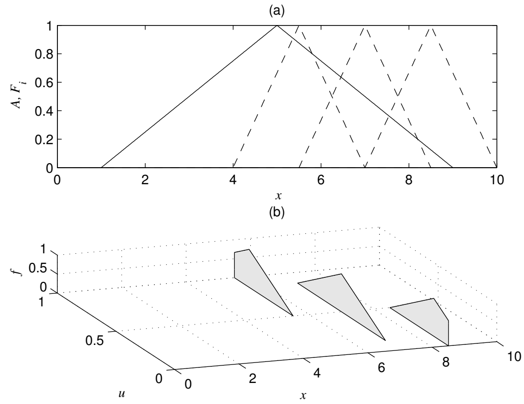

Figure 2: Construction of fuzzy-rough sets: a) A^k — antecedent membership function (*solid line*), μ_{F_1}, μ_{F_2}, μ_{F_3} — three realizations of non-singleton premise membership functions (*dashed lines*), b) $f_{x'}(u)$ — corresponding secondary membership functions constituting $f(u,x')$.

2.2 Triangular Fuzzification of Gaussians

The next case of triangular fuzzification involves antecedents $A_{k,n}$ in the form of Gaussian fuzzy sets, i.e. $\mu_{A_{k,n}}(x_n) = \exp\left(-\frac{1}{2}\left(\frac{x_n - m_{k,n}}{\delta_{k,n}}\right)^2\right)$ while F_n are triangular $\mu_{F_n}(x_n) = \min\left(\Big/ \frac{x_n - x'_n + \Delta_n}{\Delta_n}, \frac{x'_n - x_n + \Delta_n}{\Delta_n} \Big/\right)$, where Δ_n and $\delta_{k,n}$ denote spread values. As a result, the secondary membership function of the type-2 fuzzy set induced by the fuzzy-rough approximation can be presented as follows:

$$f_{k,n}(u, x'_n) = \begin{cases} \Bigg/ \frac{-\left|m_{k,n} + \delta_{k,n}\sqrt{-2\ln u} - x'_n\right| + \Delta_n}{\Delta_n} \Bigg/ & \text{if } m_{k,n} \leq x'_n \\ \Bigg/ \frac{-\left|m_{k,n} - \delta_{k,n}\sqrt{-2\ln u} - x'_n\right| + \Delta_n}{\Delta_n} \Bigg/ & \text{otherwise} \end{cases}. \qquad (16)$$

The obtained expression represents fragments of inverted Gaussian functions; however, for $\delta_{k,n} \gg \Delta_n$ these fragments can be linearly approximated, hence, the secondary membership function does not deviate significantly from the triangular function.

With respect to the primary domain, the principal membership function trivially is described by the non-fuzzified antecedent fuzzy set

$$\widehat{\mu}_{A_{k,n}}(x'_n) = \mu_{A_{k,n}}(x'_n).$$

The upper membership function is composed of two Gaussians connected by a unity kernel $[m_{k,n} - \Delta_n, m_{k,n} + \Delta_n]$, i.e.

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$$\overline{\mu}_{A_{k,n}}(x'_n) = \begin{cases} 1 & \text{if } x'_n \in [m_{k,n} - \Delta_n, m_{k,n} + \Delta_n] \\ \max \begin{pmatrix} \exp\left(-\frac{1}{2}\left(\frac{x_n - m_{k,n} - \Delta_n}{\delta_{k,n}}\right)^2\right), \\ \exp\left(-\frac{1}{2}\left(\frac{x_n - m_{k,n} + \Delta_n}{\delta_{k,n}}\right)^2\right) \end{pmatrix} \\ \text{otherwise} \end{cases} . \quad (17)$$

The lower membership function is a subnormal piecewise Gaussian, i.e.

$$\underline{\mu}_{A_{k,n}}(x'_n) = \min \begin{pmatrix} \exp\left(-\frac{1}{2}\left(\frac{x_n - m_{k,n} - \Delta_n}{\delta_{k,n}}\right)^2\right), \\ \exp\left(-\frac{1}{2}\left(\frac{x_n - m_{k,n} + \Delta_n}{\delta_{k,n}}\right)^2\right) \end{pmatrix}, \quad (18)$$

where the peak value at $x'_n = m_{k,n}$ is

$$h_{k,n} = \mu_{A_{k,n}}(m_{k,n} + \Delta_n) = \exp\left(-\frac{1}{2}\left(\frac{\Delta_n}{\delta_{k,n}}\right)^2\right). \quad (19)$$

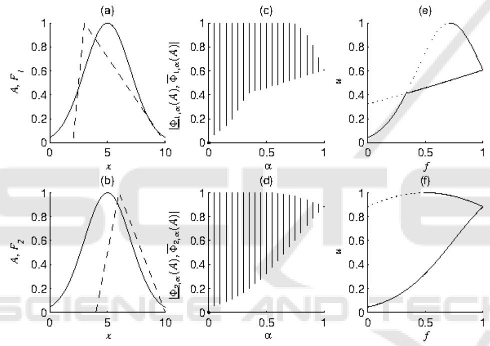

Figure 3: Construction of fuzzy-rough sets: (a,b) μ_A — original Gaussian antecedent membership function (*solid lines*), μ_{F_i} — triangular fuzzy partitions (*dashed lines*); (c,d) $\left[\underline{\Phi}_{i,\alpha}(A), \overline{\Phi}_{i,\alpha}(A)\right]$ — α-cut representations of fuzzy-rough sets according to the definition; (e,f) f — secondary membership functions of fuzzy-rough sets (*bold solid lines*).

Note that the presented derivation gives results which are intuitive formulations in the earliest works on type-2 fuzzy logic systems (Karnik et al., 1999; Mendel, 2001).

3 GENERAL FUZZY LOGIC CLASSIFIER

Consider a type-2 fuzzy logic system of N inputs in a vector form $\mathbf{x}$, and single output y (Mendel, 2001). The rule set is formed by K rules

$$\widetilde{R}_k\colon \text{IF } \widetilde{A}' \text{ is } \widetilde{A}_k \text{ THEN } \widetilde{B}' \text{ is } \widetilde{B}_k,$$

where $\widetilde{A}'$ is a type-2 fuzzified N-dimensional input $\mathbf{x}$, $\widetilde{B}'$ is a type-2 conclusion fuzzy set, $\widetilde{A}_k$ is an N-dimensional antecedent fuzzy set of type-2, and $\widetilde{B}_k$ is a consequent fuzzy set, $k = 1, \ldots, K$. We can interpret relations $\widetilde{R}_k$ either as conjunctions realized in general by type-2 t-norms, or as material implications of type-2 (Gera and Dombi, 2008).

The conclusion of the system is that y is $\widetilde{B}'$, which is an aggregation of all single rule conclusions calculated by the compositional rule of inference $\widetilde{B}'_k = \widetilde{A}' \circ (\widetilde{A}_k \mapsto \widetilde{B}_k)$, i.e.,

$$\mu_{\widetilde{B}'_k}(y) = \sup_{\mathbf{x} \in \mathbf{X}} \left\{ \widetilde{T}\left(\mu_{\widetilde{A}'}(\mathbf{x}), \widetilde{R}\left(\mu_{\widetilde{A}_k}(\mathbf{x}), \mu_{\widetilde{B}_k}(y)\right)\right) \right\}, \quad (20)$$

which in its simplest form of extended sup-min composition was first presented by (Dubois and Prade, 1980).

Fuzzification of $\mathbf{x}'$, which is the merits of this paper, can be defined as a mapping from real input space $\mathbf{X} \subset \mathbb{R}^n$ to type-2 fuzzy subsets of $\mathbf{X}$. However, in many basic cases (also examined in this paper) fuzzification functions can modify antecedent functions instead of fuzzifying inputs. Therefore, input values $\mathbf{x}'$ still may be represented by singleton type-2 fuzzy sets, and consequently, composition (20) simplifies itself to the following expression

$$\mu_{\widetilde{B}'_k}(y) \quad = \quad \widetilde{R}\left(\mu_{\widetilde{A}_k}(\mathbf{x}'), \mu_{\widetilde{B}_k}(y)\right). \quad (21)$$

If we apply conjunction relations, we expect the aggregated conclusion to be $\widetilde{B}' = \bigcup_{k=1}^{R} \widetilde{B}'_k$, i.e.

$$\mu_{\widetilde{B}'}(y) = \mathop{\widetilde{S}}_{k=1}^{K} \mu_{\widetilde{B}'_k}(y), \quad (22)$$

where $\widetilde{S}$ is a type-2 t-conorm. Otherwise, if we use type-2 material implications, we expect that $\widetilde{B}' = \bigcap_{k=1}^{R} \widetilde{B}'_k$, i.e.

$$\mu_{\widetilde{B}'}(y) = \mathop{\widetilde{T}}_{k=1}^{K} \mu_{\widetilde{B}'_k}(y). \quad (23)$$

3.1 Algebraic Operations on Triangular Type-2 Fuzzy Sets

In (Starczewski, 2013), we have defined a so-called regular t-norm on a set of triangular fuzzy truth numbers

$$\mu_{\mathrm{T}_{n=1}^{N} F_n}(u) = \max\left(0, \min\left(\lambda(u), \rho(u)\right)\right), \quad (24)$$

where

$$\lambda(u) \quad = \quad \begin{cases} \frac{u-l}{m-l} & \text{if } m > l \\ \text{singleton}(u-m) & \text{if } m = l, \end{cases} \quad (25)$$

$$\rho(u) \quad = \quad \begin{cases} \frac{r-u}{r-m} & \text{if } r > m \\ \text{singleton}(u-m) & \text{if } m = r, \end{cases} \quad (26)$$

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and

$$l = \operatorname*{T}_{n=1}^{N} l_n, \quad (27)$$

$$m = \operatorname*{T}_{n=1}^{N} m_n, \quad (28)$$

$$r = \operatorname*{T}_{n=1}^{N} r_n. \quad (29)$$

This formulation allows us to use ordinary t-norms for upper, principal and lower memberships independently. Moreover, we have proved the function given by (24) operating on triangular and normal fuzzy truth values is a t-norm of type-2.

3.2 Triangular Centroid Type Redution

The first step transforming a type-2 fuzzy conclusion into a type-1 fuzzy set is called a type reduction. In classification, we perform only type reduction without the second step of final defuzzification, which relies on the extended centroid defuzzification

$$\mu_B(y) = \sup_{y=\frac{\sum_{k=1}^{K} y_k u_{ki}}{\sum_{k=1}^{K} u_{ki}}} \min_{k=1,\dots,K} f_k(u_{ki}). \quad (30)$$

In (Starczewski, 2014), we have obtained exact type-reduced sets for triangular type-2 fuzzy conclusions as a set of ordered discrete primary values y_k and their secondary membership functions

$$f_k(u_k) = \Big/ \min\left(\frac{u_k - \underline{\mu}_k}{\widehat{\mu}_k - \underline{\mu}_k}, \frac{\overline{\mu}_k - u_k}{\overline{\mu}_k - \widehat{\mu}_k}\right) \Big/ \quad (31)$$

for $k = 1,\dots,K$. The secondaryties are specified by upper, principal and lower membership grades, $\overline{\mu}_k > \widehat{\mu}_k > \underline{\mu}_k$, $k = 1,2,\dots,K$. The well known Karnik-Mendel algorithm (Karnik et al., 1999) is used here to determine an interval centroid fuzzy set $[y_{\min}, y_{\max}]$ for the interval-valued fuzzy set constituted by the upper and lower membership grades. There is a selection of type-reduction algorithms (Greenfield and Chiclana, 2013); however, the Karnik-Mendel algorithm is chosen here becuase it results in an interval. Additionally, let y_{pr} be a centroid of the principal membership grades calculated by

$$y_{\mathrm{pr}} = \sum_{k=1}^{K} \frac{\widehat{\mu}_k y_k}{\widehat{\mu}_k}. \quad (32)$$

The exact centroid of the triangular type-2 fuzzy set is characterized by the following membership function:

$$\mu(y) = \begin{cases} \frac{y - y_{\mathrm{left}}(y)}{(1-q_{\mathrm{l}}(y))y + q_{\mathrm{l}}(y) y_{\mathrm{pr}} - y_{\mathrm{left}}(y)} & \text{if } y \in [y_{\min}, y_{\mathrm{pr}}] \\ \frac{y - y_{\mathrm{right}}(y)}{(1-q_{\mathrm{r}}(y))y + q_{\mathrm{r}}(y) y_{\mathrm{pr}} - y_{\mathrm{right}}(y)} & \text{if } y \in [y_{\mathrm{pr}}, y_{\max}] \end{cases}, \quad (33)$$

where the parameters are

$$q_{\mathrm{l}}(y) = \frac{\sum_{k=1}^{K} \widehat{\mu}_k}{\sum_{k=1}^{K} \overleftarrow{\mu}_k(y)}, \quad (34)$$

$$q_{\mathrm{r}}(y) = \frac{\sum_{k=1}^{K} \widehat{\mu}_k}{\sum_{k=1}^{K} \overrightarrow{\mu}_k(y)}, \quad (35)$$

and

$$y_{\mathrm{left}}(y) = \frac{\sum_{k=1}^{K} \overleftarrow{\mu}_k(y) y_k}{\sum_{k=1}^{K} \overleftarrow{\mu}_k(y)},$$

$$y_{\mathrm{right}}(y) = \frac{\sum_{k=1}^{K} \overrightarrow{\mu}_k(y) y_k}{\sum_{k=1}^{K} \overrightarrow{\mu}_k(y)},$$

with

$$\overleftarrow{\mu}_k(y) = \begin{cases} \overline{\mu}_k & \text{if } y_k \le y \\ \underline{\mu}_k & \text{otherwise} \end{cases},$$

$$\overrightarrow{\mu}_k(y) = \begin{cases} \overline{\mu}_k & \text{if } y_k \ge y \\ \underline{\mu}_k & \text{otherwise} \end{cases}.$$

An example of our type reduction algorithm is presented in Figure 4. Note that only three output values are subject to interpretation.

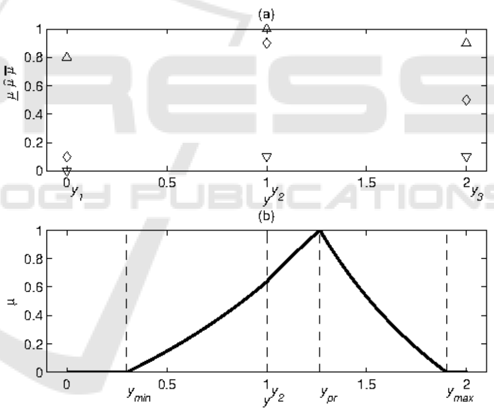

Figure 4: Centroid of fuzzy-valued fuzzy sets with triangular secondary membership functions: (a) △ – upper, ◊ – principal and ▽ – lower membership grades; (b) centroid fuzzy set

3.3 Type Reduction in Classification

Note that in classification y_k are either equal to 0 or to 1. Therefore, we propose the following procedure (Nowicki, 2009). Let us consider the fuzzy classifier defined by the equation

$$y_j = \frac{\sum_{\substack{k=1 \\ k:\, y_{j,k}=1}}^{K} \widetilde{\mu}_{A_k}(\mathbf{x})}{\sum_{k=1}^{K} \widetilde{\mu}_{A_k}(\mathbf{x})} \quad (36)$$

where $\widetilde{\mu}_{A_k}(\mathbf{x})$ is a type-2 or fuzzy-rough approximation of a fuzzy set. If we limit this set with its upper

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and lower membership functions, $\underline{\mu}_{A_k}(\mathbf{x})$ and $\overline{\mu}_{A_k}(\mathbf{x})$, respectively, the single-rule membership of an object to the j-th class is binary

$$y_{j,k} = \begin{cases} 1 & \text{if } \mathbf{x} \in C_j \\ 0 & \text{if } \mathbf{x} \notin C_j \end{cases} \quad (37)$$

for all rules $k = 1, \ldots, K$ and all classes $j = 1, \ldots, J$. The lower and upper approximations of the membership of object $\mathbf{x}$ to class C_j is given by

$$y_{\min}(j) = \frac{\sum_{\substack{k=1 \\ k:\, y_{j,k}=1}}^{K} \overleftarrow{\mu}_{A_k}(\mathbf{x})}{\sum_{k=1}^{K} \overleftarrow{\mu}_{A_k}(\mathbf{x})} \quad (38)$$

and

$$y_{\min}(j) = \frac{\sum_{\substack{k=1 \\ k:\, y_{j,k}=1}}^{K} \overrightarrow{\mu}_{A_k}(\mathbf{x})}{\sum_{k=1}^{K} \overrightarrow{\mu}_{A_k}(\mathbf{x})} \quad (39)$$

where

$$\overleftarrow{\mu}_{A_k}(\mathbf{x}) = \begin{cases} \underline{\mu}_{A_k}(\mathbf{x}) & \text{if } y_{j,k} = 1 \\ \overline{\mu}_{A_k}(\mathbf{x}) & \text{if } y_{j,k} = 0 \end{cases} \quad (40)$$

and

$$\overrightarrow{\mu}_{A_k}(\mathbf{x}) = \begin{cases} \underline{\mu}_{A_k}(\mathbf{x}) & \text{if } y_{j,k} = 0 \\ \overline{\mu}_{A_k}(\mathbf{x}) & \text{if } y_{j,k} = 1 \end{cases} \quad (41)$$

This first-step defuzzification issue with binary memberships of objects to classes, in comparison to the standard Karnik-Mendel type reduction algorithm, does not require any arrangement of $y_{j,k}$.

In addition to the boundary computation, we should have in mind, the calculation of the principal approximation of the membership of object $\mathbf{x}$ to class C_j, i.e.,

$$y_{\mathrm{pr}}(j) = \frac{\sum_{\substack{k=1 \\ k:\, y_{j,k}=1}}^{K} \widehat{\mu}_{A_k}(\mathbf{x})}{\sum_{k=1}^{K} \widehat{\mu}_{A_k}(\mathbf{x})}\,. \quad (42)$$

The decision of the classifier is very dependent on the interpretation of the three output variables obtained, albeit linearly ordered. If we choose a threshold value at $\frac{1}{2}$, we may enumerate four different decisions:

$$\begin{cases} \text{certain classification} & \text{if } y_{\min} \geq \frac{1}{2} \text{ and } y_{\max} > \frac{1}{2} \\ \text{certain rejection} & \text{if } y_{\min} < \frac{1}{2} \text{ and } y_{\max} \leq \frac{1}{2} \\ \text{likely classification} & \text{if } y_{\min} < \frac{1}{2} \text{ and } y_{\mathrm{pr}} \geq \frac{1}{2} \\ \text{likely rejection} & \text{otherwise.} \end{cases} \quad (43)$$

4 NUMERICAL SIMULATIONS

Exemplary simulations were carried out in the following order:

1. A fuzzy logic system with nonfuzzified inputs was trained on exact data (laboratory training environment), with the use of Back Propagation method. Gaussian antecedent membership functions and binary singleton consequents (as in classification). were applied in rules fired by the algebraic Cartesian product. Such system were used as a principal subsystem of the type-2 system constructed in the next step.
2. Triangular fuzzification membership functions were chosen with symmetric spread values Δ_i for all or particular singular inputs x_i. Triangular fuzzification was applied in the form of fuzzy-rough sets. As a result, a triangular type-2 fuzzy logic classifier was constructed.
3. Input data were corrupted by white additive noise with a triangular distribution and spread values identical to the fuzzification functions had. With noisy data the type-2 fuzzy classifier was tested, as it were in the real-time environment. The 10-folds cross-validation method was applied.

The classification abilities of fuzzy-rough classifiers were analyzed on modified datasets chosen from the UCI repository (Dua and Graff, 2017). The Iris flower is the standard task for classification and pattern recognition studies; however, in our cases, the dataset was corrupted with triangular additive noise. Table 1 presents the results for the 4–rule system, in which singular inputs were fuzzified and corresponding input data were uncertain, while in Table 2, all inputs were fuzzified and all data were corrupted. Correct classifications are counted whenever a sample is either certainly classified or certainly rejected. Likely correct classification signifies a good suggestion, when a likely classification is indicated for the positive desired output or a likely rejection is indicated for the false desired output. A confused likely incorrect classification label is connected with the cases of likely incorrect classification under expected classification, or likely correct classification under expected rejection.

Even on this such elementary example, we can observe that the number of misclassifications, i.e., when the classifier totally has made mistakes, is close to 0 in all cases. Although the standard type-1 fuzzy logic classifier is slightly better in correct classifications (treated as certain) than the corresponding fuzzified type-2 fuzzy classifier (based on the fuzzy-rough approach), the number of misclassifications for the type-1 system (calculated as a complement of the correct classification) is extremely greater that the number of misclassifications made by the fuzzified system. Obviously the number of correct classifications is decreasing with increasing uncertainty; however,

Table 1: Iris classification with triangular fuzzy-rough fuzzification; additional triangular noise applied to particular inputs Δ_i, $i = 1, \ldots, 4$.

Δ_1	Type-1 FLC Correct	Triangular Fuzzification Corr./ Likely Corr. / Likely Incorr. /Incorr.
0.2	0.974	0.966/0.011/0.006/0.017
0.5	0.965	0.897/0.079/0.020/0.004
1.0	0.941	0.673/0.296/0.026/0.005
2.0	0.863	0.428/0.512/0.055/0.005
5.0	0.548	0.419/0.442/0.130/0.009
Δ_2		
0.2	0.972	0.950/0.027/0.014/0.010
0.5	0.963	0.807/0.166/0.024/0.003
1.0	0.928	0.456/0.504/0.036/0.005
2.0	0.746	0.422/0.470/0.102/0.006
5.0	0.394	0.387/0.403/0.205/0.004
Δ_3		
0.2	0.972	0.967/0.010/0.004/0.019
0.5	0.967	0.896/0.079/0.013/0.012
1.0	0.920	0.744/0.214/0.035/0.008
2.0	0.843	0.469/0.461/0.060/0.010
5.0	0.644	0.406/0.443/0.133/0.018
Δ_4		
0.2	0.955	0.913/0.059/0.022/0.006
0.5	0.918	0.663/0.293/0.038/0.006
1.0	0.822	0.401/0.506/0.081/0.012
2.0	0.668	0.325/0.504/0.146/0.025
5.0	0.402	0.291/0.459/0.200/0.050

Table 2: Iris classification with triangular fuzzy-rough fuzzification; additional triangular noise applied to all inputs.

Δ_i	Type-1 FLC Correct	Triangular Fuzzification Corr./ Likely Corr. / Likely Incorr. /Incorr.
0.2	0.952	0.750/0.222/0.023/0.005
0.5	0.890	0.407/0.538/0.050/0.006
1.0	0.719	0.278/0.580/0.127/0.015
2.0	0.401	0.175/0.549/0.236/0.041
5.0	0.054	0.035/0.627/0.321/0.017

the number of likely correct classifications is always much greater than the number of likely incorrect classifications. Having only singular inputs tuned for uncertain data, we can count on high percentages of correct classifications (greater than 29% in the worst case). Unfortunately high uncertainty high uncertainty for all input attributes can diminish the number of correct classifications below 4%.

The classifier for Wisconsin Breast Cancer (removed instances with missing values) employed 3 rules. The results are shown in Table 3 (for particular inputs corrupted), and in Table 4 (for all inputs corrupted). The tests carried out confirmed the general properties of fuzzy-roughly fuzzified classifiers.

Table 3: WBC-based classification with triangular fuzzy-rough fuzzification; additional triangular noise applied to particular inputs Δ_i, $i = 1, \ldots, 9$.

Δ_1	Type-1 FLC Correct	Triangular Fuzzification Corr./ Likely Corr. / Likely Incorr. /Incorr.
2	0.955	0.918/0.060/0.011/0.011
3	0.918	0.857/0.117/0.016/0.010
5	0.847	0.791/0.172/0.026/0.011
Δ_2		
2	0.757	0.449/0.527/0.014/0.010
3	0.728	0.379/0.594/0.020/0.007
5	0.664	0.327/0.641/0.026/0.006
Δ_3		
2	0.942	0.900/0.080/0.010/0.010
3	0.871	0.849/0.129/0.013/0.009
5	0.793	0.801/0.175/0.019/0.005
Δ_4		
2	0.956	0.889/0.090/0.012/0.009
3	0.885	0.847/0.129/0.017/0.007
5	0.801	0.806/0.168/0.022/0.004
Δ_5		
2	0.969	0.894/0.085/0.010/0.011
3	0.924	0.851/0.121/0.012/0.016
5	0.840	0.798/0.167/0.014/0.021
Δ_6		
2	0.924	0.817/0.157/0.019/0.007
3	0.839	0.625/0.340/0.029/0.006
5	0.738	0.453/0.499/0.044/0.004
Δ_7		
2	0.958	0.886/0.095/0.011/0.008
3	0.918	0.844/0.136/0.014/0.006
5	0.833	0.794/0.181/0.019/0.006
Δ_8		
2	0.891	0.836/0.136/0.012/0.016
3	0.815	0.809/0.155/0.018/0.018
5	0.731	0.780/0.181/0.024/0.015
Δ_9		
2	0.925	0.859/0.114/0.022/0.005
3	0.849	0.830/0.141/0.024/0.005
5	0.742	0.808/0.158/0.028/0.006

Table 4: WBC-based classification with triangular fuzzy-rough fuzzification; triangular noise applied to all inputs.

Δ_i	Type-1 FLC Correct	Triangular Fuzzification Corr./ Likely Corr. / Likely Incorr. /Incorr.
2	0.546	0.373/0.535/0.075/0.017
3	0.327	0.173/0.655/0.155/0.017
5	0.145	0.078/0.657/0.250/0.015

The number of incorrect is not much greater 10% in all cases. The standard type-1 fuzzy classifier in comparison to the proposed type-2 classifier not always is better in performing correct classifications (for fuzzi-

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fied either 3rd, or 4th, or 7th, or 8th input the percentage trends are even reversed). The number of type-1 approach misclassifications is much greater than the number of type-2 approach misclassifications. made by the fuzzified system. The number of likely correct classifications is always much greater than the number of likely incorrect classification. Having only singular inputs fuzzified, we can count on satisfactory percentages of correct classifications (between 33% and 92%), while highly noised all inputs result with the number of correct classification below 8%.

5 CONCLUSIONS

The specificity of triangular fuzzifications in fuzzy classifiers allows us to analyze data at a deeper level of interpretation, which comes from the simultanuous use of principal, maximal and minimal fuzzy-rough approximations of data processed within the system. Instead of the standard yes-or-no classification, we obtain groups of classified objects with the four labels of confidence: certain classification, likely certain classification, likely certain rejection, definitely certain rejection. Continuing the example of medical diagnosis, we may differentiate a support for the four types of classifications. For the certain classification of a medical disease, we should urgently contact a patient with a doctor or ER care. For likely certain classifications, we may perform expensive laboratory tests to confirm or exclude the diagnosis. In cases of rather certain rejections, medical laboratory test may be more economical and can be extended over time. For certain rejections, patients can sleep calmly until their scheduled visits to the doctor. Similar methodologies can be realized by hierarchical automatic classifiers working on basic or standard, or expensive, in particular cases, data.

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