Signal Estimation with Random Parameter Matrices and Time-correlated Measurement Noises

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Abstract: This paper is concerned with the least-squares linear estimation problem for a class of discrete-time networked systems whose measurements are perturbed by random parameter matrices and time-correlated additive noise, without requiring a full knowledge of the state-space model generating the signal process, but only information about its mean and covariance functions. Assuming that the measurement additive noise is the output of a known linear system driven by white noise, the time-differencing method is used to remove this time-correlated noise and recursive algorithms for the linear filtering and fixed-point smoothing estimators are obtained by an innovation approach. These estimators are optimal in the least-squares sense and, consequently, their accuracy is evaluated by the estimation error covariance matrices, for which recursive formulas are also deduced. The proposed algorithms are easily implementable, as it is shown in the computer simulation example, where they are applied to estimate a signal from measured outputs which, besides including time-correlated additive noise, are affected by the missing measurement phenomenon and multiplicative noise (random uncertainties that can be covered by the current model with random parameter matrices). The computer simulations also illustrate the behaviour of the filtering estimators for different values of the missing measurement probability.

1 INTRODUCTION

The signal estimation problem in networked systems is usually based on measurements that are perturbed not only by additive noises, but also by other stochastic disturbances from multiple sources, which might be caused, for example, by the degradation of the measuring devices or the presence of multiplicative noises. Such random disturbances are usually inherent to the network itself (network-induced uncertainties) and they must be taken into account in the design of both the observation models and the estimation algorithms, so as to get proper estimations. For this reason, the study of the estimation problem in this kind of systems with one or several network-induced uncertainties has become a hot research topic over the last years (see e.g. (Gao and Chen, 2014), (Chen et al., 2015), (Tian et al., 2016), (Caballero-Águila et al., 2017), (Zhao et al., 2018), (Liu et al., 2018) and (Yang et al., 2019)).

The introduction of random parameter matrices in the mathematical model of the measured outputs provides a global frame to deal with some of the most common network-induced uncertainties (multiplicative noise, sensor gain degradation or missing measurements, among others). This fact is boosting the rise of several estimation algorithms in networked systems with random parameter matrices under different assumptions about the noises and the processes involved (see e.g. (Yang et al., 2016), (Sun et al., 2017), (Wang and Zhou, 2017), (Caballero-Águila et al., 2018), (Han et al., 2018) and (Caballero-Águila et al., 2019)).

Another relevant issue when addressing the estimation problem is the presence of non-white additive noises perturbing the sensor measurements. Conventional estimation algorithms usually provide accurate estimations when the additive noise is either white or correlated on a finite-time interval. However, we often come across more general situations involving sequentially time-correlated measurement noise, which is usually the output of a linear system with white

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noise. The design of recursive estimation algorithms in this class of systems with time-correlated noise is addressed, for instance, in (Liu, 2016), (Li et al., 2017) or (Liu et al., 2017) and the time-differencing approach is conventionally applied to deal with this kind of noise correlation.

Bearing these considerations in mind, this paper addresses the least-squares linear filtering and fixed-point smoothing problems of discrete-time stochastic signals under the following conditions:

Covariance Information. The state-space model generating the signal process is not necessarily known and only information about its mean and covariance functions are required in its stead.

Random Parameter Matrices. The measured outputs can be corrupted by random uncertainties, which are modelled in a comprehensive way by random parameter matrices.

Time-correlated Measurement Noises. The additive noise in the measurements is described by a discrete-time linear model perturbed by white noise.

The use of the time-differencing approach allows us to transform the original measured data into an equivalent set of transformed measurements, that are perturbed by white noise, and the problem is then reduced to find the optimal estimators based on this new set of observations. For this purpose, the innovation approach is used and recursive algorithms are derived for the filtering and fixed-point smoothing problems.

The remainder of this paper is divided into four sections. The measurement model and the problem formulation are described in Section 2, where the time-differencing approach is also detailed. Section 3 discusses the derivation of the filtering and fixed-point smoothing algorithms, based on the innovation technique. Section 4 illustrates the application of such algorithms by a numerical simulation example, where the performance of the estimators is assessed in terms of their error variances. Finally, some concluding remarks are included in Section 5.

2 PROBLEM FORMULATION

The goal of this paper is to find a recursive algorithm for the least-squares (LS) linear filtering and fixed-point smoothing estimators of a discrete-time random signal from noisy measurements which are perturbed by random parameter matrices and time-correlated additive noise vectors. The estimation problem will be addressed using covariance information; that is, we will assume that the evolution model of the signal to be estimated is unknown and only information about the mean and covariance functions of the signal and the processes involved in the observation model are available.

More precisely, consider the following observation model with random parameter matrices:

$$z_k = H_k x_k + v_k, \quad k \geq 1, \tag{1}$$

where, at each sampling time k, $x_k \in \mathbb{R}^{n_x}$ is the signal vector to be estimated, $z_k \in \mathbb{R}^{n_z}$ is the measurement vector at time k. The following assumptions on the processes involved in (1) are required:

(i) $\{x_k\}_{k\geq 1}$, the signal process, has zero mean and its covariance function is factorized as

$$E\left[x_k x_s^T\right] = \Lambda_k \Psi_s^T, \quad s \leq k,$$

where, for $k \geq 1$, $\Lambda_k, \Psi_k \in \mathbb{R}^{n_x \times n}$ are known matrices.

(ii) $\{H_k\}_{k\geq 1}$ is a sequence of independent random parameter matrices, whose entries $h_{k,pq}$, for $p = 1,\ldots,n_z$ and $q = 1,\ldots,n_x$, have known means, $E[h_{k,pq}]$, and second-order moments, $E[h_{k,pq}h_{k,p'q'}]$, for $p,p' = 1,\ldots,n_z$ and $q,q' = 1,\ldots,n_x$.

(iii) $\{v_k\}_{k\geq 1}$ is a zero-mean time-correlated noise process such that

$$v_k = A_{k-1}v_{k-1} + \beta_{k-1}, \quad k \geq 1, \tag{2}$$

where $\{A_k\}_{k\geq 0}$ are known matrices, $\{\beta_k\}_{k\geq 0}$ is a zero-mean white noise with covariance matrices $E[\beta_k \beta_k^T] = B_k$, and v_0 is a random vector with zero mean and covariance matrix V_0.

(iv) The random vector v_0 and the processes $\{x_k\}_{k\geq 1}$, $\{H_k\}_{k\geq 1}$ and $\{\beta_k\}_{k\geq 0}$ are mutually independent.

2.1 Time-differencing Approach

To address the LS estimation problem from the measured outputs (1) perturbed by time-correlated additive noise, such measurements are transformed with the aim of removing the correlated noise. For that purpose, the time-differencing approach (Liu, 2016) is used to define a new set of measurements, whose additive noise is not time-correlated, according to the following model:

$$y_k = z_k - A_{k-1}z_{k-1}, \quad k \geq 2; \qquad y_1 = z_1. \tag{3}$$

Then, if we substitute (1) into the above equation and we use (2) for v_k, we obtain

$$\begin{aligned} &y_k = H_k x_k - A_{k-1}H_{k-1}x_{k-1} + \beta_{k-1}, \quad k \geq 2; \\ &y_1 = z_1. \end{aligned} \tag{4}$$

Remark 1. Note that, since y_h, for $h = 2,\ldots,L$, is obtained from z_{h-1} and z_h, the sets $\{y_1,\ldots,y_L\}$ and $\{z_1,\ldots,z_L\}$ are equivalent in the sense that they can be obtained one from the other by linear transformations. Consequently, LS linear estimator of x_k based on the original measurements $\{z_1,\ldots,z_L\}$ given in (1) is just equal to that based on the new measurements $\{y_1,\ldots,y_L\}$ given in (4). So, in order to address this estimation problem, the first and second-order statistical properties of the process $\{y_k\}_{k\geq 1}$ are necessary.

Remark 2. The assumptions on the model guarantee that the process $\{y_k\}_{k\geq 1}$ given in (4) has zero mean and its covariance matrices $\Sigma_k^y \equiv E\left[y_k y_k^T\right]$, $k \geq 1$, satisfy

$$\begin{array}{l} \Sigma_k^y = \Sigma_{k,k}^{Hx} - \Sigma_{k,k-1}^{Hx} A_{k-1}^T - A_{k-1}\Sigma_{k-1,k}^{Hx} \\ \quad +A_{k-1}\Sigma_{k-1,k-1}^{Hx} A_{k-1}^T + B_k, \;\; k \geq 2; \\ \Sigma_1^y = \Sigma_1^{Hx} + A_0 V_0 A_0^T + B_0, \end{array} \tag{5}$$

where $\Sigma_{k,s}^{Hx} \equiv E\left[H_k x_k x_s^T H_s^T\right]$ is given by

$$\Sigma_{k,s}^{Hx} = E[H_k \Lambda_k \Psi_s^T H_s^T] = \begin{cases} \overline{H}_k \Lambda_k \Psi_s^T \overline{H}_s^T, & s < k, \\ E[H_k \Lambda_k \Psi_k^T H_k^T], & s = k, \end{cases} \tag{6}$$

with $\overline{H}_k \equiv E[H_k]$, $k \geq 1$, and the (p,p')-th entries of the above matrices $E[H_k \Lambda_k \Psi_k^T H_k^T]$ are calculated by

$$\left(E\left[H_k \Lambda_k \Psi_k^T H_k^T\right]\right)_{pp'} = \sum_{q=1}^{n_x}\sum_{q'=1}^{n_x} E[h_{k,pq} h_{k,p'q'}](\Lambda_k \Psi_k^T)_{qq'}, \; p,p' = 1,\ldots,n_z.$$

3 LS LINEAR ESTIMATION ALGORITHMS

Given the set of measurements $\{y_1,\ldots,y_L\}$ in (4), the aim in this Section is to design recursive algorithms for the optimal estimators $\widehat{x}_{k/L}$ within the class of linear estimators of the signal x_k, based on these measurements, using the LS optimality criterion. Specifically, recursive algorithms for the filter, $\widehat{x}_{k/k}$, and smoother, $\widehat{x}_{k/k+N}$, at the fixed point k for any $N \geq 1$, will be obtained.

3.1 Preliminary Results

Since the LS linear estimator of the signal, $\widehat{x}_{k/L}$, is the orthogonal projection of the signal x_k over the linear space spanned by the observations $\{y_1,\ldots,y_L\}$, which generally are non-orthogonal vectors, to obtain the estimation algorithms we will use a innovation approach (Kailath et al., 2000). According to such approach, the observation process $\{y_k\}_{k\geq 1}$ is transformed into an equivalent one, named *innovation process*, of orthogonal vectors $\{\mu_k\}_{k\geq 1}$, defined by $\mu_k = y_k - \widehat{y}_{k/k-1}$, where $\widehat{y}_{k/k-1}$ is the orthogonal projection of y_k onto the linear space generated by $\{\mu_1,\ldots,\mu_{k-1}\}$, with $\widehat{y}_{1/0} = E[y_1] = 0$.

Therefore, the LS linear estimator, $\widehat{\alpha}_{k/L}$, of any random vector α_k based on the observations $\{y_1,\ldots,y_L\}$, can be calculated as a linear combination of the corresponding innovations, $\{\mu_1,\ldots,\mu_L\}$. Namely, denoting $\Pi_h = E[\mu_h \mu_h^T]$, the LS linear estimator $\widehat{\alpha}_{k/L}$ is expressed as the following linear combination of the innovations:

$$\widehat{\alpha}_{k/L} = \sum_{h=1}^{L} E[\alpha_k \mu_h^T]\Pi_h^{-1}\mu_h. \tag{7}$$

3.1.1 Observation Predictor

Starting from expression (4) of the observations y_k, and taking into account that H_{k-1} is correlated with the innovation μ_{k-1}, to obtain the predictor $\widehat{y}_{k/k-1}$ the observations y_k are rewritten as follows:

$$y_k = H_k x_k - A_{k-1}\overline{H}_{k-1} x_{k-1} + W_{k-1}, \;\; k \geq 2, \tag{8}$$

where $W_k = \beta_k - A_k\left(H_k - \overline{H}_k\right)x_k$, $k \geq 1$.

Then, according to the projection theory, we have:

$$\widehat{y}_{k/k-1} = \overline{H}_k \widehat{x}_{k/k-1} - A_{k-1}\overline{H}_{k-1}\widehat{x}_{k-1/k-1} + \widehat{W}_{k-1/k-1}.$$

Now, to obtain the estimator $\widehat{W}_{k/k}$ we use the general expression (7). Since H_k is independent of $\mu_1,\ldots,\mu_{k-1}$, it is easy to see that $E[W_k \mu_h^T] = 0$, for $h < k$, hence, denoting $\mathcal{W}_k \equiv E\left[W_k \mu_k^T\right]$, from (7) we have that $\widehat{W}_{k/k} = \mathcal{W}_k \Pi_k^{-1}\mu_k$, $k \geq 1$. So, the observation predictor $\widehat{y}_{k/k-1}$ satisfy:

$$\begin{array}{l} \widehat{y}_{k/k-1} = \overline{H}_k \widehat{x}_{k/k-1} - A_{k-1}\overline{H}_{k-1}\widehat{x}_{k-1/k-1} \\ \quad + \mathcal{W}_{k-1}\Pi_{k-1}^{-1}\mu_{k-1}, \;\; k \geq 2. \end{array} \tag{9}$$

Now, we derive an expression for $\mathcal{W}_k = E\left[W_k \mu_k^T\right]$. Since $E[W_k \mu_h^T] = 0$, for $h < k$, it is clear that $\mathcal{W}_k = E\left[W_k y_k^T\right]$. Using (8) for y_k, we have that $E\left[W_k y_k^T\right] = E\left[W_k x_k^T H_k^T\right]$, and, from the definition of W_k, we obtain that $\mathcal{W}_k = E\left[W_k \mu_k^T\right]$ is calculated by

$$\mathcal{W}_k = -A_k\left(\Sigma_{k,k}^{Hx} - \overline{H}_k \Lambda_k \Psi_k^T \overline{H}_k^T\right), \;\; k \geq 1, \tag{10}$$

where $\Sigma_{k,k}^{Hx}$ is given in (6).

3.2 LS Linear Filtering Algorithm

The starting points to derive the proposed LS linear recursive algorithm for the filtering estimators, $\widehat{x}_{k/k}$, are the general expression (7) for the estimators as linear combination of the innovations, along with expression (9) for the one-stage observation predictor. Also, a recursive formula for the filtering error covariance matrices, $\Sigma_{k/k} \equiv E\left[(x_k - \widehat{x}_{k/k})(x_k - \widehat{x}_{k/k})^T\right]$, is obtained in this Section. These covariance matrices are used to assess the accuracy of the filtering estimators $\widehat{x}_{k/k}$ when the LS criterion is used.

3.2.1 Signal Filtering Estimators

From the general expression (7), obtaining the signal filter, $\widehat{x}_{k/k} = \sum_{h=1}^{k} \mathcal{X}_{k,h}\Pi_h^{-1}\mu_h,\ k \geq 1$, requires calculating the coefficients

$$\mathcal{X}_{k,h} \equiv E\left[x_k\mu_h^T\right] = E\left[x_k y_h^T\right] - E\left[x_k\widehat{y}_{h/h-1}^T\right],\ 1 \leq h \leq k.$$

Expression (4) for y_h and the separable form of the signal covariance, specified in assumption *(i)*, lead easily to

$$\begin{array}{l} E[x_k y_h^T] = \Lambda_k \left(\overline{H}_h\Psi_h - A_{h-1}\overline{H}_{h-1}\Psi_{h-1}\right)^T,\ \ k \geq 2; \\ E[x_k y_1^T] = \Lambda_k\overline{H}_1\Psi_1. \end{array} \tag{11}$$

From now on, the following operator will be used for notational simplicity:

$$\begin{array}{l} \overline{\mathcal{H}}_{\Upsilon_k} \equiv \overline{H}_k\Upsilon_k - A_{k-1}\overline{H}_{k-1}\Upsilon_{k-1},\ \ k \geq 2; \\ \overline{\mathcal{H}}_{\Upsilon_1} \equiv \overline{H}_1\Upsilon_1, \end{array} \tag{12}$$

and it will be applied to the matrices $\Upsilon_k = \Lambda_k$ and $\Upsilon_k = \Psi_k$ that define the signal covariance function (see assumption (i)).

Then, from (11) and (12), it is clear that

$$E\left[x_k y_h^T\right] = \Lambda_k\overline{\mathcal{H}}_{\Psi_h}^T,\ 1 \leq h \leq k,$$

and, using (9) for $\widehat{y}_{h/h-1}$, together with (7) for $\widehat{x}_{h/h-1}$ and $\widehat{x}_{h-1/h-1}$, the following expression for the filter coefficients is obtained:

$$\begin{array}{l} \mathcal{X}_{k,h} = \Lambda_k\overline{\mathcal{H}}_{\Psi_h}^T - \sum_{j=1}^{h-1} \mathcal{X}_{k,j}\Pi_j^{-1}\Big(\overline{H}_h\mathcal{X}_{h,j} + A_{k-1}\overline{H}_{h-1}\mathcal{X}_{h-1,j}\Big)^T \\ \qquad -\mathcal{X}_{k,h-1}\Pi_{h-1}^{-1}\mathcal{W}_{h-1}^T, \quad 2 \leq h \leq k; \\ \mathcal{X}_{k,1} = \Lambda_k\overline{\mathcal{H}}_{B_1}^T. \end{array}$$

Hence, if we define a function $\mathcal{E}_h$ satisfying

$$\begin{array}{l} \mathcal{E}_h = \overline{\mathcal{H}}_{\Psi_h}^T - \sum_{j=1}^{h-1} \mathcal{E}_j\Pi_j^{-1}\mathcal{E}_j^T\overline{\mathcal{H}}_{\Lambda_h}^T - \mathcal{E}_{h-1}\Pi_{h-1}^{-1}\mathcal{W}_{h-1}^T,\ \ h \geq 2; \\ \mathcal{E}_1 = \overline{\mathcal{H}}_{\Psi_1}^T, \end{array}$$

we obtain that the coefficients $\mathcal{X}_{k,h}$ can be expressed as follows:

$$\mathcal{X}_{k,h} = \Lambda_k\mathcal{E}_h, \quad 1 \leq h \leq k.$$

So, by defining the vectors

$$e_k \equiv \sum_{h=1}^{k} \mathcal{E}_h\Pi_h^{-1}\mu_h,\ \ k \geq 1,$$

and using (7) for $\widehat{x}_{k/k}$, it is clear that the signal filtering estimators are given by

$$\widehat{x}_{k/k} = \Lambda_k e_k, \quad k \geq 1.$$

Now, using this expression and defining

$$K_k^e \equiv E\left[e_k e_k^T\right] = \sum_{h=1}^{k} \mathcal{E}_h\Pi_h^{-1}\mathcal{E}_h^T,\ \ k \geq 1,$$

a formula for the filtering error covariance matrix $\Sigma_{k/k}$ is derived. Actually, from the Orthogonal Projection Lemma (OPL), we rewrite $\Sigma_{k/k} = E\left[x_k x_k^T\right] - E\left[\widehat{x}_{k/k}\widehat{x}_{k/k}^T\right]$; then, since $E\left[\widehat{x}_{k/k}\widehat{x}_{k/k}^T\right] = \Lambda_k K_k^e\Lambda_k^T$, using assumption (i) for $E\left[x_k x_k^T\right]$, we have

$$\Sigma_{k/k} = \Lambda_k\left(\Psi_k - \Lambda_k K_k^e\right)^T, \quad k \geq 1.$$

Bearing in mind the preceding results, using the OPL to write the innovation covariance matrix as

$$\Pi_k = E[y_k y_k^T] - E[\widehat{y}_{k/k-1}\widehat{y}_{k/k-1}^T],$$

and taking into account that $E[\widehat{y}_{k/k-1}e_{k-1}^T] = \overline{\mathcal{H}}_{\Psi_k}^T - \mathcal{E}_k$, the recursive filtering algorithm below can be deduced without trouble.

3.2.2 Recursive Filtering Algorithm

Under hypotheses (i)-(iv), the LS linear filter, $\widehat{x}_{k/k}$, and the corresponding error covariance matrix, $P_{k/k}$, are given by

$$\begin{array}{c} \widehat{x}_{k/k} = \Lambda_k e_k,\ \ k \geq 1 \\ \Sigma_{k/k} = \Lambda_k\left(\Psi_k - \Lambda_k K_k^e\right)^T,\ \ k \geq 1, \end{array}$$

where the vectors e_k and the matrices $K_k^e = E[e_k e_k^T]$ are recursively obtained from

$$e_k = e_{k-1} + \mathcal{E}_k\Pi_k^{-1}\mu_k,\ \ k \geq 1; \quad e_0 = 0,$$

$$K_k^e = K_{k-1}^e + \mathcal{E}_k\Pi_k^{-1}\mathcal{E}_k^T,\ \ k \geq 1;\ \ K_0^e = 0,$$

and the matrices $\mathcal{E}_k = E[e_k\mu_k^T]$ satisfy

$$\begin{array}{l} \mathcal{E}_k = \overline{\mathcal{H}}_{\Psi_k}^T - K_{k-1}^e\overline{\mathcal{H}}_{\Lambda_k}^T - \mathcal{E}_{k-1}\Pi_{k-1}^{-1}\mathcal{W}_{k-1}^T,\ \ k \geq 2; \\ \mathcal{E}_1 = \overline{\mathcal{H}}_{\Psi_1}^T. \end{array}$$

The innovations, μ_k, are given by

$$\begin{array}{l} \mu_k = y_k - \overline{\mathcal{H}}_{\Lambda_k}e_{k-1} - \mathcal{W}_{k-1}\Pi_{k-1}^{-1}\mu_{k-1},\ \ k \geq 2; \\ \mu_1 = y_1, \end{array}$$

and the innovation covariance matrices, Π_k, are obtained by

$$\begin{array}{l} \Pi_k = \Sigma_k^y - \overline{\mathcal{H}}_{\Lambda_k}(\overline{\mathcal{H}}_{\Psi_k}^T - \mathcal{E}_k) \\ \qquad -\mathcal{W}_{k-1}\Pi_{k-1}^{-1}\left(\overline{\mathcal{H}}_{\Lambda_k}\mathcal{E}_{k-1} + \mathcal{W}_{k-1}\right)^T,\ \ k \geq 2; \\ \Pi_1 = \Sigma_1^y. \end{array}$$

The vectors y_k are given in (3), and the matrices Σ_k^y and $\mathcal{W}_k$ are calculated by expressions (5) and (15), respectively. Finally, the notations $\overline{\mathcal{H}}_{\Lambda_k}$ and $\overline{\mathcal{H}}_{\Psi_k}$ are defined in (12).

3.3 LS Linear Smoothing Algorithm

The goal in this Section is to derive a recursive algorithm for the LS linear smoothing estimators, $\widehat{x}_{k/k+N}$, at the fixed point k, for any $N \geq 1$, as well as, a recursive formula for the smoothing error covariance matrices, $\Sigma_{k/k+N} \equiv E\left[(x_k - \widehat{x}_{k/k+N})(x_k - \widehat{x}_{k/k+N})^T\right]$.

3.3.1 Recursive Fixed-point Smoothing Algorithm

Under hypotheses (i)-(iv), for each $k \geq 1$, the LS linear fixed-point smoothers, $\widehat{x}_{k/k+N}$, $N \geq 1$, are calculated by

$$\widehat{x}_{k/k+N} = \widehat{x}_{k/k+N-1} + \mathcal{X}_{k,k+N}\Pi^{-1}_{k+N}\mu_{k+N}, \quad N \geq 1, \tag{13}$$

with initial condition given by the filter, $\widehat{x}_{k/k}$.

The matrices $\mathcal{X}_{k,k+N} \equiv E\left[x_k\mu^T_{k+N}\right]$ are recursively obtained by

$$\begin{aligned}\mathcal{X}_{k,k+N} = {} & \left(\Psi_k - \mathcal{M}_{k,k+N-1}\right)\overline{\mathcal{H}}^T_{\Lambda_{k+N}} \\ & -\mathcal{X}_{k,k+N-1}\Pi^{-1}_{k+N-1}\mathcal{W}^T_{k+N-1}, \quad N \geq 1; \\ \mathcal{X}_{k,k} = {} & \Lambda_k\mathcal{E}_k,\end{aligned} \tag{14}$$

where $\mathcal{M}_{k,k+N} \equiv E[x_ke^T_{k+N}]$ satisfy the following recursive formula

$$\begin{aligned}\mathcal{M}_{k,k+N} &= \mathcal{M}_{k,k+N-1} + \mathcal{X}_{k,k+N}\Pi^{-1}_{k+N}\mathcal{E}^T_{k+N}, \quad N \geq 1; \\ \mathcal{M}_{k,k} &= \Lambda_k K^e_k.\end{aligned} \tag{15}$$

The fixed-point smoothing error covariance matrices, $\Sigma_{k/k+N}$, are obtained by

$$\Sigma_{k/k+N} = \Sigma_{k/k+N-1} - \mathcal{X}_{k,k+N}\Pi^{-1}_{k+N}\mathcal{X}^T_{k,k+N}, \quad N \geq 1,$$

with initial condition given by the filtering error covariance matrix $\Sigma_{k/k}$.

3.3.2 Smoothing Algorithm Derivation

Using the general expression (7), the smoothing estimators are written as $\widehat{x}_{k/k+N} = \sum_{h=1}^{k+N} \mathcal{X}_{k,h}\Pi^{-1}_h\mu_h$, $N \geq 1$; hence, it is clear that, starting from the filter, $\widehat{x}_{k/k}$, the fixed-point smoothing estimators are recursively obtained by (13).

To obtain the recursive relation (14) for $\mathcal{X}_{k,k+N} = E\left[x_ky^T_{k+N}\right] - E\left[x_k\widehat{y}^T_{k+N/k+N-1}\right]$, $N \geq 1$, we proceed as follows:

– On the one hand, the assumption (i) together with (12), yield
$$E\left[x_ky^T_{k+N}\right] = \Psi_k\overline{\mathcal{H}}^T_{\Lambda_{k+N}}.$$

– On the other, using that
$$\widehat{y}_{k/k-1} = \overline{\mathcal{H}}_{\Lambda_k}e_{k-1} + \mathcal{W}_{k-1}\Pi^{-1}_{k-1}\mu_{k-1},$$
it is clear that
$$\begin{aligned}E\left[x_k\widehat{y}^T_{k+N/k+N-1}\right] = {} & E\left[x_ke^T_{k+N-1}\right]\overline{\mathcal{H}}^T_{\Lambda_{k+N}} \\ & +\mathcal{X}_{k,k+N-1}\Pi^{-1}_{k+N-1}\mathcal{W}^T_{k+N-1}, \quad N \geq 1.\end{aligned}$$

Therefore, denoting $\mathcal{M}_{k,k+N} = E\left[x_ke^T_{k+N}\right]$, expression (14) holds and, using the recursive relation for the vectors e_k, given in the filtering algorithm, the recursive expression (15) for the matrices $\mathcal{M}_{k,k+N}$ is also straightforward.

Finally, using (13) for the smoothers $\widehat{x}_{k/k+N}$, the recursive formula for the fixed-point smoothing error covariance matrices, $\Sigma_{k/k+N}$, is immediately deduced, and the smoothing algorithm is proven. □

4 COMPUTER SIMULATION RESULTS

This section analyzes a numerical simulation example to illustrate the application of the recursive filtering and fixed-point smoothing algorithms proposed in the current paper.

Specifically, we consider that the signal to be estimated is a scalar signal $\{x_k\}_{k\geq 1}$ which is generated by the following first-order autoregressive model

$$x_{k+1} = 0.95x_k + \omega_k, \quad k \geq 1,$$

where $\{\omega_k\}_{k\geq 1}$ is a zero-mean white Gaussian noise with constant variance $Var[\omega_k] = 0.1$, for all k. The autocovariance function of this signal is

$$E[x_kx_s] = 1.025641 \times 0.95^{k-s}, \quad s \leq k$$

which, in accordance with the assumption (i) on the theoretical model, can be factorized in a semi-degenerate kernel form, taking, for example,

$$\Lambda_k = 1.025641 \times 0.95^k, \quad \Psi_k = 0.95^{-k}.$$

The measured outputs of this signal are assumed to be described by the following model with missing measurements and multiplicative noise:

$$z_k = \theta_k\,(1 + 0.9\varepsilon_k)\,x_k + v_k, \quad k \geq 1,$$

where $\left\{\theta_k\right\}_{k\geq 1}$ is a sequence of independent Bernoulli random variables with constant probabilities, $P\left(\theta_k = 1\right) = \overline{\theta}$. Hence $1 - \overline{\theta}$ is the probability that the signal is absent in the measurements; that is, the missing measurement probability. The multiplicative noise, $\left\{\varepsilon_k\right\}_{k\geq 1}$, is a zero-mean Gaussian white process with unit variance.

The additive noise, $\{v_k\}_{k\geq 1}$, is a zero-mean time-correlated noise process such that

$$v_k = 0.4v_{k-1} + \beta_{k-1}, \quad k \geq 1,$$

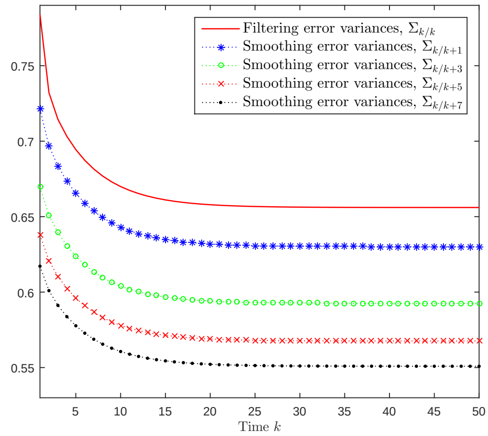

Figure 1: Error variance comparison of the filter and smoothers, when $\overline{\theta} = 0.5$.

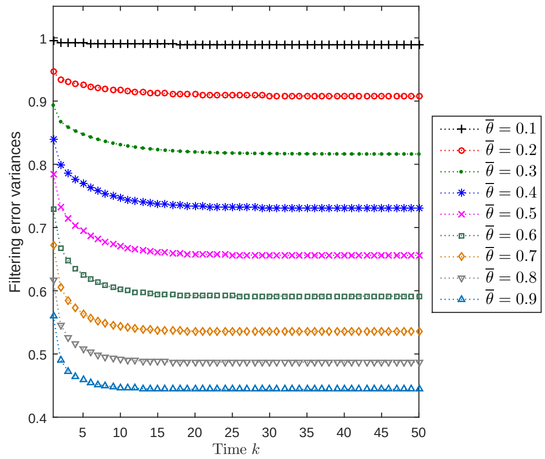

Figure 2: Filtering error variances versus $\overline{\theta}$.

where $\{\beta_k\}_{k\geq 0}$ is also a zero-mean Gaussian white process with unit variance, and v_0 is a standard gaussian random variable.

Finally, we assume that the processes involved in the system model satisfy the independence hypotheses assumptions imposed on the theoretical model.

The proposed algorithms have been implemented in a MATLAB program to obtain the filtering and fixed-point smoothing estimators, as well as the corresponding estimation error variances. Fifty iterations of this program have been run to show the feasibility and effectiveness of the proposed estimation algorithms. The estimation accuracy has been examined by analyzing the error variances for different probabilities, $\overline{\theta}$, of the Bernoulli variables modelling the missing measurement phenomenon.

Performance of the Filtering and Smoothing Estimators. The performance of the proposed LS linear estimators, measured by the estimation error variances, has been assessed when $\overline{\theta} = 0.5$. The error variances of both filtering and smoothing estimators are displayed in Figure 1 which shows that the error variances corresponding to the smoothers are less than those of the filters (and, consequently, the performance of the smoothing estimators is better), as it could be expected. From this figure it is also gathered that the estimation accuracy of the smoothers at each fixed-point, k, becomes better as the number of available observations increases.

Influence of the Probability $\overline{\theta}$*.* Analogous results to those shown in Figure 1 are obtained for other values of the probability $\overline{\theta}$ that the signal is present in the measurements. A global analysis of the filtering error variances versus the probability $\overline{\theta}$ is presented in Figure 2. From this figure, it is inferred that as $\overline{\theta}$ decreases (and, consequently, the missing measurement probability, $1-\overline{\theta}$, increases) the filtering error variance becomes greater and, hence, as expected, worse estimations are obtained. Similar results are deduced for the fixed-point smoothers.

5 CONCLUSIONS

Based on the LS optimality criterion, optimal algorithms have been designed for the linear filtering and fixed-point smoothing estimators of discrete-time random signals using measured outputs with random parameter matrices and time-correlated additive noise. The use of random parameter matrices yields a widely applicable measurement model, which is suitable to cope with different network-induced uncertainties in the sensor measurements. The additive measurement noise is assumed to obey a dynamic linear equation corrupted by white noise. As it is usual in these situations, the time-differencing approach has been adopted to define, at each sampling time, an equivalent measurement, which is a linear combination of two consecutive measurements. After this transformation, we get an equivalent set of measurements, where the time-correlation of the noise has been eliminated. Since the LS estimators of the signal based on the original set of observations is equal to that based on the new set of transformed observations, the original estimation problem is simplified and reduced to that of designing recursive algorithms for the filtering and fixed-point smoothing estimators of the signal based on the transformed measurements. The algorithm design has been carried out by an innovation approach and without requiring the knowledge of the signal evolution model, but only the first and second-order moments of the processes involved in the measurement model. To conclude, some computer simulations have shown how the proposed algorithms are applicable to some common engineering problems, involving missing measurements and multiplicative noise, which satisfy the system model under consideration.

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