Optimal Control to Limit the Propagation Effect of a Virus Outbreak on a Network

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Abstract: The aim of this paper is to propose an optimal control strategy to face the propagation effects of a virus outbreak on a network; a recently proposed model is integrated and analysed. Depending on the specific model caracteristics, the epidemic spread could be more or less dangerous leading to a virus free or to a virus equilibrium. Two possible controls are introduced: a test on the computers connected in a network and the antivirus. In a condition of limited resources the best allocation strategy should allow to reduce the spread of the virus as soon as possible.

1 INTRODUCTION

The computer virus spread represents an important issue since all the connected devices are susceptible of being infected. A computer virus is a code that in general can modify normal operations, as well as damage files, and attack other computers. It can be transmitted by downloading files from internet, by using external devices, or by e-mails. The implications of this threat involve the field of Cyber Security, that has been recently defined as a *game between defender and attacker*, (Karunanithi et al., 2018). Since the computers and the internet connection are becoming more and more widespread, a computer virus is able to disrupt the productivity and cause billions of damages, (Zhu and Yang, 2012); therefore, a large amount of resources has been dedicated to blocking the spread of viruses.

There are many analogies with diseases epidemic spread and also the nomenclature is similar. The basic model is the *SIR* one, where S stands for susceptible, indicating the *subjects*, in this case the computers, free from virus but that can be infected, I stands for the set of all the computers infected and infectious and R represents the set of all recovered computers.

The computer virus may have a latent period during which the device, while being infected, is still not able to infect other computers, (Peng et al., 2013); the delay with which a virus breaks out is an important parameter to be taken into account when implementing a control strategy. From the early *1980s* the problem of computer virus detection and removal has been faced, taking into account the characteristics of this kind of infection: the latency, the parasitism, the hiding and the infectiousness (Hu et al., 2015). The modeling of the computers virus dynamics can vary depending on the specific scenario considered that suggests a more or less complex partition of the population. In the recent paper (Fatima et al., 2018) a susceptible-latent-breakingout-quarantine-susceptible computer virus dynamics is proposed and implemented, showing the positive effects of the quarantine. The scenario considered in this paper is similar to the one in (Xu and Ren, 2016); the population of computers connected in a network is partitioned into four classes. Besides the classes of susceptible S and recovered R computers, that are the devices not infected that can become infected and those that can not get the infection respectively, there are two classes of infected computers. The first one, E, is the most dangerous, since it is assumed that the virus has not yet manifested itself but the computers in this class can infect the devices they get in touch with; the second class, I, contains the computers infected and in which the virus has broken itself out. This is a common condition that could include infected e-mails or viruses that break out with a delay. It appears important to become aware of the presence of a virus as soon as possible, and successively to apply the suitable antivirus. This is the rationale for the choices of the two strategies proposed: u_1 represents the test to be performed on the computers which are

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supposed free of virus, i.e. S and E, while u_2 represents the antivirus to be applied on the known infected computers I, once the infection has broken out.

In this paper a variation of the model proposed in (Xu and Ren, 2016) is introduced and analysed, by considering the reproduction number and the possible existence of virus equilibrium. Optimal control strategies are proposed to face a possible epidemic spread. The paper is organized as follows; Section 2 is divided into three parts; in the first two a new model describing the computer virus spread is analysed considering the reproduction number, the equilibrium points and their stability. In the third, the optimal control strategy is proposed to allocate efficiently the limited available resources. In Section 3 numerical results show the plausibility of the model and the effectiveness of the introduced actions. Conclusions and future developments are presented in Section 4.

2 MATERIALS AND METHODS

In this Section a new model of virus spread in a computer network is described and analysed. The inspiration is the model proposed in the paper in (Xu and Ren, 2016); the total set of computers connected the internet is partitioned into four classes: S, the compartment of susceptible computers: they are the uninfected computers connected to the network; E, the compartment of exposed computers: they are the infected computers where all viruses are latent; I, the compartment of infected computers: they are the infected computers where all the viruses are currently breaking out; R, the compartment of recovered computers: they are the recovered computers that have run the antivirus software.

It is assumed that new computers are connected to the network with rate b; a susceptible computer in S, when having a connection with a computer in the exposed class E, can get the virus with rate β. As an effect, it can become infected, moving to I, with a probability p, or latent, moving to E, with a probability $1-p$. A latent computer virus breaks out with rate γ, and the computers are removed from the net with rate μ; moreover, it is possible that the virus is temporarily suppressed with probability $\varepsilon > 0$. The proposed model is a variation of the one in (Xu and Ren, 2016), having modeled the infection interaction between susceptible and exposed computers by the term βSE and not βSI; another difference is the impossibility for an infected device to become recovered without an external control.

The control actions introduced in this paper to limit the spread of a virus consist in a prevention campaign u_1, such as to induce people to test the condition and the vulnerability of their computer, and in the antivirus action u_2; they are weighted by σ_1 and σ_2 representing the unit cost efficiency. The virus test u_1 is applied on all the susceptible computers in S and its result influences the evolution of number of the exposed computers. The proposed model completed by the control actions is:

$$\dot{S} = b - \beta SE - \mu S \tag{1}$$

$$\dot{E} = (1-p)\beta SE + \varepsilon I - (\gamma+\mu)E - \sigma_1\left(\frac{E}{S+E}\right)u_1 \tag{2}$$

$$\dot{I} = p\beta SE + \gamma E - (\varepsilon+\mu)I + \sigma_1\left(\frac{E}{S+E}\right)u_1 - \sigma_2 I u_2 \tag{3}$$

$$\dot{R} = -\mu R + \sigma_2 I u_2 \tag{4}$$

The controls u_i are assumed bounded:

$$0 \le u_i(t) \le U_i^M, i = 1,2 \tag{5}$$

being U_i^M the corresponding possible maximum value of u_i. Obviously, all the state variables S, E, I and R are non negative, representing the number of the computers in the four identified different conditions; then, the same must hold for the initial conditions

$$S(0) = S_0 \ge 0 \qquad E(0) = E_0 \ge 0$$
$$I(0) = I_0 \ge 0 \qquad R(0) = R_0 \ge 0 \tag{6}$$

2.1 The Model Analysis

The proposed model is now analysed to understand its dynamical behavior. Two related aspects are now considered; the determination of the equilibrium points and the reproduction number. To determine the equilibrium points, the equation (4) of the model is not informative and therefore only the first three are considered from now on. For sake of notation, the model (1)–(3) is represented as follows:

$$\dot{X} = f(X,U) \tag{7}$$

with state $X = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}^T = \begin{pmatrix} S & E & I \end{pmatrix}^T$ and control input $U = \begin{pmatrix} u_1 & u_2 \end{pmatrix}^T$ In (7), $f = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}^T$, where the f_i are the *r.h.s.* functions of equations (1)–(3). The equilibrium points are determined by assuming null the control inputs u_i and considering the equation $f(X) = 0$. One solution, that can be referred as the *disease free equilibrium*, is

$$X_{e1} = \begin{pmatrix} \frac{b}{\mu} & 0 & 0 \end{pmatrix}^T \tag{8}$$

To determine the other equilibrium points, if they exist, it is useful to introduce the auxiliary variable $Z = \beta E + \mu$; from $f_1 = 0$

$$S = \frac{b}{Z}, \tag{9}$$

can be obtained, whereas from $f_3 = 0$ one has

$$I = \frac{E(p\beta b + \gamma Z)}{Z(\varepsilon + \mu)} \tag{10}$$

Therefore, the existence of S and I as equilibrium state components depends on the existence of E. By substituting expressions (9) and (10) in $f_2 = 0$, one obtains $E = 0$, the disease free equilibrium, and

$$Z = \frac{b\beta(\varepsilon + \mu - p\mu)}{\mu(\varepsilon + \mu + \gamma)} \tag{11}$$

By recalling the definition of Z, one obtains:

$$E = \frac{b(\varepsilon + \mu - p\mu)}{\mu(\varepsilon + \mu + \gamma)} - \frac{\mu}{\beta} = \frac{bA}{\mu B} - \frac{\mu}{\beta} \tag{12}$$

once

$$A = \varepsilon + \mu(1-p) \qquad B = \varepsilon + \mu + \gamma \tag{13}$$

are defined. If $E > 0$, that is

$$\frac{bA}{\mu B} > \frac{\mu}{\beta} \tag{14}$$

then a second equilibrium point is obtained:

$$X_{e2} = \begin{pmatrix} \frac{\mu B}{\beta A} \\ \frac{bA}{\mu B} - \frac{\mu}{\beta} \\ \frac{(bA\beta - \mu^2 B)(p\mu B + A\gamma)}{\mu B\beta A(\mu+\varepsilon)} \end{pmatrix} \tag{15}$$

It is useful to connect the existence of the second equilibrium point with the rate β at which computers become infected. From (14) it can be deduced that if

$$\beta > \frac{\mu^2 B}{bA} = \beta_b \tag{16}$$

then system (7) has two equilibrium points, X_{e1} and X_{e2}; if $\beta < \beta_b$, the solution is not feasible since the components assume negative values; for $\beta = \beta_b$ the two solutions X_{e1} and X_{e2} coincide. The value β_b is the bifurcation one.

Now local stability of the equilibrium points is investigated computing the Jacobian matrix of the system

$$J = \begin{pmatrix} -\beta E - \mu & -\beta S & 0 \\ (1-p)\beta E & (1-p)\beta S - (\gamma + \mu) & \varepsilon \\ p\beta E & p\beta S + \gamma & -(\varepsilon + \mu) \end{pmatrix} \tag{17}$$

and evaluating it in X_{e1} and, if it exists, in X_{e2}. As far as the point X_{e1} in (8) is concerned, the evaluation of (17) gives

$$J(X_{e1}) = \begin{pmatrix} -\mu & -\beta\frac{b}{\mu} & 0 \\ 0 & (1-p)\beta\frac{b}{\mu} - (\gamma+\mu) & \varepsilon \\ 0 & p\beta\frac{b}{\mu} + \gamma & -(\varepsilon+\mu) \end{pmatrix} \tag{18}$$

One eigenvalue of (18) is clearly $\lambda_1 = -\mu$; the other two are the roots of the polynomial equation

$$\lambda^2 + \left((\varepsilon + \gamma + 2\mu) - (1-p)\beta\frac{b}{\mu}\right)\lambda + \mu(\varepsilon + \gamma + \mu) - \beta\frac{b}{\mu}(\varepsilon + \mu(1-p)) = 0 \tag{19}$$

The two solutions, written as function of all the parameters appearing in the equation, do not allow an easy analysis. However, once the bifurcation value (16) is computed, it can be interesting to study the sign of the roots of (19) in a neighbourhood of the bifurcation condition. The two coefficients can be evaluated setting $\beta = \beta_b \pm \delta$, with $0 < \delta \ll 1$. By using the expression (16) for β_b, as well as the notations in (13), the polynomial equation (19) becomes

$$\lambda^2 + \left(\varepsilon\frac{B}{A} + \mu \mp (1-p)\frac{b}{\mu}\delta\right)\lambda + \mp(\varepsilon + \mu(1-p))\frac{b}{\mu}\delta = 0 \tag{20}$$

It can be noted that if $\beta = \beta_b - \delta$ is considered, the two coefficients of (20) are

$$\varepsilon\frac{B}{A} + \mu + (1-p)\frac{b}{\mu}\delta \quad > \quad 0 \tag{21}$$

$$(\varepsilon + \mu(1-p))\frac{b}{\mu}\delta \quad > \quad 0 \tag{22}$$

Then, for the Descartes' rule of signs, the two solutions have negative real part and the equilibrium point is locally asymptotically stable. Once $\beta = \beta_b + \delta$ is assumed, the coefficient (22) becomes

$$-(\varepsilon + \mu(1-p))\frac{b}{\mu}\delta < 0 \tag{23}$$

while the coefficient of λ changes its sign as δ varies. However, no matter what the sign of the coefficient of λ is: the polynomial equation has one positive and one negative real solutions; consequently, the equilibrium point X_{e1} results unstable. This behaviour follows the classical case of generic epidemic spread: there is a bifurcation condition which separates the case of only one asymptotically stable equilibrium point, corresponding to the disease free condition, and the case of two equilibria, one again the disease free condition which becomes unstable, and the second equilibrium point, corresponding to the so called endemic condition, locally asymptotically stable. Also in this case it is possible to verify that the equilibrium point X_{e2} is locally asymptotically stable, when it exists. Since the conditions obtained are function of all the parameters involved, their analytical expressions are quite complicated and not useful to bring to clear conclusions. In the numerical simulations, after the definition of the parameters values, the cases $\beta < \beta_b$ and $\beta > \beta_b$ are verified and illustrated.

2.2 The Reproduction Number

In epidemic spread the *reproduction number* $\mathcal{R}$, as recalled in (Driessche, 2002) and (Driessche, 2017), is a useful parameter to describe the ability of an infectious disease to invade a population. In this case the population is the set of all computers connected in the network and $\mathcal{R}$ is now evaluated by the *next generation matrix*. More precisely, from the dynamical model (1)– (3), considering only the compartments containing the infected subjects, the dynamical evolutions (2) and (3) can be split into two parts, collected in the two vectors:

$$K = \begin{pmatrix} (1-p)\beta SE \\ p\beta SE \end{pmatrix} \qquad Y = \begin{pmatrix} (\mu+\gamma)E - \varepsilon I \\ (\varepsilon+\mu)I - \gamma E \end{pmatrix}$$

The matrix K accounts for the rate of appearance of new infections in the compartments E and I, whereas Y describes the rate of other transitions between them. Now, the jacobian of K and Y (with respect to E and I) evaluated at the free disease equilibrium X_{e1} yields respectively:

$$F = \begin{pmatrix} \frac{(1-p)\beta b}{\mu} & 0 \\ \frac{p\beta b}{\mu} & 0 \end{pmatrix} \qquad V = \begin{pmatrix} \mu+\gamma & -\varepsilon \\ -\gamma & \varepsilon+\mu \end{pmatrix}$$

The next generation matrix is FV^{-1}, that is:

$$FV^{-1} = \frac{\beta b \begin{pmatrix} (\mu+\varepsilon)(1-p) & \varepsilon(1-p) \\ p(\varepsilon+\mu) & p\varepsilon \end{pmatrix}}{\mu[(\mu+\gamma)(\varepsilon+\mu)-\varepsilon\gamma]}$$

The reproduction number is defined as the spectral radius of the matrix FV^{-1}; in this case it is quite easy to compute it, getting

$$\mathcal{R} = \frac{(\varepsilon+\mu-p\mu)\beta b}{\mu[(\mu+\gamma)(\varepsilon+\mu)-\varepsilon\gamma]} = \frac{A\beta b}{\mu^2 B} \tag{24}$$

Following the classical definition of reproduction number, when $\mathcal{R} < 1$ the epidemic spread is characterised by a low contagious effect which tends to keep limited the number of infected units, asymptotically going to the disease free equilibrium; in the cases of $\mathcal{R} > 1$ the number of infected machines grows, going asymptotically to an endemic condition where the more $\mathcal{R}$ is great the more are the infected machines. In the present case, comparing expression (24) with (16), it is possible to find the relationship

$$\mathcal{R} = \frac{\beta}{\beta_b} \tag{25}$$

between the reproduction number and the bifurcation condition, so that $\mathcal{R} > 1$ coincides with the condition of existence of the second equilibrium point X_{e2}, (14).

2.3 The Optimal Control Problem Solution

In the proposed model (1)–(4) two limited controls are introduced, u_1 and u_2, aiming at a fast virus detection and a suitable antivirus action; with the first control the computers in the infected latent condition (E) are detected, whereas with the second one the antivirus allows the recovery of the infected computers in I. The available resources are generally bounded, from economic, logistic and practical point of view; this implies the need of suitable allocation strategy. In the similar contest of epidemic diseases, the natural framework to face efficiently the virus spread with limited resources is the optimal control theory. A cost index is introduced; the idea is to choose the control in such a way that the total number of infected computers (i.e. computers in E and in the I classes) is minimized, taking into account the resources limitations (5). For sake of notation, they are rewritten as

$$\begin{array}{ll} q_1 = -u_1 \le 0 & q_2 = u_1 - u_1^M \le 0 \\ q_3 = -u_2 \le 0 & q_4 = u_2 - u_2^M \le 0 \end{array} \tag{26}$$

The proposed cost index that interprets the discussed need is:

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left(A_1E^2 + A_2I^2 + B_1u_1^2 + B_2u_2^2\right) dt \tag{27}$$

where A_i and B_j, $i, j = 1,2$ are the weights of the states, E and I, and of the controls u_1 and u_2 respectively. The optimal control problem can be solved by using classical techniques of calculus of variation; the Hamiltonian function is defined as:

$$\begin{aligned} H &= \frac{1}{2}\left(A_1E^2 + A_2I^2 + B_1u_1^2 + B_2u_2^2\right) \\ &\quad +\lambda^T(t)F(X,U) \end{aligned} \tag{28}$$

where $\lambda(t) = \begin{pmatrix}\lambda_1(t) & \lambda_2(t) & \lambda_3(t)\end{pmatrix}^T$ is the costate. The necessary conditions of optimality are given by:

$$\dot{\lambda}_i = -\frac{\partial H}{\partial X_i}, \quad \lambda_i(t_f) = 0 \quad i = 1,2,3 \tag{29}$$

$$0 = \frac{\partial H}{\partial u_j} + \sum_{k=1}^{4} \frac{\partial q_k}{\partial u_j}\eta_k, \quad j = 1,2 \tag{30}$$

$$\eta_j q_j = 0, \quad \eta_j \ge 0, \quad j = 1,\dots,4 \tag{31}$$

Note that in (29) the independence of q_j, $j = 1,\dots,4$ from the state X has been used. By taking into account the conditions (31) the possible 2^4 cases are reduced to 4. By solving conditions (30) and taking into account the constraints (26) (5), along with (29), the optimal controls u_i, $i = 1,2$ are determined:

$$u_1 = \max\{\min\{\frac{(\lambda_2-\lambda_4)\sigma_1 E}{(S+E)B_1}, u_1^M\}, 0\} \tag{32}$$

$$u_2 = \max\{\min\{\frac{(\lambda_3 - \lambda_4)\sigma_2 I}{B_2}, u_2^M\}, 0\} \qquad (33)$$

Note that the optimal controls require the solution of the costate equations (29) in addition to the state equation (7) with the initial conditions (6).

3 NUMERICAL RESULTS AND DISCUSSION

In this Section, a numerical analysis is performed to study the proposed computer virus spread model and the optimal control strategies. For the choices of the model parameters the value proposed in (Xu and Ren, 2016) are assumed:

$$\begin{aligned} &b = 5; \quad \beta = 0.15; \quad \mu = 0.3; \\ &\varepsilon = 0.01; \quad \gamma = 0.4 \end{aligned} \qquad (34)$$

The initial conditions are set equal to

$$X_0 = \begin{pmatrix} 100 & 5 & 0 & 0 \end{pmatrix}^T \qquad (35)$$

thus assuming a ”population” of 105 devices, among which 5 are in the E condition of being infected and infectious, without knowing it. The control period is set equal to 10 *unit of time*.

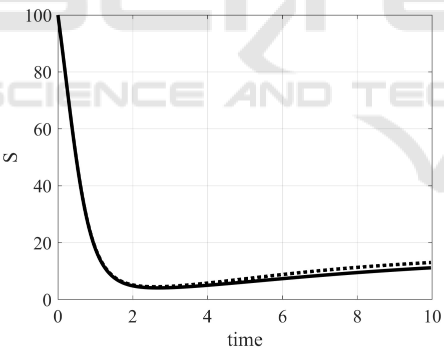

Figure 1: Case 1, $\beta = 0.15$: evolution of the number of susceptible computers (continuous line: free evolution; dotted line: optimal control is applied).

Two cases are considered, depending on the value of the probability p. As *Case 1* it is chosen $p = 0.8$, thus assuming that a computer, after being infected, with high probability enters in the I class, meaning that the infection is immediately known, and therefore it can not infect other computers. With the chosen values, $\mathcal{R} < 1$ is obtained, thus meaning that only the equilibrium point X_{e1} exists:

$$X_{e1} = \begin{pmatrix} 17 & 0 & 0 & 0 \end{pmatrix}^T \qquad (36)$$

By evaluating the corresponding Jacobian (18) and its eigenvalues, the stability of (36) can be verified. In

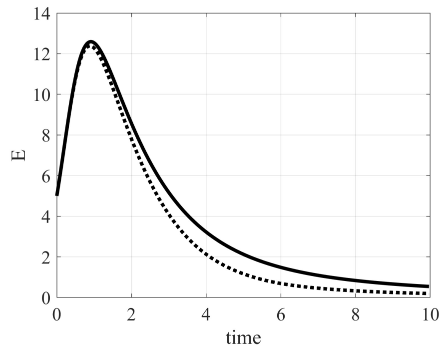

Figure 2: Case 1, $\beta = 0.15$: evolution of the number of exposed computers (continuous line: free evolution; dotted line: optimal control is applied).

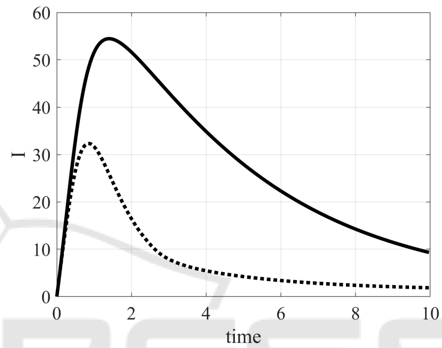

Figure 3: Case 1, $\beta = 0.15$: evolution of the number of infected computers (continuous line: free evolution; dotted line: optimal control is applied).

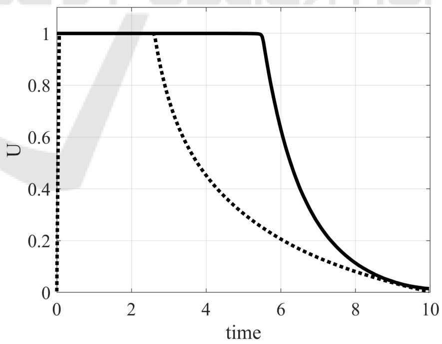

Figure 4: Case 1, $\beta = 0.15$: evolution of the optimal control strategies (continuous line: u_1; dotted line: u_2).

this case the parameters are such that for $\beta = 0.15$ it results $\beta_b = 0.1826$ and then $\beta < \beta_b$, thus confirming that the system is below the bifurcation condition with the chosen parameter values. By changing only the value of β into 0.25, one gets $\beta > \beta_b$ and a completely different situation it is obtained: $\mathcal{R} = 1.37 > 1$ and therefore also the second equilibrium point X_{e2} exists:

$$X_{e2} = \begin{pmatrix} 12 & 0 & 4 & 0 \end{pmatrix}^T \qquad (37)$$

By considering the Jacobian (17) calculated in the two

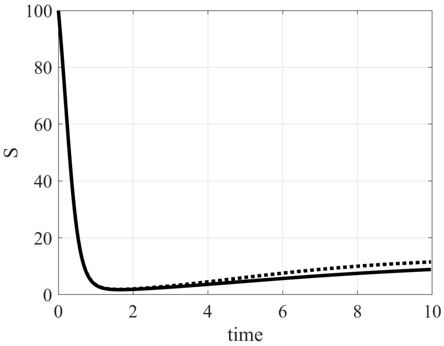

Figure 5: Case 1, $\beta = 0.25$: evolution of the number of susceptible computers (continuous line: free evolution; dotted line: optimal control is applied).

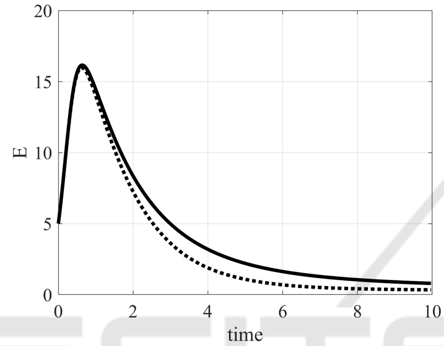

Figure 6: Case 1, $\beta = 0.25$: evolution of the number of exposed computers (continuous line: free evolution; dotted line: optimal control is applied).

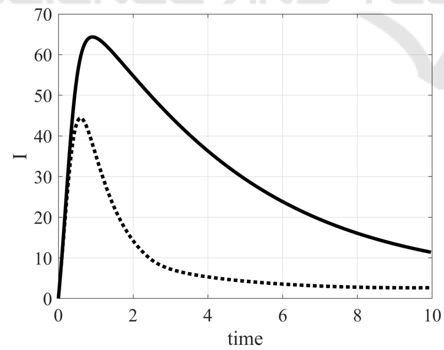

Figure 7: Case 1, $\beta = 0.25$: evolution of the number of infected computers (continuous line: free evolution; dotted line: optimal control is applied).

points X_{e1}, (36), and X_{e2}, (37), and evaluating the corresponding eigenvalues, one has that for X_{e1} the Jacobian has one eigenvalue with positive real part, and therefore it is unstable, while for X_{e2} all the eigenvalues have negative real parts, thus guaranteeing its local asymptotic stability. These results confirm what has been previously stated in Section 2. The *Case 2* regards the choice $p = 0.2$; this means that after the infection most of the computers become laten; of

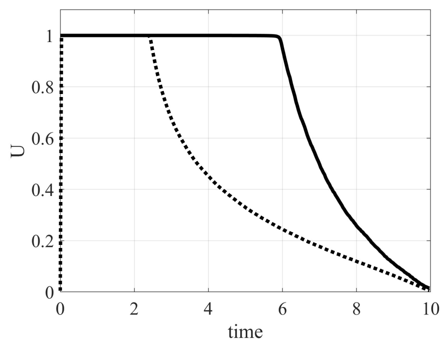

Figure 8: Case 1, $\beta = 0.25$: evolution of the optimal control strategies (continuous line: u_1; dotted line: u_2).

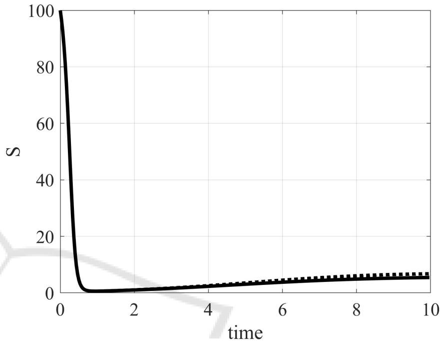

Figure 9: Case 2, $\beta = 0.15$: evolution of the number of susceptible computers (continuous line: free evolution; dotted line: optimal control is applied).

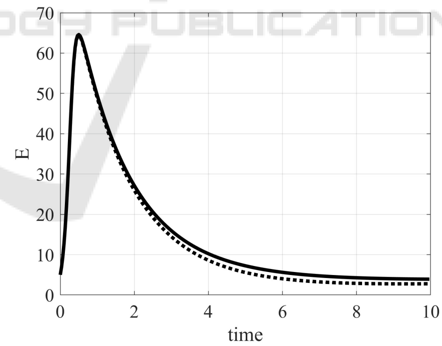

Figure 10: Case 2, $\beta = 0.15$: evolution of the number of exposed computers (continuous line: free evolution; dotted line: optimal control is applied).

course this is the most dangerous situation. In this case $\beta_b = 0.051$; by choosing $\beta = 0.15$, $\mathcal{R} = 2.53 > 1$ is obtained. Therefore the second equilibrium point X_{e2} exists and assumes the value

$$X_{e2} = \begin{pmatrix} 6 & 4 & 7 & 0 \end{pmatrix}^T \tag{38}$$

Since $\mathcal{R} > 1$ the equilibrium point X_{e1}, given again by (36), is unstable, as it can be verified by calculating the eigenvalues of (18), whereas the same analysis applied to the second equilibrium point X_{e2} in (38)

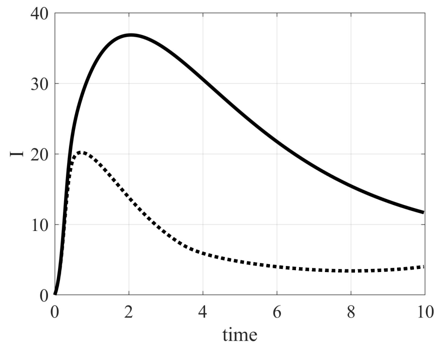

Figure 11: Case 2, $\beta = 0.15$: evolution of the number of infected computers (continuous line: free evolution; dotted line: optimal control is applied).

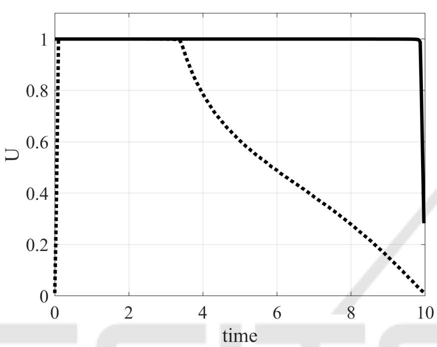

Figure 12: Case 2, $\beta = 0.15$: evolution of the optimal control strategies (continuous line: u_1; dotted line: u_2).

guarantees its stability.

If the parameter β is reduced to $\beta = 0.015$, that is to a value lower than β_b, $\mathcal{R} = 0.25 < 1$ is obtained. Therefore, a unique stable equilibrium point, X_{e1}, is expected, as it can be directly verified by considering, as usual, the eigenvalues of (18).

The optimal control actions are determined by minimizing (27); the values $A_1 = 10, A_2 = 0.1, B_1 = 1$ and $B_2 = 10$ are chosen for the weights, whereas for the boundaries (5) of u_1 and u_2, $u_1^M = u_2^M = 1$ are assumed. This means that the main aim is to minimize the number of exposed computers, being the most dangerous; therefore the limitations of the resources in the cost index is less restrictive when referring to the u_1, rather than u_2. In Figs. 1–3 the evolutions of the state in *Case 1* with the choice $\beta = 0.15$ are shown; they are the effects of the optimal control depicted in Fig. 4, that proposes the maximum allowed value 1 for u_1 up to about time 5.5, whereas the second control u_2 must be at its maximum value up to time 3. The effects of these controls are evident in the almost halved number of infected, Fig. 3. As seen, the same *Case 1* analysed with $\beta = 0.25$ yields a different kind of solution with two equilibrium points; the evolutions of the state along with the optimal controls are shown in Figs. 5–8. Due to the more dangerous situation, the number of exposed and of the infected devices in free evolution is significantly higher than in the previous case, Fig. 7, reaching for the infected computers almost the peak of 65 computers versus 55 of the previous case; also the number of exposed computers reaches higher value (16 versus 13) compensated with the controls, Fig. 6. As far as the control is concerned, in this case a larger effort, up to time 6, is requested for the control u_1 that at the end of the control time 10 has not yet reached its minimum allowed value; similar considerations can be applied to the optimal control u_2, that after keeping the maximum value up to time 3, it decreases without reaching the minimum value at time 10.

In *Case 2*, after the infection, a larger number of computers becomes exposed, being able to infect other devices; it can be expected a stronger control action to face this dangerous situation, see Fig. 12 in case of $\beta = 0.15$: for the u_1 action, the maximum effort is required for almost all the time period, whereas the maximum value for u_2 is required almost up to time 4. The results are shown in Figs. 9–11. As expected, the number of infected computers increases sensibly in absence of any action but the optimal control is able to face the spread, Fig. 11. The same *Case 2* is analysed assuming $\beta = 0.015$; this means that the virus is not so dangerous as in the previous case; coherently, the optimal control does not require the maximum value for both u_1 and u_2, see Fig. 16: the best action for u_1 is the maximum value up to time 7, whereas for the control u_2 it is not required the maximum value. This is reasonable, since the epidemic is not spreading and therefore there is not a great need of anti–virus action. The reduced capability of the virus to spread can be evidenced also noting that the evolution of the susceptible devices does not vary significantly without and with the control, Fig. 13, being the epidemic situation not so dangerous.

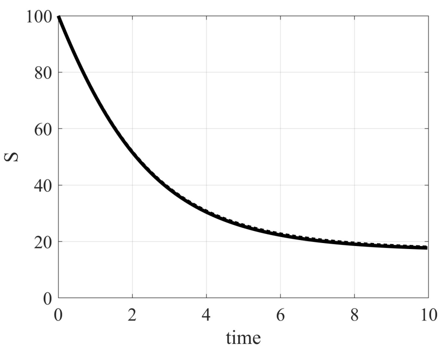

Figure 13: Case 2, $\beta = 0.015$: evolution of the number of susceptible computers (continuous line: free evolution; dotted line: optimal control is applied).

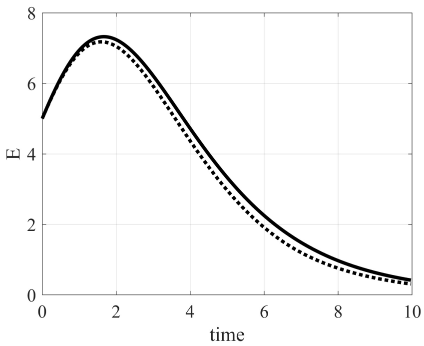

Figure 14: Case 2, $\beta = 0.015$: evolution of the number of exposed computers (continuous line: free evolution; dotted line: optimal control is applied).

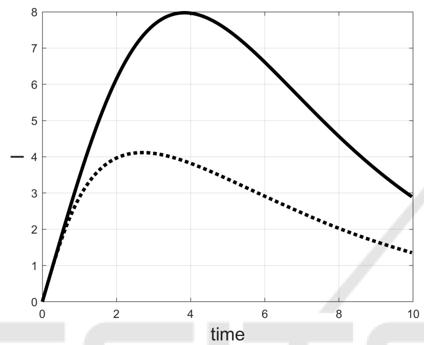

Figure 15: Case 2, $\beta = 0.015$: evolution of the number of infected computers (continuous line: free evolution; dotted line: optimal control is applied).

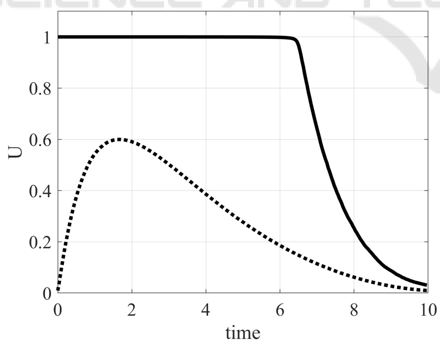

Figure 16: Case 2, $\beta = 0.015$: evolution of the optimal control strategies (continuous line: u_1; dotted line: u_2).

4 CONCLUSIONS

The computer virus spread represents a problem in a globalized world more and more connected, being able to cause economic and social damages. By using the same modeling of epidemic disease spread, it is possible to describe the dynamics of the computer virus and to act in the most efficient way, in order to avoid as much as possible the obvious remedy, the isolation of the infected computers. A simple dynamical model to describe the computer virus spread is proposed and optimal strategies are developed. The first results appear satisfactory and the effort will be devoted to apply the discussed model and the control scheme to a real scenario.

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