Compensation of Mismatched Disturbances for Nonlinear Plants with Distributed Time-delay

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Abstract: The paper deals with the robust algorithm for compensation of unknown mismatched disturbances depending on a state vector of plants with distributed time-delay. The algorithm based on generalization of a backstepping method and disturbance compensation method. The proposed control system compensates disturbances with required accuracy. The simulation results illustrate the efficiency and robustness of the suggested control system. System". MIMO System".

1 INTRODUCTION

The paper is devoted to a new robust scheme for compensation of unknown mismatched disturbances. It is known, that the backstepping method is effective method for control of plant under mismatched condition.

The first backstepping method is proposed in (Kokotovic, 1992). In more detail the backstepping method is presented in (Fradkov, Miroshnik, and Nikiforov, 1999; Khalil, 2002). Currently there are a lot of modification of the backstepping algorithm.

In (Chang and Cheng, 2010) a methodology of designing the block backstepping controller for a class of multi-input systems with mismatched perturbations is proposed. Some adaptive mechanisms are embedded both in the virtual input controller and in the backstepping controllers such that some part knowledge of the upper bound of perturbation is not required.

The paper (Ma, Schilling, and Schmid, 2005) is concerned with the adaptive sliding-mode control of a class of nonlinear systems in nonlinear parametric-pure-feedback form with mismatched uncertainties. Backstepping design procedure is applied, which leads to a new adaptive sliding-mode control. Gaussian radial-basis-function networks are used to approximate the unknown system dynamics. More nodes are added to the networks progressively in order to improve the transient behaviour. With ideal sliding mode, asymptotic stability is reached.

In (Xu and Min, 2010) for a class of strict-feedback nonlinear systems with mismatched uncertainties, an adaptive backstepping fuzzy controller design is presented. By applying backstepping design strategy and online approaching uncertainties with fuzzy approximator, the control inputs and adaptive tuning rules are derived from the Lyapunov stability theory. To deal with the problem of extreme expanded operation quantity of backstepping method, a nonlinear tracking differentiator is introduced. By choosing suitable design parameters, the developed control scheme guarantees that all the signals of the closed-loop system are uniformly ultimately bounded and the system tracking error can reach to a very small region around zero.

However, the above mentoined algorithms cannot compensate disturbances with distributed time-delay. For disturbances compensation there are a lot of methods, for example (Bobtsov and Kremlev, 2005; Tsykunov, 2007). The main idea of

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disturbances compensation consists representation of uncontrollable disturbances as a new function in the control system which we can use for design of the control system. Among of them the auxiliary loop algorithm (Tsykunov, 2007.) is one of the effectiveness algorithm. The auxiliary loop algorithm based on parallel reference model (auxiliary loop) to the plant. The auxiliary loop is used for obtaining the uncertainties acting on the plant. The idea of this method consists in implementing an auxiliary loop with desired parameters parallel to the plant. The difference between the output of the plant and the output of the auxiliary loop gives a function which depends on parametric and external disturbances. This function gives the control law that guarantees required accuracy of the control system. The proposed algorithm provides the tracking by the output of the plant of the reference output with the required accuracy. However, mismatched disturbances may control system get unstable. It will be demonstrated in Section 4.

In this paper we propose a new modified backstepping algorithm with mismatched disturbances compensation (MBADC), where disturbances are presented nonlinear functions depending on external bounded disturbances, parametric uncertainties and state vector with distributed time-delay. This algorithm is a generalization of results (Khalil, 2002; Tsykunov, 2007) for robust control of nonlinear plants under mismatched parametric uncertainties and external disturbances. It is assumed that input signal and state vector of the plant are available for measurement. The proposed algorithm guarantees stabilization by the plant state vector with the required accuracy.

The paper is organized as follows. The problem statement is presented in Section 2. The MBADC for control of nonlinear plants under mismatched perturbations is proposed in Section 3. In Section 4 the efficient of MBADC is illustrated by modeling of an unstable nonlinear plant. Also, in Section 4 the comparison of the simulation results for the MBADC, the backstepping algorithm and auxiliary loop algorithm are presented. Concluding remarks are given in Section 5. Appendix A gives the proof of the MBADC.

2 PROBLEM STATEMENT

Consider the plant model with distributed time-delay in the form

$$\begin{aligned}
&\dot{x}_1(t) = x_2(t) \\
&+\varphi_1\left(x_2(t),\dots,x_n,\ \int_{-h_{11}}^{0} x_2(t+\theta)d\theta,\dots,\ \int_{-h_{1n}}^{0} x_n(t+\theta)d\theta,\ f,t \right), \\
&\dot{x}_2(t) = x_3(t) \\
&+\varphi_2\left(x_3(t),\dots,x_n(t),\ \int_{-h_{21}}^{0} x_3(t+\theta)d\theta,\dots,\ \int_{-h_{2n}}^{0} x_n(t+\theta)d\theta,\ f,t \right), \\
&\vdots \\
&\dot{x}_n(t) = bu(t) + \varphi_2(\ f,t),
\end{aligned} \qquad (1)$$

where $x(t) = [x_1(t), x_1(t), \dots, x_n(t)]^{\mathrm{T}}$ is a state vector, $u(t) \in R$ is an input, $\varphi_i(\cdot) \in R$, $i = 1, 2, \dots, n$ are unknown functions depending on parametric uncertainties and external disturbances, f is a vector of unknown parameters, $b > 0$ is unknown constant and $[f^{\mathrm{T}}, b]^{\mathrm{T}} \in \Xi$, Ξ is a known bounded set, h_{ij}, $i, j = 1, 2, \dots, n$ are unknown time delay.

We assume that signal $x(t)$ is available for measurement.

Additionally assume that the functions $\varphi_i(\cdot) \in R$, $i = 1, 2, \dots, n$ are bounded or bounded on t and f, and Lipchitz in $x(t)$, $\int_{-h_{ij}}^{0} x_2(t+\theta)d\theta$, $\int_{-h_{ij}}^{0} x_2(t+\theta)d\theta$, ..., $\int_{-h_{ij}}^{0} x_n(t+\theta)d\theta$, $i, j = 1, 2, \dots, n$.

The problem is to synthesis a control law such that the goal condition holds

$$|x(t)| < \delta \ \text{ for } t > T, \qquad (2)$$

where $\delta > 0$ is a prespecified required accuracy, $T > 0$ is a transient time, $|\cdot|$ is Euclidean norm of the corresponding vector.

3 MAIN RESULT

Let us denote

$$\varphi_i = \varphi_i\left(x(t),\ \int_{-h_{i1}}^{0} x_1(t+\theta)d\theta,\dots,\ \int_{-h_{in}}^{0} x_n(t+\theta)d\theta,\ f,t \right).$$

The synthesis of the control system is split into n steps. Auxiliary loops and auxiliary controls will are designed on $1, \dots, n-1$ steps for compensation of unknown function φ_i. The control law $u(t)$ will be introduced on the n-th step.

Step 1. Introduce the first auxiliary loop in the form

$$\dot{z}_1(t) = -c_1 z_1(t) + x_2(t), \qquad (3)$$

where $c_1 > 0$ is a coefficient chosen by a designer.

Taking into account the first equation of system (1) and equation (3), rewrite the function $\xi_1(t) = x_1(t) - z_1(t)$ as follows

$$\dot{\xi}_1(t) = c_1 z_1(t) + \varphi_1. \tag{4}$$

From (4) it follows that the function $\varphi_1(\cdot)$ may be rewritten in the form

$$\varphi_1 = \dot{\xi}_1(t) - c_1 z_1(t). \tag{5}$$

Substituting (5) to the first equation of (1), we have

$$\dot{x}_1(t) = x_2(t) + \dot{\xi}_1(t) - c_1 z_1(t). \tag{6}$$

Assume that the function $x_2(t)$ is a control signal in (6) and define $x_2(t)$ in the form $x_2(t) = v_1(t)$. Since the function $\dot{\xi}_1(t)$ is not available for measurement, we introduce the first auxiliary control law $v_1(t)$ as follows

$$v_1(t) = -c_1\xi_1(t) - \hat{\dot{\xi}}_1(t), \tag{7}$$

where $\hat{\dot{\xi}}_1(t)$ is an estimate of the function $\dot{\xi}_1(t)$. Substituting (7) to (6), we get

$$\dot{x}_1(t) = -c_1 x_1(t) + \eta_1(t), \tag{8}$$

where $\eta_1(t) = \dot{\xi}_1(t) - \hat{\dot{\xi}}_1(t)$ is an error estimate.

Step i $(2 \le i \le n-1)$. Since $v_i(t)$ is not real control law, we consider the i-th error function $e_{i-1}(t) = x_i(t) - v_{i-1}(t)$ as follows

$$\dot{e}_{i-1}(t) = x_{i+1}(t) + \tilde{\varphi}_i(t), \tag{9}$$

where $\tilde{\varphi}_i(t) = \varphi_i - \dot{v}_{i-1}(t)$.

Introduce the i-th auxiliary loop in the form

$$\dot{z}_i(t) = -c_i z_i(t) + x_i(t), \tag{10}$$

where $c_i > 0$ is a coefficient chosen by a designer. Taking into account (9) and (10), rewrite the function $\xi_i(t) = e_{i-1}(t) - z_i(t)$ as follows

$$\dot{\xi}_i(t) = c_i z_i(t) + \tilde{\varphi}_i(t). \tag{11}$$

From (11) it follows that the function $\tilde{\varphi}_i(t)$ may be rewritten in the form

$$\tilde{\varphi}_i(t) = \dot{\xi}_i(t) - c_i z_i(t). \tag{12}$$

Substituting (12) to (9), we have

$$\dot{e}_{i-1}(t) = x_{i+1}(t) + \dot{\xi}_i(t) - c_i z_i(t). \tag{13}$$

Assume that function $x_{i+1}(t)$ is a control signal in (13) and define $x_{i+1}(t)$ in the form $x_{i+1}(t) = v_i(t)$. Since the function $\dot{\xi}_i(t)$ is not available for measurement, we introduce the second auxiliary control law $v_i(t)$ as follows

$$v_i(t) = -c_i\xi_i(t) - \hat{\dot{\xi}}_i(t), \tag{14}$$

where $\hat{\dot{\xi}}_i(t)$ is an estimate of the function $\dot{\xi}_i(t)$. Substituting (14) to (13), we get

$$\dot{e}_{i-1}(t) = -c_i e_{i-1}(t) + \eta_i(t), \tag{15}$$

where $\eta_i(t) = \dot{\xi}_i(t) - \hat{\dot{\xi}}_i(t)$ is an error estimate.

Step n. Since $v_n(t)$ is not real control law, consider the $n-1$-th error function $e_{n-1}(t) = x_n(t) - v_{n-1}(t)$ in the form

$$\dot{e}_{n-1}(t) = bu(t) + \tilde{\varphi}_n(t), \tag{16}$$

where $\tilde{\varphi}_n(t) = \varphi_n - \dot{v}_{n-1}(t)$.

Introduce the n-th auxiliary loop in the form

$$\dot{z}_n(t) = -c_n z_n(t) + \alpha u(t), \tag{17}$$

where $c_n > 0$ and $\alpha > 0$ are coefficients chosen by a designer.

Taking into account (16) and (17), rewrite the function $\xi_n(t) = e_{n-1}(t) - z_n(t)$ as follows

$$\dot{\xi}_n(t) = c_n z_n(t) + \tilde{\varphi}_n(t) + (b-\alpha)u(t), \tag{18}$$

From (18) it follows that the function $\tilde{\varphi}_n(t)$ may be rewritten in the form

$$\tilde{\varphi}_n(t) = \dot{\xi}_n(t) - c_n z_n(t). \tag{19}$$

Substituting (19) to (16), we get

$$\dot{e}_{n-1}(t) = \dot{\xi}_n(t) - c_n z_n(t) + \alpha u(t). \tag{20}$$

Since the function $\dot{\xi}_n(t)$ is not available for measurement, we introduce the control law $u(t)$ as follows

$$u(t) = -\frac{1}{\alpha}\left(c_n\xi_n(t) + \hat{\dot{\xi}}_n(t)\right), \tag{21}$$

where $\hat{\dot{\xi}}_n(t)$ is an estimate of the function $\dot{\xi}_n(t)$. Substituting (21) to (20), we get

$$\dot{e}_{n-1}(t) = -c_n e_{n-1}(t) + \eta_n(t), \tag{22}$$

where $\eta_n(t) = \dot{\xi}_n(t) - \hat{\dot{\xi}}_n(t)$ is an error estimate.

The signals $\dot{\xi}_i(t)$, $i = 1, 2, ..., n$ are not available for measurement, because it depends on derivatives of $x(t)$. Therefore, we introduced the estimate functions $\hat{\dot{\xi}}_i(t)$, $i = 1, 2, ..., n$ of the functions $\dot{\xi}_i(t)$,

$i = 1, 2, ..., n$ at each step. For implementation of $\dot{\hat{\xi}}_i(t)$, $i = 1, 2, ..., n$ use the following observers

$$\dot{\hat{\xi}}_{i-1}(t) = -\mu^{-1}\hat{\xi}_{i-1}(t) + \mu^{-1}\dot{\xi}_{i-1}(t),\ i = 1, 2, \ldots, n, \quad (23)$$

where $\mu > 0$ is enough small number.

Theorem: There exist constants $c_i > 0$, $i = 1, 2, ..., n$ and $\mu_0 > 0$ such that for $\mu \le \mu_0$ the control system consisting of auxiliary loops (3), (10), (17), auxiliary control laws (7), (14), control law (21), observers (23) provides goal (2) for plant (1).

The proof of Theorem is given in Appendix.

It is shown that the proposed algorithm based on multi-agent system design, where each equation is associated as appropriate agent. The additional investigation have shown that the proposed algorithm is effective for dynamical networks with unknown distributed time-delays and mismatched parametric and external disturbances.

4 EXAMPLE

Consider the plant model with distributed time-delay in the form

$$\dot{x}_1(t) = x_2(t) + \varphi_1\left(x_2(t), \int_{-h_{12}}^{0} x_2(t+\theta)d\theta,\ f, t\right), \quad (24)$$
$$\dot{x}_2(t) = bu(t) + \varphi_2(f, t),$$

where $f = [f_1, f_2]^{\mathrm{T}}$ The set Ξ is defined by the following inequalities:

$$-10 \le f_1 \le 10,\ -10 \le f_2 \le 10.$$

The problem is to synthesis the control system providing goal condition (2).

Let us design the control system. Consider auxiliary loops in the following form

$$\dot{z}_1 = -c_1 z_1 + x_2,\ \ \dot{z}_2 = -c_2 z_2 + \alpha u, \quad (25)$$

where $c_1 = c_2 = 1$ and $\alpha = 1$.

Introduce the auxiliary control law $v_1(t)$ and the control law $u(t)$ as follows

$$v_1 = -c_1\xi_1 - \dot{\hat{\xi}}_1,\ \ u = -\frac{1}{\alpha}\left(c_2\xi_2 + \dot{\hat{\xi}}_2\right), \quad (26)$$

where $\xi_1(t) = x_1(t) - z_1(t)$, $\xi_2(t) = e_1(t) - z_2(t)$ and $e_1(t) = x_2(t) - v_1(t)$.

Introduce the observers in the forms

$$\dot{\hat{\xi}}_1(t) = \frac{p}{\mu p + 1}\xi_1(t),\ \ \dot{\hat{\xi}}_2(t) = \frac{p}{\mu p + 1}\xi_2(t), \quad (27)$$

where $p = d / dt$, $\mu = 0.01$.

Let all initial conditions be zero in the control system.

Let us choose the functions of (24) as follows

$$\varphi_1 = f_1(1 + 0.5\sin t)x_1(t)\int_{-h_{12}}^{0} x_1(t+\theta)d\theta,\ \ h_{12} = 1, \quad (28)$$
$$\varphi_2 = f_2\left[\left(x_1^2(t) + x_2^2(t)\right)\sin 1.3t + \int_{-h_{22}}^{0} x_2(t+\theta)d\theta,\ \ h_{22} = 2.\right.$$

For comparison we also consider the synthesis of control systems using the backstepping algorithm (Khalil, 2002) and the auxiliary loop algorithm (Tsykunov, 2007). According to (Khalil, 2002) the backstepping algorithm is presented by the following equations

$$v_1 = -c_1 x_1 - \varphi_1,$$
$$u = -c_2 e_1 - b^{-1}\left[\varphi_2 + c_1(x_2 + \varphi_1) + \dot{\varphi}_1\right], \quad (29)$$

where $e_1(t) = x_2(t) - v_1(t)$. Algorithm (29) depends on functions φ_i, $i = 1, 2$, therefore, implementations of (29) requires that functions φ_i, $i = 1, 2$ must be known.

According to (Tsykunov, 2007), the auxiliary loop algorithm is presented by the following equations:

equation of the auxiliary loop

$$\dot{x}_a(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_a(t) + u(t), \quad (30)$$

equation of the control law

$$u(t) = \begin{bmatrix} -1 & -2 \end{bmatrix}\varepsilon(t) - \frac{p}{\mu p + 1}\begin{bmatrix} 0 & 1 \end{bmatrix}\varepsilon(t), \quad (31)$$

where $\varepsilon(t) = x(t) - x_a(t)$. Algorithm (30), (31) can compensate only function φ_2 and coefficient b and cannot compensate function φ_1. Therefore, algorithm (30), (31) may be unstable for appropriate values of the function φ_1.

Consider two cases.

Case 1. Let $f_1 = 1$ and $f_2 = 1$ in (24) and functions φ_i, $i = 1, 2$ are known.

Case 2. Let $f_1 = 2$ and $f_2 = 3$ in (24) and functions φ_i, $i = 1, 2$ are unknown.

In Fig. 1-6 the transients are presented for the state vector $x(t)$ which is obtained by proposed MBADC, the backstepping algorithm and the auxiliary loop algorithm for each of two cases. In Fig.1-6 black curve and red curve correspond to the signals $x_1(t)$ and $x_2(t)$ respectively.

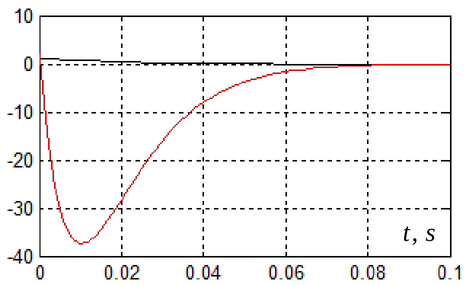

Figure 1: The transients of the vector state $x(t)$ with MBADC for case 1.

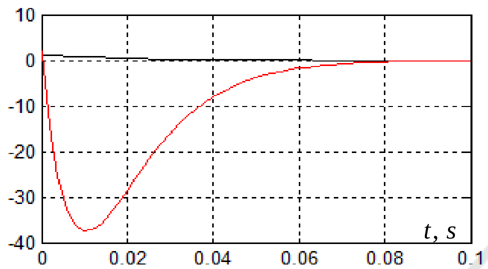

Figure 2: The transients of the vector state $x(t)$ with MBADC for case 2.

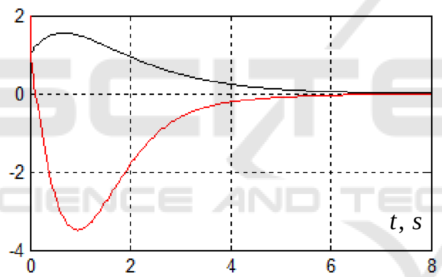

Figure 3: The transients of the vector state $x(t)$ with the backstepping algorithm for case 1.

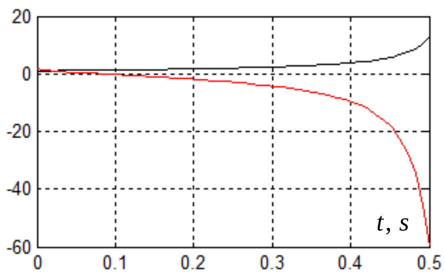

Figure 4: The transients of the vector state $x(t)$ with the backstepping algorithm for case 2.

The analysis of simulation results shows that the proposed MBADC guaranties the fulfillment of goal (2). The backstepping algorithm is stable if all functions in (24) are known, but the control system is sensitive to parametric uncertainty and external disturbances.

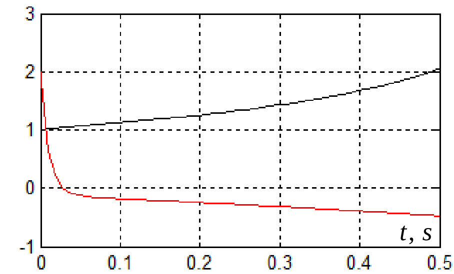

Figure 5: The transients of the vector state $x(t)$ with the auxiliary loop algorithm for case 1.

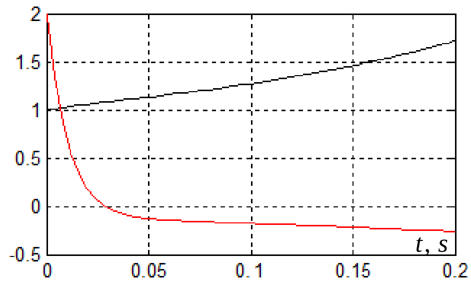

Figure 6: The transients of the vector state $x(t)$ with the auxiliary loop algorithm for case 2.

The auxiliary loop algorithm may be unstable even we known all functions in the plant model.

We also can note that the proposed algorithm (25)-(27) compensates parametric uncertainties and external disturbances with the required accuracy $\delta = 0.04$ achieved after 1 s for any uncertainties from the set Ξ. Additional investigation under saturated control input is shown that the proposed algorithm is stable while algorithms (Khalil, 2002; Tsykunov, 2007) lose stability. These results are similar for multi-agent systems.

5 CONCLUSIONS

The paper describes the robust algorithm for compensation of mismatched disturbances with distributed time-delay. The synthesis of control system based on the backstepping algorithm and the auxiliary loop algorithm. The proposed algorithm guarantees stabilization of the plant state vector with the required accuracy. We also compare the proposed algorithm with the backstepping algorithm and auxiliary loop algorithm. The simulation results illustrate the efficiency and robustness of the suggested control system.

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APPENDIX

Lemma (Furtat, 2014; Furtat and Gushchin, 2019). Let the system be described by the following differential equation

$$\dot{x} = f(x, \mu_1, \mu_2, t), \tag{32}$$

where $x \in R^{s_1}$, $\mu = \text{col}(\mu_1, \mu_2) \in R^{s_2}$, $f(x, \mu_1, \mu_2, t)$ is Lipchitz continuous function in x. Let (34) have a bounded closed set of attraction $\Omega = \{x \mid P(x) \le C\}$ for $\mu_2 = 0$, where $P(x)$ is piecewise-smooth, positive definite function in R^{s_1}. In addition let there exist some numbers $C_1 > 0$ and $\bar{\mu}_1 > 0$ such that the following condition holds

$$\sup_{|\mu_1| \le \bar{\mu}_1} \left[\left\langle \left[\frac{\partial P(x)}{\partial x} \right]^{\mathrm{T}}, f(x, \mu_1, 0, t) \right\rangle \middle| \; P(x) = C \right] \le -C_1.$$

Then there exists $\mu_0 > 0$ such that the system (32) has the same set of attraction Ω for $\mu_2 \le \mu_0$.

Proof of Theorem. Taking into account (30), rewrite the equations for the error estimates $\eta_i(t) = \dot{\xi}_i(t) - \hat{\dot{\xi}}_i(t)$, $i = 1, 2, ..., n$ as follows

$$\dot{\eta}_i(t) = -\mu^{-1}\eta_i(t) + \ddot{\xi}_i(t), \; i = 1, 2, ..., n. \tag{33}$$

Rewrite (8), (15), (22) and (32) as the following system

$$\begin{aligned} &\dot{x}_1(t) = -c_1 x_1(t) + \eta_1(t), \\ &\dot{e}_j(t) = -c_{j+1} e_j(t) + \eta_{j+1}(t), \quad j = 1, 2, \ldots, n-1, \\ &\mu_1 \dot{\eta}_i(t) = -\eta_i(t) + \mu_2 \ddot{\xi}_i(t), \; i = 1, 2, \ldots, n, \end{aligned} \tag{34}$$

where $\mu_1 = \mu_2 = \mu$. To analyze system (34) the following Lemma is needed.

Let us check conditions of Lemma. Consider system (34) for $\mu_2 = 0$. Let $P(x) = V(t)$, where $V(t)$ is Lyapunov function defined in the form

$$V(t) = 0.5 x_1^2(t) + 0.5 \sum_{j=1}^{n-1} e_i^2(t) + 0.5 \sum_{i=1}^{n} \eta_i^2(t). \tag{35}$$

Take the derivative of $V(t)$ along the trajectories (34), we get

$$\begin{aligned} &\dot{V}(t) = -c_1 x_1^2(t) + x_1(t)\eta_1(t) \\ &+ \sum_{j=1}^{n-1} \left[-c_{j+1} e_j^2(t) + e_j(t)\eta_{j+1}(t) \right] - \sum_{i=1}^{n} \mu_1^{-1} \eta_i^2(t). \end{aligned} \tag{36}$$

Find upper bounds for the fourth term of (36):

$$e_j \eta_{j+1} \le 0.5 \mu_0^{-1} e_j^2 + 0.5 \mu_0 \eta_{j+1}^2, \quad j = 1, 2, \ldots, n-1. \tag{37}$$

Substituting (37) to (36), we get

$$\dot{V} \le -c_1 x_1^2 - \sum_{j=1}^{n-1} \hat{c}_{j+1} e_j^2 - \mu_1^{-1} \eta_1^2 - \sum_{i=2}^{n} \hat{d}_i \eta_i^2, \quad (38)$$

where $\hat{c}_{j+1} = c_{j+1} - 0.5\mu_0^{-1}$ and $\hat{d}_i = \mu_1^{-1} - 0.5\mu_0$. Obviously, there exist coefficients c_{j+1}, $j = 1, 2, ..., n\text{-}1$, μ_1, and μ_0 such that $\hat{c}_{j+1} > 0$, $\hat{d}_i > 0$ and system (34) is asymptotically stable.

Taking into account to (35), rewrite (38) as follows

$$\dot{V}(t) \le -\beta V(t), \quad (39)$$

where $\beta = 2 \min \{c_1, \hat{c}_2, ..., \hat{c}_n, \mu_1^{-1}, \hat{d}_2, ..., \hat{d}_n\}$.

Solving inequality (46) with respect to $V(t)$, we get

$$V(t) \le V(0) e^{-\beta t}. \quad (40)$$

From (40) it follows that solutions of system (33) are exponentially tend to zero.

Proof boundendess of all signals in the closed-loop system.

Taking into account (23), rewrite the first equation of (4) in the following form

$$\xi_1(t) = \frac{(p + c_1)(\mu_1 p + 1)}{\mu_1 p^2} x_1(t).$$

Since the function $x_1(t)$ is asymptotically stable than the functions $\xi_1(t)$, $\dot{\xi}_1(t)$, $\ddot{\xi}_1(t)$ and $\int_{-h_{1i}}^{0} x_2(t+\theta) d\theta$, $i = 1, 2, ..., n$ are bounded. From boundedness of $\eta_1(t)$ it follows that the signal $\hat{\xi}_1(t)$ is bounded. Taking into account (23) and system (34), the proof of boundedness of the signals $\xi_i(t)$, $\dot{\xi}_i(t)$, $\ddot{\xi}_i(t)$ and $\hat{\xi}_i(t)$, $i = 2, 3, ..., n$ is same. Therefore, from (7), (14) and (21) the signals $v_i(t)$, $i = 1, 2, ..., n-1$ and $u(t)$ are bounded. Hence, the function $x(t)$ is bounded. From (3), (10) and (17) it follows that the functions $z_i(t)$, $i = 1, 2, ..., n$ are bounded. Therefore, the functions φ_i and $\tilde{\varphi}_i$ are bounded. Consequently, all signals in the closed-loop system are bounded.

According to Lemma there exist $\mu_0 > 0$ such that for $\mu_1 \le \mu_0$ and $\mu_2 \le \mu_0$ the attraction set is the same as for $\mu_2 = 0$. However, system (34) is not asymptotically stable for $\mu_2 \ne 0$. It will be has some attraction set. Let us find the set of attraction of system (34) for $\mu_2 \ne 0$. Taking into account result (38), take derivative in time of (35) along trajectories (33) for $\mu_1 = \mu_2 = \mu_0$

$$\dot{V} \le -c_1 x_1^2 - \sum_{j=1}^{n-1} \hat{c}_{j+1} e_j^2 - \mu_0^{-1} \eta_1^2 - \sum_{i=2}^{n} \tilde{d}_i \eta_i^2 + \sum_{l=1}^{n} \eta_l \ddot{\xi}_l, \quad (41)$$

where $\hat{d}_i = \mu_0^{-1} - 0.5\mu_0$.

Use the following upper bounds:

$$\eta_l \ddot{\xi}_l \le 0.5\mu_0^{-1} \eta_l^2(t) + 0.5\mu_0 \ddot{\xi}_l^2 \le 0.5\mu_0^{-1} \eta_l^2(t) + \mu_0 \chi, \quad (42)$$

where $\chi = 0.5 \sup_{t,\, l=1, 2, ..., n} \{\ddot{\xi}_l^2\}$.

Taking into account (42), rewrite (41) in the form

$$\dot{V} \le -c_1 x_1^2 - \sum_{j=1}^{n-1} \hat{c}_{j+1} e_j^2 - 0.5\mu_0^{-1} \eta_1^2 - d \sum_{i=2}^{n} \eta_i^2 + n\mu_0 \chi, \quad (43)$$

where $d = \mu_0^{-1} - \mu_0$

Taking into account (35), rewrite (43) as follows

$$\dot{V}(t) \le -\gamma V(t) + n\mu_0 \chi, \quad (44)$$

where $\gamma = 2 \min \{c_1, \hat{c}_2, ..., \hat{c}_n, 0.5\mu_0^{-1}, d\}$.

Solving inequality (44) with respect to $V(t)$, we get

$$V(t) \le e^{-\gamma t} V(0) + \left(1 - e^{-\gamma t}\right) \gamma^{-1} n\mu_0 \chi. \quad (45)$$

From (45) we can note that goal (2) holds. The theorem is proved.