Discrete-time Adaptive Regulation of Systems with Uncertain Upper-bounded Input Delay: A State Substitution Approach

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Abstract: This paper proposes a discrete-time adaptive regulation approach for scalar linear time-invariant systems with unknown, constant input time delay that has a known upper-bound, without explicitly estimating the time delay. To cope with the unknown time delay, a state substitution is made that results in a delay free system that simplifies the control law design. In addition, the proposed approach does not require that the system have stable open-loop zeros. A stability analysis shows that the proposed regulator drives the system state to zero asymptotically and simulation results are shown to verify the approach.

1 INTRODUCTION

Processes with time delays at the input (i.e. delayed control action) are encountered in many situations requiring the use of feedback control, and these delays may have significant consequences if a controller is not designed to compensate for them (Richard, 2003) (Gu and Niculescu, 2003). For example, communications delays in bilateral teleoperation can cause the force feedback loop to become unstable, possibly creating a hazard for the remote operator (Abidi et al., 2016). In visual servoing, the computational complexity of visual information processing may insert a significant delay into the control loop that could affect stability (Bjerkeng et al., 2014). Pneumatic actuators are commonly used in soft robots, where long pneumatic lines and the compressibility of air can introduce actuator delays (Skorina et al., 2017). In all these applications it is clear that a practical controller must be able to cope with uncertainty in the plant parameters, including the time delay.

There are two ways to approach the problem of compensating for input time delay. The first approach explicitly recognises the need to predict the future state of the system. To illustrate this, consider a discrete-time time-invariant system $x_{k+1} = f(x_k, u_{k-d})$ where x_k is the state and u_{k-d} is the time-delayed control input. Shifting this d steps ahead, i.e. $x_{k+d+1} = f(x_{k+d}, u_k)$, it becomes obvious that in order to affect the state at $k+d+1$ it would be desirable to have knowledge of the future state at $k+d$. A rigorous argument for the case of linear time invariant plants with a single input delay is found in (Mirkin and Raskin, 2003). Examples of predictor-based control laws for linear and nonlinear plants are (Manitius and Olbrot, 1979) (Abidi, 2014) and (Abidi and Postlethwaite, 2019) respectively.

The second approach, commonly known as Artstein's model reduction (Artstein, 1982), applies specifically to linear, possibly time-varying systems with input delay. A state substitution is used to transform the original system into an equivalent delay-free system, for which it is easier to derive a control law. The control law for the former can then be found by reversing the substitution. This in fact leads to the same control laws as the first approach, but it simplifies the derivation in the general case where there may be multiple lumped and/or distributed input delays. As this paper will show, it is also a useful starting point for generalising to the case of uncertain plant parameters and time delay.

To illustrate the method with a simple example from (Richard, 2003), consider the linear system with input delay $\dot{x}(t) = Ax(t) + Bu(t-d)$. Introducing a state substitution

$$z(t) = x(t) + \int_{t-d}^{t} e^{A(t-d-\tau)} Bu(\tau) d\tau \qquad (1)$$

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yields the equivalent delay-free system in $z(t)$ given by

$$\dot{z}(t) = Az(t) + e^{-Ad}Bu(t) \quad (2)$$

The control law for the latter is simply a state-feedback law $u(t) = Kz(t)$ where K is chosen to stabilise $(A, e^{-Ad}B)$. The control law for the original system is then found by substituting the definition of $z(t)$ to get

$$u(t) = Kx(t) + K\int_{t-d}^{t} e^{A(t-d-\tau)}Bu(\tau)d\tau \quad (3)$$

To cope with uncertainty in the plant parameters, including the time delay, two paradigms are available. The first paradigm is to make the controller robust to the uncertainty. In delay-independent truncated predictor feedback (Wei and Lin, 2018), a state-feedback controller is used with gains selected by a Lyapunov equation based method which does not require knowledge of the delay. However, this work assumes plant parameters are known, and for unstable plants the amount of delay that the method can handle is limited (Wei and Lin, 2017). A more generally applicable but mathematically involved alternative is to employ the framework of robust control theory (Zhong, 2006).

The other paradigm to consider is adaptive control. Early work on adaptive controllers for time delay systems only addressed uncertainty in the parameters but not the time delay (Ortega and Lozano, 1988)(Niculescu and Annaswamy, 2003). The reason is that adaptive laws rely on the plant representation being linear in the uncertain parameters, whereas the time delay appears inside the argument of the control input. Krstic (Krstic, 2009) overcomes this by expressing the plant dynamics in terms of the entire input history over the delay interval (given by the function $u(x,t)$ where x parameterises a point on the interval), and modelling the delay as a transport PDE. Thus, the time delay can be estimated along with other plant parameters, and used to compute a predictor-based control law (Bresch-Pietri and Krstic, 2009). However, the resulting adaptive laws for both time delay and parameter estimation are complicated.

Most time-delay controllers have been formulated in a continuous-time setting, but they will almost certainly be implemented on a digital computer. Discretisation of continuous-time control laws, especially those for time-delay compensation, is fraught with numerical pitfalls (Mirkin, 2004)(Zhong, 2004). It may be more straightforward to design controllers in discrete-time, for example the discrete-time APC (Abidi et al., 2017) (Abidi and Xu, 2015). This is a model-reference adaptive controller that achieves reference trajectory tracking on a plant with an unknown, constant, upper-bounded time delay. However, the model-tracking error does not vanish asymptotically, and its bound is dependent on the mismatch between the delay upper-bound assumed by the controller and the true delay. The adaptive law also contains parameters that may be difficult to tune in practice.

This paper proposes a discrete-time adaptive regulator for a scalar, linear time-invariant system with an unknown, constant input time delay that has a known upper-bound, without explicitly estimating the time delay. Using an approach similar to Artstein's model reduction, a state substitution is devised that allows the plant dynamics to be expressed in a delay-free form, which facilitates derivation of a control law. In order to apply the model reduction technique to the case of unknown time delay, the plant dynamics and the state substitutes are expressed in a manner that is 'agnostic' to the specific value of the time delay. This also makes it possible to estimate the plant parameters using recursive least squares, even without knowledge of the time delay. A stability analysis shows that the proposed regulator drives the plant state to zero asymptotically.

2 PROBLEM DEFINITION

Consider the scalar system in continuous-time with input delay given as

$$\dot{x}(t) = ax(t) + bu(t-\tau) \quad (4)$$

where the state is $x \in \mathbb{R}$, the input is $u \in \mathbb{R}$, the system parameters $a, b \in \mathbb{R}$ are uncertain parameters, and the constant time delay $\tau \in \mathbb{R}$ is uncertain but has a known upper-bound, τ_p, such that $\tau \leq \tau_\mathrm{p}$.

Sampling this system at uniform time intervals T (where in general the time delay τ may not be an integer multiple of T) gives a discrete-time system described by

$$x_{k+1} = \phi x_k + \gamma_1 u_{k-d} + \gamma_2 u_{k-d-1} \quad (5)$$

where $\phi, \gamma_1, \gamma_2 \in \mathbb{R}$ are uncertain parameters, and $d \in [0, p] \subset \mathbb{Z}^+$ is the uncertain constant delay known to be at most p time-steps long. It is not necessary to ensure that the sampled system has stable zeros, i.e., if the system is written in the form $u_{k-d} = \frac{1}{\gamma_1}(-\gamma_2 u_{k-d-1} + x_{k+1} - \phi x_k)$ then the ratio $\frac{\gamma_2}{\gamma_1}$ need not be inside the unit-circle.

Assumption 1: The upper-bound on the delay in time-steps, p, satisfies $pT \leq \tau_\mathrm{p} \leq (p+1)T$.

Assumption 2: There exists a $\phi_{\min} > 0$ such that $\phi \geq \phi_{\min}$.

Assumption 3: There exists a $\gamma_{\min} > 0$ such that $\gamma_1 + \gamma_2 \geq \gamma_{\min}$.

The regulation problem is to find a bounded control input u_k which will drive the system state to zero asymptotically, i.e. $\lim_{k\to\infty} x_k = 0$, while keeping all system signals bounded.

Remark 1. *S*ince the parameter ϕ is computed as $\phi = e^{aT}\forall a \in \mathbb{R}$ then it is reasonable to assume a positive lower bound such that $0 < \phi_{\min} \leq \phi$.

3 MAIN RESULT

In this section the design of the control law and adaptive law is presented which is then followed by a rigorous stability analysis of the system.

3.1 Adaptive Regulator

Consider the system (5) expressed in the form

$$x_{k+1} = \phi x_k + \sum_{i=0}^{p+1} \psi_i u_{k-i} = \theta^\top \zeta_k \tag{6}$$

where $\theta^\top \triangleq \begin{bmatrix}\phi & \psi_0 & \cdots & \psi_{p+1}\end{bmatrix} \in \mathbb{R}^{p+3}$ is the augmented parameter vector and $\zeta_k^\top \triangleq \begin{bmatrix}x_k & u_k & \cdots & u_{k-p-1}\end{bmatrix} \in \mathbb{R}^{p+3}$ is the augmented signal vector. The parameters $\psi_i \in \mathbb{R}$ are defined as

$$\psi_i = \begin{cases} \gamma_1 & i = d \\ \gamma_2 & i = d+1 \\ 0 & \text{otherwise} \end{cases} \qquad i \in [0, p+1] \tag{7}$$

Defining the variables η_k and $\hat{\eta}_k$ such that

$$\eta_{k+1} = x_{k+1} + \sum_{i=1}^{p+1} \hat{\beta}_{i,k} u_{k-i+1} \tag{8}$$

and

$$\hat{\eta}_k = x_k + \sum_{i=1}^{p+1} \hat{\beta}_{i,k} u_{k-i} \tag{9}$$

where $\hat{\beta}_{i,k} \in \mathbb{R}$. Consider now the system (6), adding and subtracting the term $\hat{\phi}_k x_k$ on the right-hand-side, it is obtained that

$$x_{k+1} = \tilde{\phi}_k x_k + \hat{\phi}_k x_k + \sum_{i=0}^{p+1} \psi_i u_{k-i} \tag{10}$$

where $\hat{\phi}_k$ is the estimate of ϕ and $\tilde{\phi}_k \triangleq \phi - \hat{\phi}_k$ is the estimation error. Substitution of (9) and (10) in (8) and adding and subtracting the term $\hat{\beta}_{0,k} u_k$ on the right-hand-side results in the delay free dynamics of the form

$$\begin{aligned}
\eta_{k+1} &= \tilde{\phi}_k x_k + \hat{\phi}_k \hat{\eta}_k + \hat{\beta}_{0,k} u_k - \hat{\beta}_{0,k} u_k - \hat{\phi}_k \sum_{i=0}^{p+1} \hat{\beta}_{i,k} u_{k-i} \\
&\quad + \sum_{i=0}^{p+1} \psi_i u_{k-i} + \sum_{i=1}^{p+1} \hat{\beta}_{i,k} u_{k-i+1} \\
&= \tilde{\phi}_k x_k + \sum_{i=0}^{p+1} \psi_i u_{k-i} - \left(\hat{\beta}_{0,k} - \hat{\beta}_{1,k}\right) u_k - \sum_{i=1}^{p} \Big(\hat{\phi}_k \hat{\beta}_{i,k} \\
&\quad - \hat{\beta}_{i+1,k}\Big) u_{k-i} - \hat{\phi}_k \hat{\beta}_{p+1,k} u_{k-p-1} + \hat{\phi}_k \hat{\eta}_k + \hat{\beta}_{0,k} u_k \\
&= \tilde{\phi}_k x_k + \sum_{i=0}^{p+1} \psi_i u_{k-i} - \sum_{i=0}^{p+1} \hat{\psi}_{i,k} u_{k-i} + \hat{\phi}_k \hat{\eta}_k + \hat{\beta}_{0,k} u_k
\end{aligned} \tag{11}$$

where the adaptive parameters $\hat{\psi}_{i,k}$ are the estimates of ψ_i and are defined as

$$\hat{\psi}_{i,k} = \begin{cases} \hat{\beta}_{0,k} - \hat{\beta}_{1,k} & i = 0 \\ \hat{\phi}_k \hat{\beta}_{i,k} - \hat{\beta}_{i+1,k} & i \in [1, p] \\ \hat{\phi}_k \hat{\beta}_{i,k} & i = p+1 \end{cases} \tag{12}$$

Note that the parameters $\hat{\beta}_{i,k} \forall i \in [0, p+1]$ are computed from the adaptative parameters as

$$\hat{\beta}_{i,k} = \begin{cases} \displaystyle\sum_{j=0}^{p+1} \hat{\phi}_k^{-j} \hat{\psi}_{j,k} & i = 0 \\ \displaystyle\sum_{j=i}^{p+1} \hat{\phi}_k^{-j+i-1} \hat{\psi}_{j,k} & i \in [1, p+1] \end{cases} \tag{13}$$

which is obtained by rearranging (12). Finally, defining $\tilde{\psi}_{i,k} \triangleq \psi_i - \hat{\psi}_{i,k}$ such that (11) is written as

$$\begin{aligned}
\eta_{k+1} &= \tilde{\phi}_k x_k + \sum_{i=0}^{p+1} \tilde{\psi}_{i,k} u_{k-i} + \hat{\phi}_k \hat{\eta}_k + \hat{\beta}_{0,k} u_k \\
&= \tilde{\theta}_k^\top \zeta_k + \hat{\phi}_k \hat{\eta}_k + \hat{\beta}_{0,k} u_k
\end{aligned} \tag{14}$$

where $\tilde{\theta}_k \triangleq \theta - \hat{\theta}_k$ is the augmented parameter estimation error vector and $\hat{\theta}_k^\top \triangleq \begin{bmatrix}\hat{\phi}_k & \hat{\psi}_{0,k} & \cdots & \hat{\psi}_{p+1,k}\end{bmatrix} \in \mathbb{R}^{p+3}$ is the augmented adaptive parameter vector, then from (14), the control law is selected as

$$u_k = -\hat{\beta}_{0,k}^{-1} \hat{\phi}_k \hat{\eta}_k \tag{15}$$

With the formulation of the control law completed, the adaptive law can be derived.

Remark 2. *Note that the inverses of $\hat{\beta}_{0,k}$ and $\hat{\phi}_k$ are needed in (15) and (13), therefore, the adaptive law must ensure that those terms are non-singular.*

Proceeding with the adaptive law derivation, consider the system (14) and the control law (15). Substitution of the control law results in the closed-loop dynamics

$$\eta_{k+1} = \tilde{\theta}_k^\top \zeta_k \tag{16}$$

From (16), the adaptive law is selected as

$$\hat{\theta}_k = \begin{cases} \mathcal{L}\left[\hat{\theta}_{k-1} + P_k \zeta_{k-1} \eta_k\right] & \forall k \in (k_0, \infty) \\ \hat{\theta}_{k_0} & \forall k \in [0, k_0] \end{cases} \tag{17}$$

$$P_k = \begin{cases} P_{k-1} - \dfrac{P_{k-1}\zeta_{k-1}\zeta_{k-1}^\top P_{k-1}}{1+\zeta_{k-1}^\top P_{k-1}\zeta_{k-1}} & \forall k \in (k_0, \infty) \\ P_{k_0} & \forall k \in [0, k_0] \end{cases} \tag{18}$$

where $k_0 \geq 0$ is the initial time-step, $P_k \in \mathbb{R}^{p+3 \times p+3}$ is the symmetric positive-definite covariance matrix. The purpose of the operator $\mathcal{L}[\cdot]$ is to keep $\hat{\phi}_k$ and $\hat{\beta}_{0,k}$ non-singular. The definition of the operator $\mathcal{L}[\cdot]$ for $\hat{\phi}_k$ is given as

$$\mathcal{L}[\hat{\phi}_k] = \begin{cases} \hat{\phi}_k & \hat{\phi}_k \in (\phi_{\min}, \infty) \\ \phi_{\min} & \hat{\phi}_k \in (-\infty, \phi_{\min}] \end{cases} \tag{19}$$

Before proceeding with the definition of the operator $\mathcal{L}[\cdot]$ for $\left[\hat{\psi}_{0,k} \ \cdots \ \hat{\psi}_{p+1,k}\right]$, consider (13) and note that for $i = 0$

$$\hat{\beta}_{0,k} = \sum_{j=0}^{p+1} \hat{\phi}_k^{-j} \hat{\psi}_{j,k}. \tag{20}$$

Since $\hat{\phi}_k > \phi_{\min}$ and ideally $\hat{\psi}_{j,k} \geq 0$, then the only possibility for a singular $\hat{\beta}_{0,k}$ is if the adaptive parameters $\hat{\psi}_{j,k} = 0, \forall j \in [0, p+1]$. Therefore, the operator $\mathcal{L}[\cdot]$ will need to be defined under two scenarios. In the first scenario, consider the case that one or more but not all adaptive parameters $\hat{\psi}_{j,k}$ are smaller than or equal to zero, which leads to the $\mathcal{L}[\cdot]$ being defined as

$$\mathcal{L}\left[\hat{\psi}_{j,k}\right] = \begin{cases} \hat{\psi}_{j,k} & \hat{\psi}_{j,k} \in (0, \infty) \\ 0 & \hat{\psi}_{j,k} \in (-\infty, 0] \end{cases} \tag{21}$$

for all $j \in [0, p+1]$. In the second scenario, consider the case when all the adaptive parameters $\hat{\psi}_{j,k}$ are smaller than or equal to zero, then the operator $\mathcal{L}[\cdot]$ is defined as

$$\mathcal{L}\left[\hat{\psi}_{j,k}\right] = \frac{\gamma_{min}}{p+2} \tag{22}$$

for all $j \in [0, p+1]$. The block diagram of the resulting closed-loop system is shown in Fig. 1. Note that in Fig. 1 the delay-operator is represented by q^{-1}.

Remark 3. *Note that since the delay d is uncertain, it is not possible to assign the lower bound on γ_1, γ_2 correctly in the case when all the values $\hat{\psi}_{j,k} \leq 0$. Therefore, the average value of the lower bound $\gamma_{\min}$ is assigned to every $\hat{\psi}_{j,k}$.*

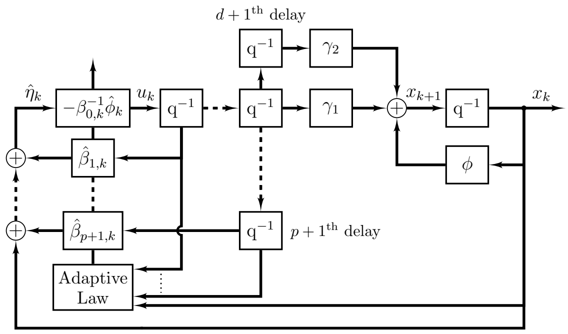

Figure 1: Block diagram depicting the proposed adaptive regulator connected to the system.

3.2 Stability Analysis

In this section, it is shown that the parameter adaptation produces bounded and convergent parameter estimates (*Lemma 1*), that the adaptive system model converges in input-output behaviour to the true system (*Lemma 2*), and that the proposed adaptive regulator drives the system state to zero with vanishing control effort (*Theorem 3*).

Lemma 1. For the system (16) with the adaptive laws (17) and (18) it is true that

$$\lim_{k\to\infty} \frac{\eta_k^2}{1+\zeta_{k-1}^T P_{k-1}\zeta_{k-1}} = 0 \tag{23}$$

Furthermore, it is also true that the parameter estimate $\hat{\theta}_k$ is bounded, hence the parameter estimation error $\tilde{\theta}_k$ is also bounded.

Proof. Consider the positive function

$$V_k = \tilde{\theta}_k^\top P_k^{-1} \tilde{\theta}_k \tag{24}$$

The backward difference ΔV_k is

$$\Delta V_k = V_k - V_{k-1} = \left[\tilde{\theta}_k^\top P_k^{-1}\tilde{\theta}_k - \tilde{\theta}_{k-1}^\top P_{k-1}^{-1}\tilde{\theta}_{k-1}\right] \tag{25}$$

Using the fact that, (Abidi and Xu, 2008),

$$\left(\theta - \mathcal{L}\left[\hat{\theta}_k\right]\right)^\top \left(\theta - \mathcal{L}\left[\hat{\theta}_k\right]\right) \leq \left(\theta - \hat{\theta}_k\right)^\top \left(\theta - \hat{\theta}_k\right) \tag{26}$$

and that if P_k^{-1} is positive-definite then it is obtained that

$$\left(\theta - \mathcal{L}\left[\hat{\theta}_k\right]\right)^\top P_k^{-1} \left(\theta - \mathcal{L}\left[\hat{\theta}_k\right]\right) \leq \left(\theta - \hat{\theta}_k\right)^\top P_k^{-1} \left(\theta - \hat{\theta}_k\right) \tag{27}$$

Substituting the (27) and the adaptative law (17) in

(25) leads to

$$
\begin{aligned}
\Delta V_k &= V_k - V_{k-1} \\
&= \left(\tilde{\theta}_{k-1} - P_k\zeta_{k-1}\eta_k\right)^\top P_k^{-1}\left(\tilde{\theta}_{k-1} - P_k\zeta_{k-1}\eta_k\right) \\
&\quad - \tilde{\theta}_{k-1}^\top P_{k-1}^{-1}\tilde{\theta}_{k-1} \\
&= \tilde{\theta}_{k-1}^\top P_k^{-1}\tilde{\theta}_{k-1} - \tilde{\theta}_{k-1}^\top P_{k-1}^{-1}\tilde{\theta}_{k-1} - \tilde{\theta}_{k-1}^\top P_k^{-1} \\
&\quad \times (P_k\zeta_{k-1}\eta_k) - (P_k\zeta_{k-1}\eta_k)^\top P_k^{-1}\tilde{\theta}_{k-1} \\
&\quad + (P_k\zeta_{k-1}\eta_k)^\top P_k^{-1}(P_k\zeta_{k-1}\eta_k) \\
&= \tilde{\theta}_{k-1}^\top(P_k^{-1} - P_{k-1}^{-1})\tilde{\theta}_{k-1} - 2\tilde{\theta}_{k-1}^\top\zeta_{k-1}\eta_k \\
&\quad + \zeta_{k-1}^\top P_k\zeta_{k-1}\eta_k^2
\end{aligned} \tag{28}
$$

and using the fact that $P_k^{-1} = P_{k-1}^{-1} + \zeta_{k-1}\zeta_{k-1}^\top$, (Abidi et al., 2017), it is obtained that (28) is simplified to the form

$$
\begin{aligned}
\Delta V_k &= \tilde{\theta}_{k-1}^\top\zeta_{k-1}\zeta_{k-1}^\top\tilde{\theta}_{k-1} \\
&\quad - 2\tilde{\theta}_{k-1}^\top\zeta_{k-1}\eta_k + \zeta_{k-1}^\top P_k\zeta_{k-1}\eta_k^2
\end{aligned} \tag{29}
$$

Substituting in the model estimation error dynamics (16) gives

$$
\begin{aligned}
\Delta V_k &= \eta_k^2 - 2\eta_k^2 + \zeta_{k-1}^T P_k\zeta_{k-1}\eta_k^2 \\
&= \eta_k^2[-1 + \zeta_{k-1}^T P_k\zeta_{k-1}]
\end{aligned} \tag{30}
$$

Furthermore, from (18) it is obtained that

$$
\zeta_{k-1}^T P_k\zeta_{k-1} = \frac{\zeta_{k-1}^T P_{k-1}\zeta_{k-1}}{1 + \zeta_{k-1}^T P_{k-1}\zeta_{k-1}} \tag{31}
$$

Substitution of (31) in (30) finally leads to

$$
\begin{aligned}
\Delta V_k &= \eta_k^2\left[-1 + \frac{\zeta_{k-1}^T P_{k-1}\zeta_{k-1}}{1 + \zeta_{k-1}^T P_{k-1}\zeta_{k-1}}\right] \\
&= -\frac{\eta_k^2}{1 + \zeta_{k-1}^T P_{k-1}\zeta_{k-1}}
\end{aligned} \tag{32}
$$

From the result (32) it is evident that ΔV_k is always non-positive and, hence V_k, is non-increasing. Therefore, the parameter estimation error $\tilde{\theta}_k$ and the parameter estimate $\hat{\theta}_k$ are bounded. Furthermore, the value of V_k is the accumulation of changes ΔV_k to its initial value V_{k_0}

$$
V_k = V_{k_0} + \sum_{i=1}^{k-k_0} \Delta V_i \tag{33}
$$

Substituting (32) into (33) gives

$$
V_k = V_{k_0} - \sum_{i=1}^{k-k_0} \frac{\eta_i^2}{1 + \zeta_{i-1}^T P_{i-1}\zeta_{i-1}} \tag{34}
$$

and using the result in (Abidi et al., 2017) it is concluded that

$$
\lim_{k\to\infty} \Delta V_k = \lim_{k\to\infty} \frac{\eta_k^2}{1 + \zeta_{k-1}^T P_{k-1}\zeta_{k-1}} = 0 \tag{35}
$$

□

Lemma 2. From *Lemma 1*, for the signals (8) and (9) the following are true:

(a) $|x_k| \le c_0 \max_{i\in[0,p+1]} |\hat{\eta}_{k-i}|$, for some positive constant c_0.

(b) $|\hat{\eta}_k| \le d_0 + d_1 \max_{i\in[0,p+1]} |\eta_{k-i}|$, for some positive constants d_0 and d_1.

(c) $\lim_{k\to\infty} \hat{\eta}_k = \eta_k$.

(d) $\lim_{k\to\infty} \eta_k = 0$ and $\lim_{k\to\infty} \hat{\eta}_k = 0$.

Proof. Consider the signal (9) re-written as

$$
x_k = \hat{\eta}_k - \sum_{i=1}^{p+1} \hat{\beta}_{i,k} u_{k-i} \tag{36}
$$

Substitution of the control law (15), it is obtained as

$$
x_k = \hat{\eta}_k + \sum_{i=1}^{p+1} \hat{\beta}_{i,k}\hat{\beta}_{0,k-i}^{-1}\hat{\phi}_{k-i}\hat{\eta}_{k-i} \tag{37}
$$

In *Lemma 1* it is established that the adaptive parameters are bounded which implies that $\hat{\beta}_{i,k}\hat{\beta}_{0,k-i}^{-1}\hat{\phi}_{k-i}$ is bounded. Using the boundedness of the adaptive paramters in (37), it is obtained that

$$
\begin{aligned}
|x_k| &\le |\hat{\eta}_k| + \sum_{i=1}^{p+1} |\hat{\beta}_{i,k}\hat{\beta}_{0,k-i}^{-1}\hat{\phi}_{k-i}||\hat{\eta}_{k-i}| \\
&\le c_0 \max_{i\in[0,p+1]} |\hat{\eta}_{k-i}|
\end{aligned} \tag{38}
$$

for some positive constant c_0.

Consider now the difference of the two signals (9) and (8) given as

$$
\begin{aligned}
\hat{\eta}_k &= -\sum_{i=1}^{p+1}\left(\hat{\beta}_{i,k} - \hat{\beta}_{i,k-1}\right)u_{k-i} + \eta_k \\
&= \sum_{i=1}^{p+1}\left(\hat{\beta}_{i,k} - \hat{\beta}_{i,k-1}\right)\hat{\beta}_{0,k-i}^{-1}\hat{\phi}_{k-i}\hat{\eta}_{k-i} + \eta_k
\end{aligned} \tag{39}
$$

Expressing this in augmented form and defining $\bar{\beta}_{i,k} \triangleq \hat{\beta}_{0,k-i}^{-1}\hat{\phi}_{k-i}(\hat{\beta}_{i,k} - \hat{\beta}_{i,k-1})$ such that,

$$
\hat{\bar{\eta}}_k = \begin{bmatrix} \bar{\beta}_{1,k} & \bar{\beta}_{2,k} & \cdots & \bar{\beta}_{p+1,k} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \end{bmatrix} \hat{\bar{\eta}}_{k-1} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \eta_k \tag{40}
$$

where $\hat{\bar{\eta}}_{k-1}^\top \triangleq \left[\hat{\eta}_k \ \hat{\eta}_{k-1} \ \cdots \ \hat{\eta}_{k-p}\right]$. From *Lemma 1* and using the results in (Abidi et al., 2017), $\lim_{k\to\infty} \hat{\beta}_{0,k-i}^{-1}\hat{\phi}_{k-i}(\hat{\beta}_{i,k} - \hat{\beta}_{i,k-1}) = 0$ and that implies that the augmented system (40) is stable and a bound on $\hat{\eta}_k$ exists such that

$$
|\hat{\eta}_k| \le d_0 + d_1 \max_{i\in[0,p+1]} |\eta_{k-i}| \tag{41}
$$

for some positive constants d_0 and d_1.

Going back to the augmented system (40), at steady state when $k \to \infty$, the term $\hat{\beta}_{0,k-i}^{-1}\hat{\phi}_{k-i}(\hat{\beta}_{i,k} - \hat{\beta}_{i,k-1})$ will vanish resulting in an augmented system of the form

$$\hat{\hat{\eta}}_k = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \end{bmatrix} \hat{\hat{\eta}}_{k-1} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \eta_k \qquad (42)$$

which has a solution $\hat{\eta}_k = \eta_k$ when $k \to \infty$.

Finally, given the result in (35), the Key Technical Lemma states that η_k vanishes if there exists positive constants $\mu_0, \mu_1 \in \mathbb{R}$ such that $|\zeta_{k-1}| \leq \mu_0 + \mu_1 \max_{\kappa\in[0,k]} |\eta_\kappa|$. To show that this is indeed true, recall that $\zeta_k^\top \triangleq \begin{bmatrix} x_k & u_k & \cdots & u_{k-p-1} \end{bmatrix}$. It has been established that $|x_k| \leq c_0 \max_{i\in[0,p+1]} |\hat{\eta}_{k-i}|$ and, since, $\hat{\beta}_{0,k-i}^{-1}\hat{\phi}_{k-i}$ is bounded then $|u_k| \leq c_1|\hat{\eta}_k|$. Therefore, $|\zeta_{k-1}| \leq \mu_0 + \mu_1 \max_{\kappa\in[0,k]} |\eta_\kappa|$ for some positive constants μ_0 and μ_1 and, ultimately, $\lim_{k\to\infty} \eta_k = 0$. Furthermore, from (c) it has been established that $\lim_{k\to\infty} \hat{\eta}_k = \eta_k$ resulting in $\lim_{k\to\infty} \hat{\eta}_k = 0$. □

Theorem 1. *The state of the closed-loop system, as well as the control signal, approach zero asymptotically, i.e.* $\lim_{k\to\infty} x_k = \lim_{k\to\infty} u_k = 0$.

Proof. Beginning with the closed-loop controller dynamics (15) and taking the limit of the norm of both sides as $k \to \infty$, it is obtained that

$$\begin{aligned} &\lim_{k\to\infty} |u_k| \\ &= \lim_{k\to\infty} \left| -\hat{\beta}_{0,k}^{-1}\hat{\phi}_k \eta_k \right| \leq \lim_{k\to\infty} \left| \hat{\beta}_{0,k}^{-1}\hat{\phi}_k \right| |\eta_k| \\ &= 0 \end{aligned} \qquad (43)$$

Consider now the expression (8) as $k \to \infty$ which is given as

$$\begin{aligned} \lim_{k\to\infty} |x_k| &\leq \lim_{k\to\infty} |\hat{\eta}_k| + \lim_{k\to\infty} \sum_{i=1}^{p+1} |\hat{\beta}_{i,k} u_{k-i}| \\ &= 0 \end{aligned} \qquad (44)$$

which establishes the asymptotic stability of the state x_k. □

4 SIMULATION EXAMPLE

Consider an unstable system with an unstable zero, given by

$$x_{k+1} = 1.05x_k + 1u_{k-d} + 1.1u_{k-d-1} \qquad (45)$$

where the system delay is set as $d = 0,1,2,3,4,5$ while the upper bound is set as $p = 5$. The initial condition of the system is set at $x_k = 2$.

To investigate the adaptive performance of the regulator with respect to mismatch between the delay upper-bound p and the true delay d, the closed-loop system is simulated using various values of d. The adaptive regulator is initialised with $P_0 = 5 \times I_{p+3\times p+3}$ and $\hat{\theta}_0^\top = \begin{bmatrix} 0.1 & 0.4 & \cdots & 0.4 \end{bmatrix}$. In Fig. 2-4 the results are shown for the convergence of the signals $\eta_k, \hat{\eta}_k$, the control input profile and state regulation performance of the closed-loop system under various values of the delay values. As expected, x_k is regulated to zero asymptotically. As it can be seen from the results, the greater the mismatch between p and d, the longer the settling-time. Note that with increasing system delay, d, the initial divergence grows which is to be expected as the controller cannot react fast enough initially to force the system to converge.

Finally, the system is simulated with a longer system delay, d, and an upper bound set at $p = 10$. The adaptive regulator parameters are kept the same as the previous case while the system delay is increased from $d = 5$ until $d = 10$. The results are shown in Fig. 5-7. The results for this case show a similar performance to the shorter delay case. As it would be expected, the convergence speed increases as the actual delay, d, and the upper bound, p, approach each other.

5 CONCLUSIONS

In this paper, a discrete-time adaptive regulation approach was designed for a scalar system with an unknown, constant time-delay with a known upper-bound. The approach utilized a state substitution the resulted in a delay-free system which simplified the control law design. The approach is also capable of handling systems with unstable zeros. A rigorous stability proof was presented that shows the adaptive control law drives the system state to zero asymptotically. Finally, numerical simulations were shown that illustrate the ability of the adaptive control law to cope with mismatches between the delay upper-bound and the true delay in the system.

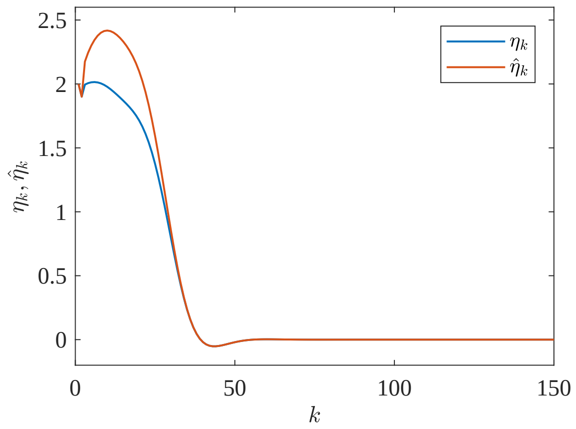

Figure 2: Asymptotic convergence of η_k and $\hat{\eta}_k$ when $d = 5$ and $p = 5$.

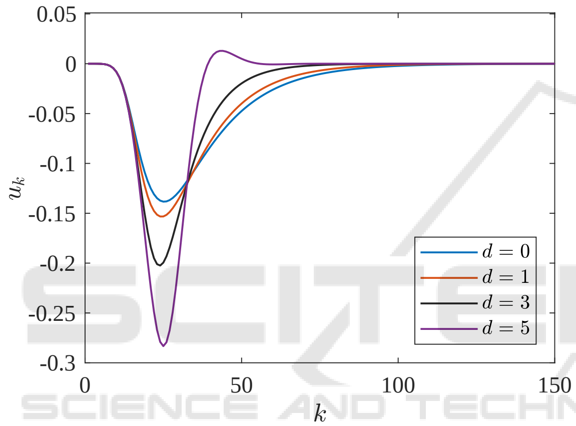

Figure 3: Control input profile on the system when $p = 5$, with various values of the actual system delay d.

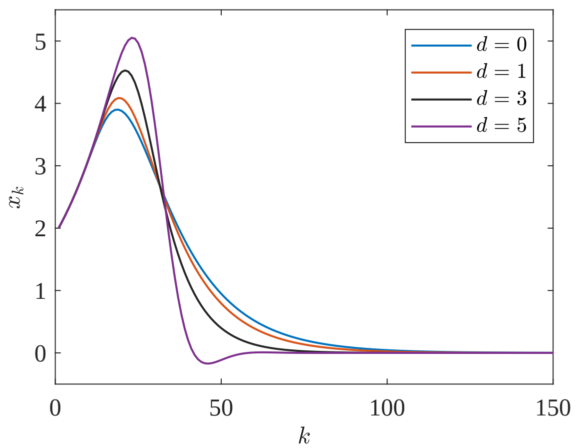

Figure 4: State regulation of the system when $p = 5$, with various values of the actual system delay d.

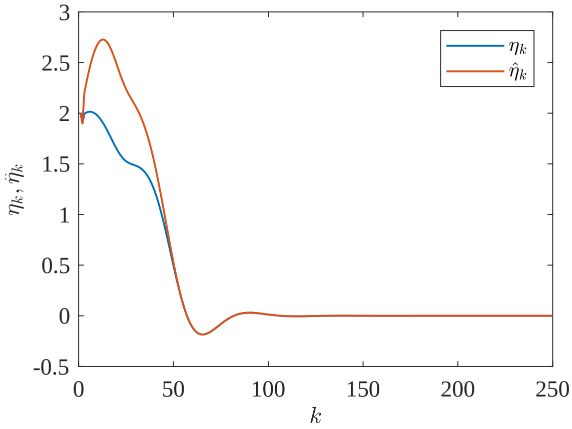

Figure 5: Asymptotic convergence of η_k and $\hat{\eta}_k$ when $d = 10$ and $p = 10$.

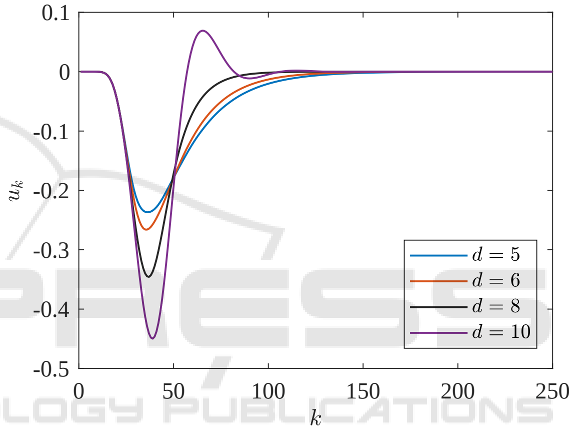

Figure 6: Control input profile on the system when $p = 10$, with various values of the actual system delay d.

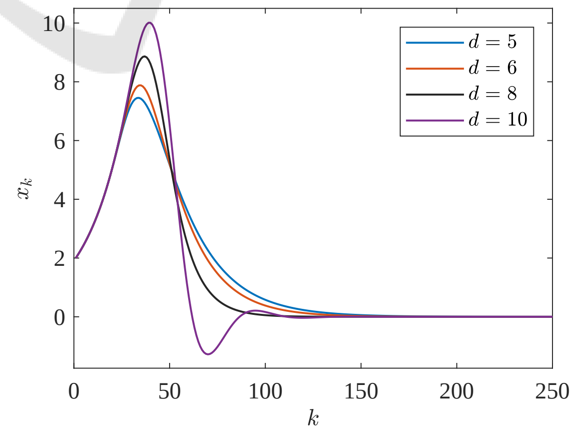

Figure 7: State regulation of the system when $p = 10$, with various values of the actual system delay d.

REFERENCES

Abidi, K. (2014). A robust discrete-time adaptive control approach for systems with almost periodic time-varying parameters. *International Journal of Robust and Nonlinear Control*, 24(1):166–178.

Abidi, K. and Postlethwaite, I. (2019). Discrete-time adaptive control for systems with input time-delay and non-sector bounded nonlinear functions. *IEEE Access*, 7:4327–4337.

Abidi, K. and Xu, J. (2008). A discrete-time periodic adaptive control approach for time-varying parameters with known periodicity. *IEEE Transactions on Automatic Control*, 53(2):575–581.

Abidi, K. and Xu, J. X. (2015). *Advanced Discrete-Time Control: Designs and Applications*. Springer.

Abidi, K., Yildiz, Y., and Annaswamy, A. (2017). Control of uncertain sampled-data systems: An adaptive posicast control approach. *IEEE Transactions on Automatic Control*, 62(5):2597–2602.

Abidi, K., Yildiz, Y., and Korpe, B. E. (2016). Explicit time-delay compensation in teleoperation: An adaptive control approach. *International Journal of Robust and Nonlinear Control*, 26(15):3388–3403.

Artstein, Z. (1982). Linear systems with delayed controls: A reduction. *IEEE Transactions on Automatic Control*, 27(4):869–879.

Bjerkeng, M., Falco, P., Natale, C., and Pettersen, K. Y. (2014). Stability analysis of a hierarchical architecture for discrete-time sensor-based control of robotic systems. *IEEE Transactions on Robotics*, 30(3):745–753.

Bresch-Pietri, D. and Krstic, M. (2009). Adaptive trajectory tracking despite unknown input delay and plant parameters. *Automatica*, 45(9):2074–2081.

Gu, K. and Niculescu, S. (2003). Survey on recent results in the stability and control of time-delay systems. *Journal of Dynamic Systems, Measurement, and Control*, 125(2):158–165.

Krstic, M. (2009). *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Birkhäuser.

Manitius, A. and Olbrot, A. (1979). Finite spectrum assignment problem for systems with delays. *IEEE Transactions on Automatic Control*, 24(4):541–552.

Mirkin, L. (2004). On the approximation of distributed-delay control laws. *Systems & Control Letters*, 51(5):331–342.

Mirkin, L. and Raskin, N. (2003). Every stabilizing dead-time controller has an observer-predictor-based structure. *Automatica*, 39(10):1747–1754.

Niculescu, S. and Annaswamy, A. M. (2003). An adaptive smith-controller for time-delay systems with relative degree $n^* \leq 2$. *Systems & Control Letters*, 49(5):347–358.

Ortega, R. and Lozano, R. (1988). Globally stable adaptive controller for systems with delay. *International Journal of Control*, 47(1):17–23.

Richard, J. (2003). Time-delay systems: An overview of some recent advances and open problems. *Automatica*, 39(10):1667–1694.

Skorina, E. H., Luo, M., Tao, W., Chen, F., Fu, J., and Onal, C. D. (2017). Adapting to flexibility: Model reference adaptive control of soft bending actuators. *IEEE Robotics and Automation Letters*, 2(2):964–970.

Wei, Y. and Lin, Z. (2017). Maximum delay bounds of linear systems under delay independent truncated predictor feedback. *Automatica*, 83:65–72.

Wei, Y. and Lin, Z. (2018). Stability criteria of linear systems with multiple input delays under truncated predictor feedback. *Systems & Control Letters*, 111:9–17.

Zhong, Q. (2004). On distributed delay in linear control laws – part i: discrete-delay implementations. *IEEE Transactions on Automatic Control*, 49(11):2074–2080.

Zhong, Q. (2006). *Robust Control of Time-Delay Systems*. Springer.