Undominated Valid Inequalities for a Stochastic Capacitated Discrete Lot-sizing Problem with Lead Times, Cancellation and Postponement

Carlos E. Testuri[1], Héctor Cancela[1] and Víctor M. Albornoz[2]

[1]*Depto. de Investigación Operativa, Instituto de Computación, Facultad de Ingeniería, Universidad de la República, J. Herrera y Reissig 565, 11300 Montevideo, Uruguay*

[2]*Depto. de Industrias, Campus Santiago Vitacura, Universidad Técnica Federico Santa María, Avda. Santa María 6400, Vitacura, Santiago, Chile*

Keywords: Stochastic Lot-sizing, Multi-stage Stochastic Integer Programming, Valid Inequalities.

Abstract: The problem addresses the expected cost minimization of meeting the uncertain demand of a product during a discrete time planning horizon. The product is supplied by selecting fixed quantity shipments that have lead times. Due to the uncertainty of demand, corrective actions, such as shipment cancellations and postponements, must be taken with associated costs and delays. The problem is modeled as an extension of the discrete lot-sizing problem with different capacities and uncertain demand, which belongs to the $\mathcal{NP}$-hard class. To improve the resolution of the problem by tightening its formulation, valid inequalities based on the (ℓ,S) inequalities approach are used. Given that the inequalities are highly dominated for most experimental instances, a scheme is established to determine undominated ones. Computational experiments are performed on the resolution of the model and variants that include subsets of undominated and representative valid inequalities for instances of several information structures of uncertainty. The experimental results allow to conclude that the inclusion of undominated and representative derived (ℓ,S) valid inequalities enable a more efficient resolution of the model.

1 INTRODUCTION

The studied problem is the minimization of the expectation of the costs incurred in decisions taken to meet the uncertain demand of a product over a finite discrete time planning horizon. To meet the demand, there are certain optional distinguishable shipments (denominated as cargoes) with a non-fractional quantity of the product that can be acquired at most once with an associated cost. The cargoes have meaningful delivery lead times within the planning horizon; so that a significant amount of time elapses between the purchase decision and the moment when the cargo is received. After meeting the demand in a given period, the remaining quantity of product is stored, keeping an inventory up to a certain capacity, to flexibly satisfy the future demand in subsequent periods. Due to the passage of time, while the uncertainty of the demand is revealed and changes, it could happen that at a given time a cargo, that was already acquired (ordered) and has not yet been received, is no longer necessary. In this case it could be decided to cancel its acquisition order or postpone its delivery; decisions, which, in turn, have minimum execution times in relation to the time of delivery and associated costs. These decisions hedge against the risk of excess inventory.

The problem can be modeled as an extension of the lot-sizing formulation (Wagner and Whitin, 1958) and particularly of the variant with variable capacity and discrete dimensioning (Nemhauser and Wolsey, 1988). For the case where the parameters are known with certainty (deterministic case), the dimensioning is continuous, and without capacity constraints or with constant capacity, the problem has efficient resolution through dynamic programming (Wagner and Whitin, 1958; Wagelmans et al., 1992). In addition, there are known formulations which determine the convex hull of the feasible region: the extended facility location formulation (Krarup and Bilde, 1977) and the (ℓ,S) valid inequalities formulation (Barany et al., 1984). The deterministic variant with discrete sizing is a generalization of the binary knapsack problem, and belongs to the $\mathcal{NP}$-hard complexity class (Bitran and Yanasse, 1982).

In the case that the parameters are random variables (stochastic variant) the problem can be formulated by stochastic programming (Birge and Lou-

Testuri, C., Cancela, H. and Albornoz, V.
Undominated Valid Inequalities for a Stochastic Capacitated Discrete Lot-sizing Problem with Lead Times, Cancellation and Postponement.
DOI: 10.5220/0007395203900397
In *Proceedings of the 8th International Conference on Operations Research and Enterprise Systems (ICORES 2019)*, pages 390-397
ISBN: 978-989-758-352-0

veaux, 1997). An adjusted extended formulation of the stochastic continuos non-capacitated problem with Wagner-Whitin conditions are not satisfied for the stochastic variant (Ahmed et al., 2003). The (ℓ,S) inequalities are also valid for the stochastic continuous variant, and they were extended to a general class that allow to define facets of the feasible set (Guan et al., 2006).

Other variants of the deterministic continuous non-capacitated lot-sizing problem model delivery time of the lots (e.g. due to production time). A variant in which demands have a compliance interval has efficient resolution by dynamic programming (Lee et al., 2001). There are two variants according to whether the lots are or are not distinguishable with respect to delivery times (Brahimi et al., 2006). For there are efficient algorithms based on dynamic programming for the distinguishable case and for the undistinguishable case when the order-delivery windows are not inclusive. For these variants, there are tight extended formulations (Wolsey, 2006). For the stochastic case, the problem can be efficiently solved when delivery windows do not intersect in time (Huang and Küçükyavuz, 2008). The distinctive features of the problem under study in the present work are cancellation and postponement corrective decisions with time delays in a stochastic setting; these aspects are novel and were not found in the literature review.

The present work is organized as follows. In Section 2 an algebraic model of the problem is presented. Valid inequalities for the model are presented in Section 3. In Section 4 experiments are established to determine utility of the valid inequalities formulation. Work is completed with Section 5, where conclusions and future work are discussed.

2 STOCHASTIC MODEL FORMULATION

Basic index sets are established according to Table 1. The planning time is represented by the set T of discrete time periods. The set C of cargoes is partitioned in two sets: the set A of already acquired cargoes – cargoes ordered in past resolutions of the model– that are pending reception, and the set P of possible cargoes to be acquired from now on. Acquisition decision could be made on possible cargoes to be acquired, and cancellation and postponement decisions could be made on cargoes that were acquired before the actual planning horizon.

The uncertain demand is represented by a discrete-time stochastic process indexed in the planning periods. The process is defined in a finite probability space. It is assumed that the demand of the first period is deterministic, and that the demands of the remaining periods are random with known distribution functions. The decisions made in a period can not anticipate the realization of the uncertainty of the next period. These decisions must simultaneously take into account all possible revelations of the demand uncertainty of the following periods. This information structure can be represented by a tree structure called *tree of scenarios* (Römisch and Schultz, 2001). This is a perfect directed tree, with the root node representing the present time at period $t=1$, and with leaf nodes identifying the future scenarios at period $t=H$.

Each node of the scenario tree describes the state of the process at a given period, and it is identified by a period and a scenario. An useful abbreviated notation is to identify the nodes by a single index n in a numerable set of nodes, N. For the first period, $t=1$, there is a unique node, denoted by 1, that represents the root of the tree. Each node $n \in N$ has an immediate time predecessor node, $p(n)$; the auxiliary node 0 is defined as the predecessor of the root node, $0 := p(1)$, such that $0 \notin N$. The period corresponding to each node n is denoted as $t(n)$. The probability of the state of each node n is denoted as π_n, such that $\sum_{n \in N | t(n)=t} \pi_n = 1$, for all $t = 1,...,H$. The t-th time predecessor of node n is defined as $p(n,t) := p(p(n,t-1))$, such that $p(n,1) := p(n)$. The nodes of the path from the root node to node n are denoted as the ordered set $P(n) := \{p(n,t(n)-1),...,p(n,1),n\}$. The set of successors of node n is defined as $S(n) := \{n' \in N, k = 1,...,H-t(n) | n = p(n',k)\}$. The set of leaf nodes is $L := \{n \in N | t(n) = H\}$.

Table 1: Basic index sets.

T	periods, $\{1,...,H\}$
A	already acquired cargoes
P	possible cargoes to be acquired
C	cargoes, $A \cup P$
N	nodes of the scenario tree
L	leaf nodes of the scenario tree

Parameters are described in Table 2. The demand volume of the product at each node n is known and denoted as d_n. The demand distribution for period $t = 1,...,H$ is represented by (d_n, π_n) such that $n \in N$ and $t(n) = t$. Due to storage constraints, the inventory of the product at the end of each period is restricted between a minimum volume, $\underline{s}$, and a maximum volume, $\bar{s}$, and there is an initial storage volume, s_0, at the beginning of the planning horizon.

The period at which an already acquired cargo

Table 2: Parameters.

d_n	demand volume at node $n \in N$
π_n	probability of node $n \in N$
s_0	initial inventory volume
$\underline{s}, \bar{s}$	min. and max. storage capacities by period
τ^c	period in which already acquired cargo $c \in A$ is received
q^c	volume of cargo $c \in C$
γ^c	delivery time of cargo $c \in P$, such that $0 \leq \gamma^c \leq H-1$
δ^c	cancellation minimum time of already acquired cargo $c \in A$, such that $0 \leq \delta^c \leq \tau^c - 1$
ε^c	postponement minimum time of already acquired cargo $c \in A$, such that $0 \leq \varepsilon^c \leq H - \tau^c$
ca^c	acquisition unit cost of cargo $c \in C$
cc^c	cancellation unit cost of cargo $c \in A$
cp^c	postponement unit cost of cargo $c \in A$
h_t	storage unit cost in period $t \in T$
a_t	already acquired volume that is received in period $t \in T$ (auxiliary deducted parameter)

$c \in A$ is received is fixed, τ^c, and it is decided in previous acquisitions (i.e. previous model resolutions). Each cargo $c \in C$ has a given volume, q^c. The achievement of decisions on cargoes have latency times measured in periods. The delivery time of a cargo $c \in P$, γ^c, establishes the length of the wait time (measured in periods) between the acquisition decision and the actual arrival of the cargo. The minimum time for cancellation of a cargo $c \in A$, δ^c, establishes the minimum number of periods prior to the delivery period at which the cargo may be cancelled. The minimum postponement time for a cargo $c \in A$, ε^c, establishes the minimum number of periods after the initial delivery period in which the posponed cargo can be received. The achievement period of decisions on acquisition, cancellation and postponement must take place within the planning horizon.

For each cargo $c \in C$ there are unit costs per volume associated with the decisions to acquire, ca^c, cancel, cc^c, and postpone, cp^c. In addition, there is a unit cost associated with storage at each period $t \in T$, h_t.

The already acquired volume that is scheduled to be received at each period is determined by the sum of the cargoes that are received in that period,

$$a_t := \sum_{\{c \in A | \tau^c = t\}} q^c, \quad \forall t \in T;$$

this is an auxiliary parameter that summarize decisions of previous model resolutions on a rolling horizon scheme.

Table 3: Derived index sets.

N^c_γ	nodes where it is possible to acquire cargo $c \in P$, $\{n \in N | t(n) \leq H - \gamma^c\}$
N^c_δ	nodes where it is possible to cancel and postpone cargo $c \in A$, $\{n \in N | t(n) \leq \tau^c - \delta^c\}$
T^c_ε	periods to where it is possible to postpone cargo $c \in A$, $\{t \in T | t \geq \tau^c + \varepsilon^c\}$

Table 4: Functions and mappings of nodes.

$t(n)$	period of node $n \in N$
$p(n)$	immediate predecessor node of node $n \in N$ in the tree
$p(n,t)$	t-th time predecessor node of node $n \in N$ in the tree
$P(n)$	nodes in the path from root node to node $n \in N$ in the tree
$S(n)$	successor nodes of node $n \in N$ in the tree

Table 5: Variables.

s_n	inventory volume at the end of the period of node $n \in N$
u_n	acquired volume incoming at node $n \in N$
v^c_n	if cargo $c \in P$ is acquired at node $n \in N^c_\gamma$, (binary)
w_n	cancelled volume outgoing of node $n \in N$
x^c_n	if already acquired cargo $c \in A$ is cancelled in node $n \in N^c_\delta$ (binary)
y_n	postponed volume incoming at node $n \in N$
z^c_{nt}	if already acquired cargo $c \in A$ is postponed in node $n \in N^c_\delta$ to period $t \in T^c_\varepsilon$ (binary)

Derived subsets of the sets of nodes and periods that are indexed in the parameters are established in Table 3 in order to facilitate the formulation. There are subsets to abbreviate the denomination of nodes where it is possible to acquire each cargo $c \in P$, N^c_γ, and where it is possible to cancel and postpone each cargo $c \in A$, N^c_δ. In addition, subsets of periods to where it is possible to postpone each cargo $c \in A$ are established, T^c_ε. The subset subscripts are part of their denomination.

Functions and mappings on the nodes of the tree are summarized in Table 4.

In the stochastic model, all decisions depend on the nodes of the tree according to Table 5. An acquisition decision on a cargo $c \in P$ at node $n \in N^c_\gamma$ is represented by binary variable v^c_n. For each of these decisions, the delivery of cargo c will be on the nodes of the subtree rooted on n with period $t(n) + \gamma^c$, $\{n' \in S(n) | t(n') = t(n) + \gamma^c\}$. A cancellation decision of an already acquired cargo $c \in A$ at node $n \in N^c_\delta$ is represented by binary variable x^c_n. For each of these decisions, the already acquired volume of cargo c that was budgeted to be delivered on the nodes of the subtree rooted at n with period τ^c, $\{n' \in S(n)|t(n') = \tau^c\}$, is cancelled. A postponement decision of an already acquired cargo $c \in A$ is modeled by a cancellation decision of the cargo in conjunction with a decision to postpone it towards period $t \in T_\varepsilon^c$, that is represented by binary variable z_{nt}^c. For each of these decisions, the already acquired volume of cargo c that was budgeted to be delivered on the nodes of the subtree rooted on n at period τ^c, $\{n' \in S(n)|t(n') = \tau^c\}$, is postponed to the nodes of the subtree rooted on n at period $t \in T_\varepsilon^c$, $\{n' \in S(n)|t(n') \in T_\varepsilon^c\}$. The amounts of inventory, acquisition, cancellation and postponement at node n are summarized and represented by the variables, s_n, u_n, w_n and y_n, respectively.

The indexes of periods in deterministic parameters or variables are reduced to the temporary realization of a node n by $t(n)$. This is the case for parameters corresponding to the already acquired volume and storage unit cost.

From the previous definitions the formulation of the multi-stage stochastic optimization model (SCS) is

$$\min \sum_{n\in N} \pi_n \Big[\sum_{\{c\in P|n\in N_\gamma^c\}} ca^c q^c v_n^c \tag{1}$$

$$+ \sum_{\{c\in A|n\in N_\delta^c\}} (cc^c - ca^c) q^c x_n^c \tag{2}$$

$$+ \sum_{\{c\in A, t\in T_\varepsilon^c|n\in N_\delta^c\}} (cp^c + ca^c - cc^c) q^c z_{nt}^c \tag{3}$$

$$+ h_{t(n)} s_n \Big], \tag{4}$$

s.t.

$$s_{p(n)} + a_{t(n)} + u_n + y_n = d_n + w_n + s_n, \quad \forall n \in N, \tag{5}$$

$$\underline{s} \le s_n \le \overline{s}, \quad \forall n \in N, \tag{6}$$

$$u_n = \sum_{\{c\in P|\gamma^c+1\le t(n)\}} q^c v_{p(n,\gamma^c)}^c, \quad \forall n \in N, \tag{7}$$

$$\sum_{n'\in P(n)} v_{n'}^c \le 1, \quad \forall c \in P, \forall n \in N|t(n) = H - \gamma^c, \tag{8}$$

$$w_n = \sum_{\{c\in A|t(n)=\tau^c\}} \left(q^c \sum_{\{n'\in P(n)|t(n')\le \tau^c-\delta^c\}} x_{n'}^c \right), \quad \forall n \in N, \tag{9}$$

$$\sum_{n'\in P(n)} x_{n'}^c \le 1, \quad \forall c \in A, \forall n \in N|t(n) = \tau^c - \delta^c, \tag{10}$$

$$x_n^c \ge z_{nt}^c, \quad \forall c \in A, \forall n \in N_\delta^c, \forall t \in T_\varepsilon^c, \tag{11}$$

$$y_n = \sum_{\{c\in A|t(n)\ge\tau^c+\varepsilon^c\}} \left(q^c \sum_{\{n'\in P(n)\cap N_\delta^c\}} z_{n',t(n)}^c \right), \quad \forall n \in N, \tag{12}$$

$$\sum_{\{n'\in P(n), t\in T_\varepsilon^c\}} z_{n't}^c \le 1, \quad \forall c \in A, \forall n \in N_\delta^c, \tag{13}$$

$$s_n, u_n, w_n, y_n \ge 0, \quad \forall n \in N,$$
$$v_n^c \in \{0,1\}, \quad \forall c \in P, \forall n \in N_\gamma^c,$$
$$x_n^c, z_{nt}^c \in \{0,1\}, \quad \forall c \in A, \forall n \in N_\delta^c, \forall t \in T_\varepsilon^c.$$

This formulation takes into account the information structure of the scenario tree. It minimizes the expectation of acquisition costs (1), cancellation costs less acquisition costs in case of cancellation (2), postponement costs plus acquisition costs minus postponement costs (3) (a postponement is modeled in conjunction with a cancellation) and storage costs (4).

Constraints (5) set the volume balance for each node. The lower and upper storage bounds at each node are determined by inequalities (6). The amount of product acquired that is received at each node is determined by acquisitions of cargoes in the possible range of the corresponding acquisition periods according to equalities (7). The constraints (8) state that each cargo is acquired at a single node at most in each path from the root node to a node whose period coincides with the latest acquisition period of the cargo. The product previously acquired that is cancelled at each node is determined by the cancellations of the nodes in the path from the root node to the node, whose cancellation periods are less than the delivery period less the cancellation time, according to (9). Constraints (10) state that each cargo to be cancelled is at a single node at most in each path from the root node to a node whose period coincides with the receiving period minus the cancellation time of the cargo. The postponement of the cargoes is modeled in conjunction with the cancellation, i.e. only cancelled cargoes can be postponed, (11). The already acquired volume that is postponed in a node is determined by the postponements of the cargoes in the nodes in the path from the root to the node for all periods superior to the period of reception plus the delay time of the node, according to (12). Constraint (13) state that each cargo to be postponed is at a single node at most in each path from the root node to a node in some period greater than the receiving period plus the time of postponement of the node.

3 VALID INEQUALITIES FOR THE STOCHASTIC MODEL

Since (SCS) belongs to the time complexity class $\mathcal{NP}$-hard, there is no known polyhedral description of the convex hull of its feasible solutions. It is nevertheless interesting to derive valid inequalities which can be used to strengthen the original formulation. In some cases adding these inequalities can directly improve the capacity of the solver to find solutions for larger instances in shorter times; even when this is not the case, they may be used within a more sophisticated solving strategy, such as branch and cut methods relying on constraint separation.

In this section, we discuss a variation of classic (ℓ,S) valid inequalities for the stochastic capacitated discrete lot-sizing problem with lead times, cancellation and postponement, and a scheme to obtain undominated valid inequalities of the variation.

3.1 Derived (ℓ,S) Valid Inequalities

A set of valid inequalities are derived por (SCS) based on the (ℓ,S) valid inequalities formulation for the deterministic uncapacitated lot-sizing problem (Barany et al., 1984) while considering the extension for the stochastic case (Guan et al., 2006). The derived inequalities establish bounds on decision variables for the nodes of possible paths in the scenario tree (Testuri et al., 2018).

Proposition 1 . *Let $\ell \in N$ and $S \subseteq P(\ell)$ then the derived (ℓ,S) inequality*

$$\sum_{n\in S} u_n \leq \sum_{n\in S} d_{n\ell}\beta(n) + \sum_{n\in S} w_{n\ell} + s_\ell, \tag{14}$$

where
$\beta(n) := \sum_{\{c\in P|\gamma^c+1\leq t(n)\}} v^c_{p(n,\gamma^c)}$,
$d_{n\ell} := \sum_{n'\in P(\ell)\setminus P(p(n))} d_{n'}$ *and*
$w_{n\ell} := \sum_{n'\in P(\ell)\setminus P(p(n))} w_{n'}$,
is valid for the feasible region of (SCS).

As shown by the authors the derived (ℓ,S) valid inequalities has alternative and equivalent inequalities without inventory variable, s_n,

$$\sum_{n\in P(\ell)\setminus S} u_n + \sum_{n\in S} d_{n\ell}\beta(n) + \sum_{n\in S} w_{n\ell} - w_{1\ell} + \sum_{n\in P(\ell)} y_n \geq d_{1\ell} - \sum_{n\in P(\ell)} a_{t(n)} - s_0, \quad \text{for all } \ell \in N, S \subseteq P(\ell). \tag{15}$$

3.2 Undominanted (ℓ,S) Valid Inequalities

Depending on the instance values of the parameters q^c and d_n, some of the derived (ℓ,S) inequalities (15) of a given subset $S \subseteq P(\ell)$, $\ell \in N$, may be dominated by other inequalities of a different subset. Therefore, a procedure was established to determine undominanted inequalities on the power set of $P(\ell)$.

Let $\boldsymbol{\chi} := [(v^c_n)_{c\in P, n\in N^c_\gamma}, (x^c_n)_{c\in A, n\in N^c_\delta}, (z^c_{nt})_{c\in A, n\in N^c_\delta, t\in T^c_\varepsilon}]$ be the composite variable. Let $b := d_{1\ell} - \sum_{n\in P(\ell)} a_{t(n)} - s_0$, for each $\ell \in N$, be the independent term of (15). Given the power set of $P(\ell)$, $S^p_\ell := \{S_1, ..., S_{K_\ell}\}$, let $\boldsymbol{\alpha}_k$ be the coefficient vector of variable $\boldsymbol{\chi}$ on the inequality (15) for subset $S_k, k \in \{1, ..., K_\ell\}$. Therefore, the inequalities (15) can be established as

$$\boldsymbol{\alpha}^T_k \boldsymbol{\chi} \geq b, \quad k \in \{1, ..., K_\ell\}, \ell \in N.$$

Given $i, j \in \{1, ..., K_\ell\}$ and that $\boldsymbol{\chi}$ is nonnegative, it is said that $\boldsymbol{\alpha}^T_i \boldsymbol{\chi} \geq b$ dominantes $\boldsymbol{\alpha}^T_j \boldsymbol{\chi} \geq b$, if the componentwise comparison of $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j$ is such that $\alpha_{i\lambda} \leq \alpha_{j\lambda}$ for each component $\lambda \in \Lambda$, and at least for one component the inequality is strict. Let S^d_ℓ be the subset of *dominant* inequalities on S^p_ℓ.

The procedure to obtain S^d_ℓ by pairwise comparison of inequalities has and upper bound of $O(K^2_\ell|\Lambda|)$ operations. If S^d_ℓ has few elements, an efficient heuristic to obtain a promising undominanted inequality candidate,

$$i^* := \text{argmin}_{i\in K_\ell} \sum_{\lambda\in\Lambda} \alpha_{i\lambda}, \tag{16}$$

takes $\Theta(K_\ell|\Lambda|)$ operations. Lets denote $S^{d*}_\ell := \{i^*\}$.

Furthermore, lets denote S^r_ℓ the case where the *power set*, S^p_ℓ, is approximated by a *representative* subset of S^p_ℓ that contains only the root node, $n = 1$.

Tree variants of the original formulation, (SCS), are generated by including to it the inequalities of the sets S^p_ℓ, S^{d*}_ℓ and S^r_ℓ, for each $\ell \in N$, establishing formulations denoted as (SCS-S^p), (SCS-S^{d*}) and (SCS-S^r), respectively.

4 COMPUTATIONAL EXPERIMENTS

This section explores the computational impact of adding three families of inequalities introduced in the previous section to the original formulation. These are the power set inequalities (S^p), the dominance reduction inequalities (S^{d*}), and the root representative inequalities (S^r). The original formulation and the three modified formulations are tested over a set of test instances, checking the quality of the obtained solutions and the computational effort invested by the solver.

In order to generate a number of diverse test instances, six scenario tree structures were considered.

Each structure, depicted in Table 6, is determined by the number of direct descendants of each node (tree arity) and the number of periods of the planning horizon. For each tree structure with arity g and H periods there are g^{H-1} escenarios and $(g^H-1)/(g-1)$ nodes.

Table 6: Size of scenario tree structures.

Arity(g)	Periods(H)	Scenarios	Nodes
2	5	16	31
2	6	32	63
2	7	64	127
3	5	81	121
3	6	243	364
3	7	729	1093

The size of each tree structure model instance (number of equations and variables) for a given distribution of cargos (C) is shown in Table 7.

Table 7: Instance size of scenario tree structures by cargo distribution.

g	H	$\|C\|(\|A\|+\|P\|)$	Eqs.	Vars.	(binary)
2	5	10 (2+ 8)	225	249	(124)
2	6	12 (3+ 9)	480	549	(296)
2	7	14 (3+11)	1.012	1.223	(714)
3	5	10 (3+ 7)	827	809	(324)
3	6	12 (4+ 8)	2.542	2.485	(1.028)
3	7	14 (4+10)	7.987	8.091	(3.718)

Thirty data instances were randomly generated for each tree structure and cargo distribution, totaling 180 instances. Each instance has an initial storage, $s_0 = 20$, and a lower and an upper bound storage, $\underline{s} = 0$ and $\bar{s} = 80$, respectively. For each cargo $c \in C$ there is an uniformly distributed volume, $q^c \sim U[10,50]$, and there are costs evenly distributed according to the operations of acquisition, $ca^c \sim U[150,250]$, cancellation, $cc^c \sim U[30,50]$, and postponement, $cp^c \sim U[5,12]$. Each already acquired cargo $c \in A$ has delivery period $\tau^c = 1$ or 2 with equal probability. Each cargo $c \in C$ has delivery time $\gamma^c = 1$, cancellation time $\delta^c = 1$ and delay time $\varepsilon^c = 1$. The unit storage cost at each period t is $h_t = 1$. For each scenario $n \in L$ (leaf node), a probability of state π_n is established from a distribution $Beta(\alpha = 2, \beta = 2)$; the probability of the remaining nodes is obtained from the sum of the probabilities of their corresponding immediate successor nodes. Finally, the demand for each node is evenly distributed, $d_n \sim U[10,50]$.

The computational implementation was performed using AMPL (Fourer et al., 2002) for the algebraic coding of the stochastic model, and GUROBI 7.5 (Gurobi Optimization, LLC, 2018) for the resolution of the instances through its branch and cut solver. The calculations were carried out on an Intel Core i7 5960X 3.5GHz computer with 20MB cache and 64GB RAM, operating with CentOS-7 Linux system.

For each instance, the original model and the variants were solved. A summary of the results of the original model and each variant is presented in Table 8, Table 9, Table 10 and Table 11, respectively for (SCS), (SCS-S^p), (SCS-S^{d*}) and (SCS-S^r).

Table 8: Average results of formulation (SCS) by tree structure and cargoes.

g-H-C	Time(s)	MIP	Nodes	Cuts	LP
2-5-10	0.68	-	6,449	125	10.31
2-6-12	13.07	-	30,706	189	18.85
2-7-14	†493.45	0.26	1,093,185	713	9.84
3-5-10	13.39	-	31,260	300	12.40
3-6-12	‡758.80	1.90	733,244	1,319	19.64
3-7-14	#900.25	5.02	27,291	1,264	23.42

(†) 12 of 30 instances reach the time limit of 900 s.
(‡) 24 of 30 instances reach the time limit of 900 s.
(#) All instances reach the time limit of 900 s.

Table 9: Average results of formulation (SCS-S^p) by tree structure and cargoes.

g-H-C	Time(s)	MIP	Nodes	Cuts	LP
2-5-10	1.11	-	2,132	88	7.44
2-6-12	17.48	-	24,885	253	11.25
2-7-14	†430.79	0.25	431,172	1,184	6.97
3-5-10	7.51	-	3,189	206	9.72
3-6-12	‡692.13	1.36	174,553	1,539	17.21
3-7-14	#900.83	12.85	5,754	?	36.60

(†) 12 of 30 instances reach the time limit of 900 s.
(‡) 19 of 30 instances reach the time limit of 900 s.
(#) All instances reach the time limit of 900 s.

Table 10: Average results of formulation (SCS-S^{d*}) by tree structure and cargoes.

g-H-C	Time(s)	MIP	Nodes	Cuts	LP
2-5-10	0.48	-	2,693	98	7.46
2-6-12	13.22	-	37,603	249	11.25
2-7-14	†435.25	0.28	1,005,992	1,024	6.97
3-5-10	5.87	-	12,281	239	9.72
3-6-12	‡720.13	1.71	513,898	1,609	17.24
3-7-14	#900.19	4.66	41,495	1,367	20.53

(†) 12 of 30 instances reach the time limit of 900 s.
(‡) 21 of 30 instances reach the time limit of 900 s.
(#) All instances reach the time limit of 900 s.

The summary shows, for each tree structure defined by arity, periods and number of cargoes, depicted at column "g-H-C", the average results of the 30 instances of the model (SCS) and its variant with the corresponding valid inequalities. The average results depicted are solver elapsed time at column "Time", solver MIP gap percentage for instances that reach the time limit of 900 s at column "MIP", solver number of nodes of solver branch and cut method at column

Table 11: Average results of formulation (SCS-S^r) by tree structure and cargoes.

g-H-C	Time(s)	MIP	Nodes	Cuts	LP
2-5-10	0.45	-	3,361	94	7.48
2-6-12	11.90	-	26,885	237	11.25
2-7-14	†443.32	0.27	1,145,992	978	6.99
3-5-10	4.87	-	11,584	264	9.74
3-6-12	‡449.50	0.73	330,166	1,546	17.20
3-7-14	$^{\#}$900.18	4.84	29,881	1,416	20.90

(†) 12 of 30 instances reach the time limit of 900 s.
(‡) 12 of 30 instances reach the time limit of 900 s.
(#) All instances reach the time limit of 900 s.

"Nodes", number of cuts added by solver's branch and cut method at column "Cuts", and the relative ratio percentage of the objective value with respect of the objective value of the linear programming relaxation of the model, at column "LP".

In the case of formulation (SCS-S^p), it can be seen that the average Time results for the tree structures (2-5-10) and (2-6-12) are worse than the corresponding to formulation (SCS). On the other hand, the average Time and MIP-gap results of the formulation for the tree structures (2-7-14), (3-5-10) and (3-6-12) are better than the corresponding to formulation (SCS). Also, the formulation reduces to 19 the number of instances of structure (3-6-12) that reach the time limit of 900 s, compared with 24 of the (SCS) formulation. Finally, except for structure (3-7-14), the formulation obtains a reduction of the LP-gap of the remaining structures compared with formulation (SCS).

In the case of formulation (SCS-S^{d*}), only the average Time results for the tree structures (2-6-12) are slightly worse than the corresponding ones of formulation (SCS). The formulation has lower LP-gap for all tree structures compared to formulation (SCS). With regards to its comparison with formulation (SCS-S^p), the formulation obtains better Time results for the tree structures (2-5-10), (2-6-12) and (3-5-10); and it obtains equal or slightly worse MIP-gap results, except for formulation (3-7-14), where it gets better result.

Formulation (SCS-S^r) obtains better Time results than formulations (SCS) and (SCS-S^p) for all tree structures. While it get better MIP-gap results for all tree structures than the formulation (SCS), it gets slightly worse MIP-gap results than formulation (SCS-S^p), except for structure (3-7-14), where it gets better result. In comparison with formulation (SCS-S^{d*}), it has slightly better Time results, and similar MIP-gap results. It reduces to 12 the number of instances of structure (3-6-12) that reach the time limit of 900 s, compared with 21 of the (SCS-S^{d*}) formulation.

5 CONCLUSIONS

A stochastic multi-stage capacitated discrete lot-sizing model formulation of the provision with lead time of the uncertain demand of a product has been proposed. The decisions on product lots are modeled with their delay time, aspect that for cancellation and postponement decisions is not covered in the previous literature. A discrete time stochastic process with finite probability, summarized in a scenario tree, is used to model the information structure of the uncertain demand. The model is formulated by stochastic programming with entities indexed by nodes of the scenario tree. The model incorporates the cancellation and postponement decisions with delay time, which implied the revision of the definitions of the variables and the restrictions to take into account the structure of the scenario tree. To tighten the formulation valid inequalities based on the (ℓ, S) inequalities approach were used. Since the inequalities are highly dominated for most experimental instances, a scheme is established to determine undominated ones. Three variants of the formulation are obtained from the inclusion of the power-set, undominated and representative valid inequations. The original formulation and the three variants are tested over a set of test instances, checking the quality of the obtained solutions and the computational effort invested by the solver. Computational experiments where carried out for several instances within a few tree structures of different sizes. Most computational experiments could be solved to optimality for the small and medium-size tree structures. The representative and undominated formulations obtains a slightly better results than the original and power set formulations for all tree structures.

ACKNOWLEDGMENTS

This work was partially supported by the Comisión Sectorial de Investigación Científica, CSIC, and the Programa de Desarrollo de las Ciencias Básicas, PEDECIBA.

REFERENCES

Ahmed, S., King, A. J., and Parija, G. (2003). A multi-stage stochastic integer programming approach for capacity expansion under uncertainty. *Journal of Global Optimization*, 26(1):3–24.

Barany, I., Roy, T., and Wolsey, L. A. (1984). *Mathematical Programming at Oberwolfach II*, chapter Uncapacitated lot-sizing: The convex hull of solutions, pages

32–43. Springer Berlin Heidelberg, Berlin, Heidelberg.

Birge, J. R. and Louveaux, F. (1997). *Introduction to Stochastic Programming*. Springer Series in Operations Research and Financial Engineering. Springer, New York.

Bitran, G. R. and Yanasse, H. H. (1982). Computational complexity of the capacitated lot size problem. *Management Science*, 28(10):1174–1186.

Brahimi, N., Dauzere-Peres, S., Najid, N. M., and Nordli, A. (2006). Single item lot sizing problems. *European Journal of Operational Research*, 168(1):1 – 16.

Fourer, R., Gay, D. M., and Kernighan, B. W. (2002). *AMPL: A Modeling Language for Mathematical Programming*. Duxbury Press.

Guan, Y., Ahmed, S., Nemhauser, G. L., and Miller, A. J. (2006). A branch-and-cut algorithm for the stochastic uncapacitated lot-sizing problem. *Mathematical Programming*, 105(1):55–84.

Gurobi Optimization, LLC (2018). Gurobi optimizer reference manual.

Huang, K. and Küçükyavuz, S. (2008). On stochastic lot-sizing problems with random lead times. *Operations Research Letters*, 36(3):303 – 308.

Krarup, J. and Bilde, O. (1977). *Plant location, Set Covering and Economic Lot Size: An $O(mn)$-Algorithm for Structured Problems*, pages 155–180. Birkhäuser Basel, Basel.

Lee, C.-Y., Çetinkaya, S., and Wagelmans, A. P. M. (2001). A dynamic lot-sizing model with demand time windows. *Management Science*, 47(10):1384–1395.

Nemhauser, G. and Wolsey, L. (1988). *Integer and Combinatorial Optimization*. John Wiley & Sons, Inc.

Römisch, W. and Schultz, R. (2001). Multistage stochastic integer programs: An introduction. In Grötschel, M., Krumke, S. O., and Rambau, J., editors, *Online Optimization of Large Scale Systems*, pages 581–600. Springer Berlin Heidelberg.

Testuri, C. E., Cancela, H., and Albornoz, V. M. (2018). Valid inequalities for a stochastic capacitated discrete lot-sizing problem with lead times, cancellation and postponement. *(Manuscript submitted for publication)*.

Wagelmans, A., Hoesel, S. V., and Kolen, A. (1992). Economic lot sizing: An $O(n \log n)$ algorithm that runs in linear time in the Wagner-Whitin case. *Operations Research*, 40:S145–S156.

Wagner, H. M. and Whitin, T. M. (1958). Dynamic version of the economic lot size model. *Management Science*, 5(1):89–96.

Wolsey, L. A. (2006). Lot-sizing with production and delivery time windows. *Mathematical Programming*, 107(3):471–489.