Rigorous Derivation of Temporal Coupled Mode Theory Expressions for Travelling and Standing Wave Resonators Coupled to Optical Waveguides

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Abstract: Temporal coupled mode theory (CMT) has so far been applied phenomenologically in the analysis of optical cavity-waveguide structures, and relies on a priori knowledge of the to-be-excited resonator mode. Thus a rigorous derivation from Maxwell's equations, and without any prior knowledge of the resonator type is needed. In this paper we derive temporal CMT of optical cavities coupled to waveguides. Starting from Maxwell's equations and considering a proper expansion of the modes of the waveguide and resonator, and using mode orthogonality, the temporal CMT for this structure is obtained. We show that this formulation is general and can be applied to both traveling wave and standing wave type resonators. The results are validated against full-wave simulations.

1 INTRODUCTION

Optical cavities are crucial components in integrated optical circuits, impacting a variety of different applications, including optical filters in multiplexers, optical sensors, enhancing light-matter interactions, increasing nonlinear effects, to just name a few. Waveguides are typically used to couple light in and out of the cavity resonator. Thus an optical cavity coupled to a waveguide structures is a very frequent scenario that occurs in integrated optical circuits. Therefore, it is always of particular interest to develop analytical methods to analyze these structures, as full-wave numerical solutions to these problems require time and computational power, which is increasing as these structures are becoming more complex or are repeated in a circuit multiple times.

An analytical method often used to describe light propagation in optical cavities coupled to waveguide structures is a variation of the Coupled Mode Theory (CMT) known as temporal CMT (TCMT). TCMT turns Maxwell's equations into a set of ordinary differential equations. This simplification in addition to providing an intuitive framework makes it suitable for the study and design of the resonance based components in integrated optical circuits.

The original CMT in spatial domain may be traced back to the early 1950's (Pierce, 1954) with application in microwaves and it was first developed for analyzing optical waveguides by Marcuse, Snyder and Yariv (Yariv, 1973) in 1970's. The method of temporal CMT was later developed by Haus (Haus, 1984; Haus and Huang, 1991; Little et al., 1997) mainly for the analysis of coupled resonators and resonator coupled to waveguides. Thereafter, several research has focused on utilizing this method for scenarios involving resonators coupled to waveguide structures (Fan, Suh and Joannopoulos, 2003; Wonjoo Suh, Zheng Wang and Shanhui Fan, 2004; Manolatou et al., 1999). In these works, by virtue of time-reversal symmetry and power conservation laws, relations of the coupling coefficient between the resonator and guide are derived. Despite the universality and fame of this approach, to the best of our knowledge, for optical cavity coupled to waveguide, temporal CMT methods still rely on phenomenological ways to find the coupling coefficients. That is, they are normally fitted to a response obtained from full-wave solutions, and not rigorously derived from Maxwell's equations, or the fields interacting, nor to the underlying structure. In addition, temporal CMT varies for traveling wave and standing wave resonators (Li et al., 2010) and one has to know which equation to use beforehand, which needs prior knowledge about the problem. The deficiency in the conventional temporal CMT

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approach which is phenomenological and requires a prior knowledge about the type of the resonator, necessitates a rigorous derivation from Maxwell's equations that without any prior knowledge works for both standing wave and traveling wave resonators. Recently some attempts have been made to derive the temporal CMT of optical cavity coupled to waveguide. One hybrid analytical-numerical approach to temporal CMT has been proposed (Agrawal et al., 2017) by expanding the electromagnetic field in terms of its modes and applying numerical methods to calculate the unknown coefficients. Another very recently proposed derivation based on implementing field equivalence principle to couple incoming electromagnetic fields of waveguide to that of the resonator has been shown (Kristensen et al., 2017).

In this paper we present a rigorous derivation of temporal CMT which works for both the standing wave and traveling wave resonator without prior knowledge of the type of resonator. We start with Maxwell's equations and by expanding electromagnetic fields in terms of the modes of the resonator and waveguide, and assuming orthogonality between them, Temporal CMT is derived. This paper is organized as follows: In section 2 we derive temporal CMT equations, we first consider resonator as perturbation and by substituting a proper expansion of modes in the Maxwell's equation and assuming mode orthogonality, we reach at a differential equation for the complex mode amplitude in waveguide, then by considering waveguide as perturbation and the same procedure, a differential equation for the complex mode amplitude of the resonator is derived. Next by solving these set of differential equations, we derive temporal CMT and provide closed form expressions for the coupling coefficients. At the end adding intrinsic loss of the resonator due to radiation is discussed. This approach is applied to the resonator with one and two modes. In section 3 examples of standing wave and traveling wave resonator are provided to assess the validity of the temporal CMT, results are compared to full wave FDTD simulations. Last we present conclusions in section 4.

2 DERIVATION

In this section, we derive temporal CMT for a resonator with one or two modes. For this purpose, electromagnetic fields in the optical cavity-waveguide structure is approximated with a superposition of the modes of its components, i.e. modes of the resonator and that of the waveguide. By implementing a perturbation approach and considering waveguide (resonator) as the unperturbed structure and evaluating the effect of adding resonator (waveguide) as perturbation, differential equations for the complex mode amplitude of the waveguide and resonator is derived. By solving these set of differential equation, temporal CMT is obtained.

2.1 One-Mode Resonator

2.1.1 Perturbation of Waveguide Modes

Consider a one-mode resonator side coupled to waveguide as shown in figure 1. Electromagnetic fields in the unperturbed waveguide are:

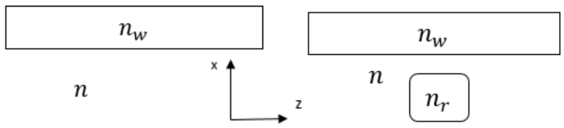

Figure 1: Refractive index distribution of unperturbed waveguide (right) and perturbed structure (left).

$$\mathbf{E_0} = \mathbf{e}_+(x,y)e^{-j\beta z}\, e^{j\omega_1 t} + c.c. \tag{1}$$

$$\mathbf{H_0} = \mathbf{h}_+(x,y)e^{-j\beta z}\, e^{j\omega_1 t} + c.c. \tag{2}$$

Where $\mathbf{e}_+(x,y)$ and $\mathbf{h}_+(x,y)$ are transverse mode profile of the waveguide and ω_1 is the operating angular frequency, and c.c. stands for complex conjugate of the same term, used for brevity. Electromagnetic fields in the coupled cavity-waveguide is approximated as:

$$\mathbf{E} = a\,\mathbf{e_r}(x,y,z) + [b_+(z)\mathbf{e}_+(x,y) + b_-(z)\mathbf{e}_-(x,y)]\, e^{j\omega_1 t} + c.c. \tag{3}$$

$$\mathbf{H} = a\,\mathbf{h_r}(x,y,z) + [b_+(z)\mathbf{h}_+(x,y) + b_-(z)\mathbf{h}_-(x,y)]\, e^{j\omega_1 t} + c.c. \tag{4}$$

Where $\mathbf{e_r}(x,y,z)$ and $\mathbf{h_r}(x,y,z)$ are resonator's mode profile and a and b_+ (b_-) are the complex mode amplitude of the resonator and forward (backward) mode of the waveguide. Electromagnetic fields of equations (1) and (2) satisfy Maxwell's equation of the unperturbed waveguide, i.e. $\nabla\times\mathbf{E_0} = -\mu_0\frac{\partial\mathbf{H_0}}{\partial t}$ and $\nabla\times\mathbf{H_0} = \varepsilon\frac{\partial\mathbf{E_0}}{\partial t}$. Where ε denotes the permittivity distribution of the unperturbed structure according to Figure 1. Similarly, taking the resonator as perturbation, a proper expansion of the modes as given by equation (3) and (4), are the solution of the equations $\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$ and $\nabla \times \mathbf{H} = (\varepsilon + \Delta\varepsilon) \frac{\partial \mathbf{E}}{\partial t}$. Where $\Delta\varepsilon$ is $\varepsilon_0({n_r}^2 - n^2)$ in the resonator, and zero elsewhere. Therefore, it's straightforward to obtain the following equation:

$$\nabla.(\mathbf{H} \times \mathbf{E_0} + \mathbf{H_0} \times \mathbf{E}) = (\varepsilon + \Delta\varepsilon)\frac{\partial \mathbf{E}}{\partial t}.\mathbf{E_0} + \mu_0 \frac{\partial \mathbf{H_0}}{\partial t}.\mathbf{H} + \varepsilon \frac{\partial \mathbf{E_0}}{\partial t}.\mathbf{E} + \mu_0 \frac{\partial\, \mathbf{H}}{\partial t}.\mathbf{H_0} \quad (5)$$

By integrating the above equation on the entire *x*-*y* plane and applying time average, high frequency components become negligible and we can write the left-hand side of the above equation as:

$$\iint_{-\infty}^{+\infty} \nabla.(\,\mathbf{H} \times \mathbf{E_0} + \mathbf{H_0} \times \mathbf{E}) dx\, dy = \iint_{-\infty}^{+\infty} \nabla_t.(\,\mathbf{H} \times \mathbf{E_0} + \mathbf{H_0} \times \mathbf{E})_t\, dx\, dy + \iint_{-\infty}^{+\infty} \frac{\partial}{\partial z}\,[a\, \mathbf{h_r} \times {\mathbf{e}_+}^* e^{j\beta z} e^{-j\omega_1 t} + b_+(z)\mathbf{h}_+ \times {\mathbf{e}_+}^* e^{j\beta z} + b_-(z)\mathbf{h}_- \times {\mathbf{e}_+}^* e^{j\beta z} + a\, {\mathbf{h}_+}^* \times \mathbf{e_r}\, e^{j\beta z}\, e^{-j\omega_1 t} + b_+(z){\mathbf{h}_+}^* \times \mathbf{e}_+ e^{j\beta z} + b_-(z){\mathbf{h}_+}^* \times \mathbf{e}_- e^{j\beta z} + c.c.\,]\, dx\, dy \quad (6)$$

Where ∇_t represents del operator in transverse coordinate. According to the two dimensional divergence theorem one has:

$$\iint_{-\infty}^{+\infty} \nabla_t.(\,\mathbf{H} \times \mathbf{E_0} + \mathbf{H_0} \times \mathbf{E})_t\, dx\, dy = \oint\, (\mathbf{H} \times \mathbf{E_0} + \mathbf{H_0} \times \mathbf{E}).\boldsymbol{dn} = 0 \quad (7)$$

here the integral is taken on the boundaries of an infinite circle and $\boldsymbol{dn}$ represents the vector normal to the boundaries. As electromagnetic fields decay by increasing distance from the structure, the above integral vanishes by integrating in the entire *x-y* plane. By assuming orthogonality between mode of the resonator and waveguide, and assuming that due to linearity, no frequency conversion occurs, therefore complex amplitude of the resonator mode is also single-frequency and in the same frequency with the waveguide mode. Therefore, equation 5 becomes as follows:

$$\frac{d}{dz} b_+(z) = -j\beta b_+(z) - \frac{j\omega_1\varepsilon_0}{4}\,\widetilde{a}\, \iint ({n_r}^2 - n^2)\mathbf{e_r}.{\mathbf{e}_+}^* dx\, dy - \frac{j\omega_1\varepsilon_0}{4} b_+(z) \iint ({n_r}^2 - n^2)\mathbf{e}_+.{\mathbf{e}_+}^* dx\, dy - \frac{j\omega_1\varepsilon_0}{4} b_-(z) \iint ({n_r}^2 - n^2)\, \mathbf{e}_-.{\mathbf{e}_+}^*\;\, dx\, dy \quad (8)$$

Where $a = \widetilde{a}\, e^{j\omega_1 t}$ and electromagnetic fields are normalized to unit power, i.e. $\frac{1}{4}\iint_{-\infty}^{+\infty} \mathbf{e}_+ \times {\mathbf{h}_+}^* + {\mathbf{e}_+}^* \times \mathbf{h}_+\, dx\, dy = 1$. The integrals are limited to resonator boundaries where $\Delta\varepsilon$ is non-zero. In the above equation, integrals represent coupling of the forward waveguide mode to the resonator, itself and backward mode, due to the perturbation. Since the integrals are limited to resonator boundaries, the first integral is dominant and we have:

$$\frac{d}{dz} b_+(z) = -j\beta b_+(z) + \kappa_+(z)\widetilde{a} \quad (9)$$

Which is the spatial coupled mode equation for the "forward mode" in the waveguide and $\kappa_+(z)$ is the corresponding coupling coefficient which can be calculated by the following equation:

$$\kappa_+(z) = -\frac{j\omega_1\varepsilon_0}{4} \iint ({n_r}^2 - n^2)\mathbf{e_r}.{\mathbf{e}_+}^*\;\, dx\, dy \quad (10)$$

Next the backword mode is considered as the electromagnetic fields in the unperturbed structure:

$$\mathbf{E_0} = \mathbf{e}_-(x,y) e^{j\beta z}\, e^{j\omega_1 t} + c.c. \quad (11)$$

$$\mathbf{H_0} = \mathbf{h}_-(x,y) e^{j\beta z}\, e^{j\omega_1 t} + c.c. \quad (12)$$

By applying a same procedure, one can obtain the spatial coupled mode equation for the backward mode in the waveguide as follows:

$$\frac{d}{dz} b_-(z) = j\beta b_-(z) + \kappa_-(z)\widetilde{a} \quad (13)$$

Where the spatial coupling coefficient of the backward mode to the resonator mode is:

$$\kappa_-(z) = \frac{j\omega_1\varepsilon_0}{4} \iint ({n_r}^2 - n^2)\mathbf{e_r}.{\mathbf{e}_-}^*\;\, dx\, dy \quad (14)$$

The mode amplitude in the input ports of the waveguide is assumed as $b_+(z_1) = S_{+1}$ and $b_-(z_2) = S_{+2}$. Therefore, solving these two equations with the mentioned boundary conditions, results in:

$$b_+(z) = S_{+1} e^{-j\beta(z - z_1)} + \int_{z_1}^{z} \kappa_+(z)\, e^{j\beta z} dz\, \widetilde{a}\, e^{-j\beta z} \quad (15)$$

$$b_-(z) = S_{+2}e^{j\beta(z-z_2)} + \int_z^{z_2} \kappa_-(z)\, e^{-j\beta z} dz\, \tilde{a}\, e^{j\beta z} \quad (16)$$

To obtain transmission and reflection coefficient, differential equations of the complex mode of the resonator is needed that is derived in the next section.

2.1.2 Perturbation of Resonator Mode

In this section the resonator is considered as the unperturbed structure (figure 2) and effect of adding a waveguide is studied to derive the differential equation of the complex mode of the resonator.

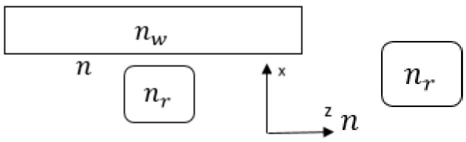

Figure 2: Refractive index distribution of unperturbed resonator (right) and perturbed structure (left).

Therefore, electromagnetic fields in the unperturbed resonator are:

$$\mathbf{E_0} = \mathbf{e_r}(x, y, z)\, e^{j\omega_0 t} + c.c. \quad (17)$$

$$\mathbf{H_0} = \mathbf{h_r}(x, y, z)\, e^{j\omega_0 t} + c.c. \quad (18)$$

Which due to radiation loss in the optical resonators, they have limited intrinsic Q-factors. as a result, modes have complex frequencies. We assume that resonator is high-Q enough to neglect this effect for now. In the next section effect of the intrinsic loss will be considered. By integrating equation (5) on the entire *x-y* plane and also from z_1 to z_2, and using divergence theorem, we have:

$$\int \nabla.(\mathbf{H} \times \mathbf{E_0} + \mathbf{H_0} \times \mathbf{E}) dr^3 \\ = \oint (\mathbf{H} \times \mathbf{E_0} + \mathbf{H_0} \times \mathbf{E}) dr^2 = 0 \quad (19)$$

Since resonator modes decay to zero at infinity, the last integral which shows integration on the transverse surfaces in infinity vanishes, and on the two surfaces in z_1 and z_2 is negligible (by choosing them far enough from the resonator).

Therefore by substituting electromagnetic fields of (17),(18) and (3),(4) in equation (5) and applying time average to omit high frequency components, one can obtain:

$$\mathrm{j}\omega_1 \tilde{a} = j(\omega_0 - \frac{\omega_1}{4}\int \Delta\varepsilon\, |\mathbf{e_r}|^2\, dr^3)\tilde{a} - \frac{j(\omega_1 - \omega_0)}{4} \\ [\int \varepsilon\, b_+(z)\mathbf{e}_+ . \mathbf{e_r}^* + \mu_0 b_+(z)\mathbf{h}_+ . \mathbf{h_r}^* dr^3] \\ - \frac{j(\omega_1 - \omega_0)}{4}[\int \varepsilon\, b_-(z)\mathbf{e}_- . \mathbf{e_r}^* \\ + \mu_0 b_-(z)\mathbf{h}_- . \mathbf{h_r}^*] dr^3] \\ - \frac{j\omega_1}{4}\int \Delta\varepsilon\, b_+(z)\mathbf{e}_+ . \mathbf{e_r}^* dr^3 \\ - \frac{j\omega_1}{4}\int \Delta\varepsilon\, b_-(z)\mathbf{e}_- . \mathbf{e_r}^* dr^3 \quad (20)$$

In the above equation the electromagnetic fields of the resonator are normalized to have unit energy i.e. $1/4 \int \varepsilon\, |\mathbf{e_r}|^2 + \mu_0 |\mathbf{h_r}|^2 dr^3 = 1$. According to figure 2, here $\Delta\varepsilon = \varepsilon_0({n_w}^2 - n^2)$ in the waveguide and is zero elsewhere, ε is the electric permittivity distribution of the unperturbed resonator. Hence the spatial coupled mode equation for forward and backward modes of the waveguide and a frequency domain equation for the complex mode amplitude of the resonator is derived. In the next section, temporal CMT of the optical coupled cavity-waveguide structure is obtained with the aid of these equations.

2.1.3 Temporal CMT of Coupled Cavity-Waveguide

By substituting $b_+(z)$ and $b_-(z)$ from (15) and (16) in (20), temporal CMT for coupled cavity-waveguide in frequency domain is obtained as follows:

$$\mathrm{j}\omega_1 \tilde{a} = j(\omega_0 - \Delta\omega_r - \Delta\omega_w)\tilde{a} + \kappa_1 S_{+1} + \kappa_2 S_{+2} - \\ \frac{1}{\tau_e}\tilde{a} \quad (21)$$

Where $\Delta\omega_r$ and $\Delta\omega_w$ are respectively the self-induced resonance frequency shift, and the resonance frequency shift due to coupling to the waveguide. κ_1 and κ_2 are coupling coefficients of incoming wave of the ports of the waveguide to the complex mode amplitude of the resonator and $1/\tau_e$ is the external decay rate of field amplitude in the resonator. By applying inverse fourier transform, one can obtain the time domain equation. The parameters in the above equation are given as follows:

$$\kappa_1 = -\frac{j\omega_1\varepsilon_0}{4}\int_{z_1}^{z_2}\iint (n_w{}^2 - n^2)\mathbf{e}_+.\mathbf{e_r}^* dx\, dy$$
$$e^{-j\beta(z-z_1)}dz - \frac{j(\omega_1-\omega_0)}{4}\int_{z_1}^{z_2}\iint \varepsilon\, \mathbf{e}_+.\mathbf{e_r}^* \qquad (22)$$
$$+\mu_0\mathbf{h}_+.\mathbf{h_r}^* dx\, dy\, e^{-j\beta(z-z_1)}dz$$

$$\Delta\omega_r = \frac{\omega_1\varepsilon_0}{4}\int (n_w{}^2 - n^2)\,|\mathbf{e_r}|^2\, d^3r \qquad (23)$$

$$\frac{1}{\tau_e} = Real\,\{f\} \qquad (24)$$

$$\Delta\omega_w = Imag\{f\} \qquad (25)$$

Where f is given by:

$$f = \int_{z_1}^{z_2}\{\frac{j\omega_1\varepsilon_0}{4}(\iint (n_w{}^2 - n^2)\mathbf{e}_+.\mathbf{e_r}^* dx\, dy)$$
$$+\frac{j(\omega_1-\omega_0)}{4}(\iint \varepsilon\,\mathbf{e}_+.\mathbf{e_r}^* + \mu_0\mathbf{h}_+.\mathbf{h_r}^* dx\, dy)\}$$
$$\int_{z_1}^{z}\kappa_+(z)e^{j\beta z}dz\,)e^{-j\beta z}\,dz$$
$$+\int_{z_1}^{z_2}\{\frac{j\omega_1\varepsilon_0}{4}(\iint (n_w{}^2 - n^2)\mathbf{e}_-.\mathbf{e_r}^* dx\, dy) \qquad (26)$$
$$+\frac{j(\omega_1-\omega_0)}{4}\int_{z_1}^{z_2}(\iint \varepsilon\,\mathbf{e}_-.\mathbf{e_r}^* + \mu_0\mathbf{h}_-.\mathbf{h_r}^* dx\, dy)\}$$
$$\int_{z}^{z_2}\kappa_-(z)e^{-j\beta z}dz\,)e^{j\beta z}\,dz$$

Transmission and Reflection coefficients are obtained according to 23 and 24, as follows:

$$T = \frac{b_+(z_2)}{b_+(z_1)} = e^{-j\beta(z_2-z_1)} + \int_{z_1}^{z_2}\kappa_+(z)e^{j\beta z}dz$$
$$e^{-j\beta z_2}\frac{\kappa_1}{j(\omega_1-\omega_0+\Delta\omega_r+\Delta\omega_w)+\frac{1}{\tau_e}} \qquad (27)$$

$$R = \frac{b_-(z_1)}{b_+(z_1)} = \int_{z_1}^{z_2}\kappa_-(z)e^{-j\beta z}dz\, e^{j\beta z_1}$$
$$\frac{\kappa_1}{j(\omega_1-\omega_0+\Delta\omega_r+\Delta\omega_w)+\frac{1}{\tau_e}} \qquad (28)$$

2.2 Dual-mode Resonator

In this section the proposed temporal CMT is generalized to the resonator with two degenerate modes. A traveling-wave resonator with clockwise (cw) and counter-clockwise (ccw) modes are considered for this purpose. This approach is general and can be applied to any other kind of dual-mode resonators. The electromagnetic fields in this structure are expanded as follows:

$$\mathbf{E} = \tilde{a}_c\;\; \mathbf{e_{r,c}}(x,y,z)e^{j\omega_1 t} + \tilde{a}_{cc}\;\; \mathbf{e_{r,cc}}(x,y,z)e^{j\omega_1 t}$$
$$+[b_+(z)\mathbf{e}_+(x,y) + b_-(z)\mathbf{e}_-(x,y)]\; e^{j\omega_1 t} + c.c. \qquad (29)$$

$$\mathbf{H} = \boldsymbol{\tilde{a}_c}\;\; \mathbf{h_{r,c}}(\boldsymbol{x,y,z})\boldsymbol{e^{j\omega_1 t}} + \boldsymbol{\tilde{a}_{cc}}\;\; \mathbf{h_{r,cc}}(\boldsymbol{x,y,z})\boldsymbol{e^{j\omega_1 t}}$$
$$+[\boldsymbol{b_+(z)}\mathbf{h}_+(\boldsymbol{x,y}) + \boldsymbol{b_-(z)}\mathbf{h}_-(\boldsymbol{x,y})]\; \boldsymbol{e^{j\omega_1 t}} + \boldsymbol{c.c.} \qquad (30)$$

By substituting these fields and the electromagnetic fields of equation (1) in (2), and applying the same procedure, one can obtain the following spatial CMT for the forward mode of the waveguide:

$$\frac{d}{dz}b_+(z) = -j\beta b_+(z) + \kappa_c^+(z)\tilde{a}_c + \kappa_{c.c}^+(z)\tilde{a}_{c.c} \qquad (31)$$

By considering electromagnetic fields of the backward mode in the unperturbed waveguide, spatial CMT for the backward mode of the waveguide is obtained as follows:

$$\frac{d}{dz}b_-(z) = j\beta b_-(z) + \kappa_{c.c}^-(z)\tilde{a}_{c.c} + \kappa_c^-(z)\tilde{a}_c \qquad (32)$$

Where κ_c^+ (κ_c^-) and $\kappa_{c.c}^+$ ($\kappa_{c.c}^-$) represent coupling of the cw to the forward (backward) mode and that of the ccw to the forward (backward) mode, which are given as follows:

$$\kappa_c^{\pm}(z) = -\frac{j\omega_1\varepsilon_0}{4}\iint (n_r{}^2 - n^2)\mathbf{e_{r,c}}.\mathbf{e}_{\pm}{}^*\;\; dx\, dy \qquad (33)$$

$$\kappa_{c.c}^{\pm}(z) = -\frac{j\omega_1\varepsilon_0}{4}\iint (n_r{}^2 - n^2)\mathbf{e_{r,cc}}.\mathbf{e}_{\pm}{}^*\;\; dx\, dy \qquad (34)$$

Solving these two equations, results in:

$$b_+(z) = S_{+1}e^{-j\beta(z-z_1)} + \int_{z_1}^{z}\kappa_c^+(z)\; e^{j\beta z}dz\,\tilde{a}_c\; e^{-j\beta z}$$
$$+\int_{z_1}^{z}\kappa_{c.c}^+(z)\; e^{j\beta z}dz\,\tilde{a}_{c.c}\; e^{-j\beta z} \qquad (35)$$

$$b_-(z) = S_{+2}e^{j\beta(z-z_2)} + \int_{z_2}^{z} \kappa_{c.c}^-(z)\, e^{-j\beta z} dz\, \tilde{a}_{c.c}\, e^{j\beta z} + \int_{z_2}^{z} \kappa_c^-(z)\, e^{-j\beta z} dz\, \tilde{a}_c\, e^{j\beta z} \quad (36)$$

Due to momentum matching, coupling between forward (backward) and ccw (cw) mode is negligible, therefore the above equations are simplified as:

$$b_+(z) = S_{+1}e^{-j\beta(z-z_1)} + \int_{z_1}^{z} \kappa_c^+(z)\, e^{j\beta z} dz\, \tilde{a}_c\, e^{-j\beta z} \quad (37)$$

$$b_-(z) = S_{+2}e^{j\beta(z-z_2)} + \int_{z_2}^{z} \kappa_{c.c}^-(z)\, e^{-j\beta z} dz\, \tilde{a}_{c.c}\, e^{j\beta z} \quad (38)$$

For obtaining the frequency domain equation for the complex mode amplitude of the resonator, the same approach as section 2.1.2 is applied and resulting equations for the cw and ccw modes are as follows:

$$\mathrm{j}\omega_1 \tilde{a}_c = j(\omega_0 - \Delta\omega_r - \Delta\omega_w)\tilde{a}_c + \kappa_1 S_{+1} - \frac{1}{\tau_e}\tilde{a}_c + \gamma \tilde{a}_{c.c} \quad (39)$$

$$\mathrm{j}\omega_1 \tilde{a}_{c.c} = j(\omega_0 - \Delta\omega_r - \Delta\omega_w{}')\tilde{a}_{c.c} - \frac{1}{\tau_e{}'}\tilde{a}_{c.c} - \gamma^* \tilde{a}_c \quad (40)$$

Parameters in the above equation are the same as those in section 2.1.3, except for substituting $\mathbf{e_r}(\mathbf{h_r})$ with $\mathbf{e_{r,c}}\ (\mathbf{h_{r,c}})$ and $\mathbf{e_{r,cc}}\ (\mathbf{h_{r,cc}})$ and also κ_+ with κ_c^+ and $\kappa_{c.c}^-$ for parameters in equation (39) and (40) respectively. A new parameter that shows coupling between cw and ccw mode is given as follows:

$$\gamma = -\frac{j(\omega_1 - \omega_0)}{4}\int_{z_1}^{z_2}\left(\iint \varepsilon\, \mathbf{e_{r,cc}}.\mathbf{e_{r,c}}^* + \mu_0 \mathbf{h_{r,cc}}.\mathbf{h_{r,c}}^* dx\, dy\right)\, dz - \frac{j(\omega_1 - \omega_0)}{4}\int_{z_1}^{z_2}\left(\iint \varepsilon_0 (n_w{}^2 - n^2)\mathbf{e_{r,cc}}.\mathbf{e_{r,c}}^* dx\, dy\right)\, dz \quad (41)$$

2.3 Intrinsic Loss

Due to coupling to radiation modes, modes of the optical resonator undergo intrinsic loss. Thus these modes are eigenmodes of Maxwell's equation with complex frequencies. We enter this effect by assuming that imaginary part of the complex frequency represents the internal decay rate of the field amplitude in the resonator. Therefore, Electromagnetic fields in the unperturbed resonator are:

$$\mathbf{E_0} = \mathbf{e_{r,c}}(x, y, z)\, e^{j(\omega_0 + \frac{j}{\tau_0})t} + c.c \quad (42)$$

$$\mathbf{H_0} = \mathbf{h_{r,c}}(x, y, z)\, e^{j(\omega_0 + \frac{j}{\tau_0})t} + c.c \quad (43)$$

Then by assuming that the resonator is high-Q enough for the mode orthogonality to be approximately valid, the resulting temporal CMT in the frequency domain becomes:

$$\mathrm{j}\omega_1 \tilde{a} = j(\omega_0 - \Delta\omega_r - \Delta\omega_w)\tilde{a} + \kappa_1 S_{+1} + \kappa_2 S_{+2} - \left(\frac{1}{\tau_e} + \frac{1}{\tau_0}\right)\tilde{a} \quad (44)$$

3 RESULTS

In this section, the derived temporal CMT is used to analyze a square resonator side-coupled to a two port waveguide, as well as a ring resonator based add drop filter (four-port structure). FDTD simulations using Lumerical are used to verify and validate the theory.

3.1 Two-port Structure with Square Resonator

Consider a square resonator with a standing wave pattern side coupled to a slab waveguide in the TE mode as shown in figure 3. The side length of the resonator, width of the waveguide and edge distance between resonator and waveguide are $= 1.54\mu m$,

$0.2\mu m$ and 0.29 μm respectively. Refractive index of the guided regions and background are 3.2 and 1 respectively.

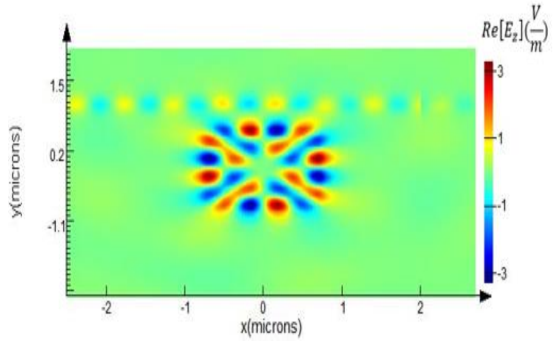

Figure 3: Electric field distribution of the TE mode of the square resonator near resonance simulated in Lumerical.

Transmission and reflection coefficients of this structure are calculated with equations derived in section 2.1.3 and are plotted in figures 4 and 5. The resonator and waveguide mode in (22) is calculated via numerical simulations. There is acceptable agreement between the proposed temporal CMT and FDTD simulations except for a slight shift between the resonance frequencies. This error is due to the approximation of mode orthogonality, or in the other words, since the resonator has limited intrinsic quality factor ($Q_0 = 4250$), its modes are not orthogonal anymore. We expect to have better results when applying the formula to a resonator with higher quality factor. The amplitude and phase of the complex mode of the resonator are calculated and plotted in figure 6.

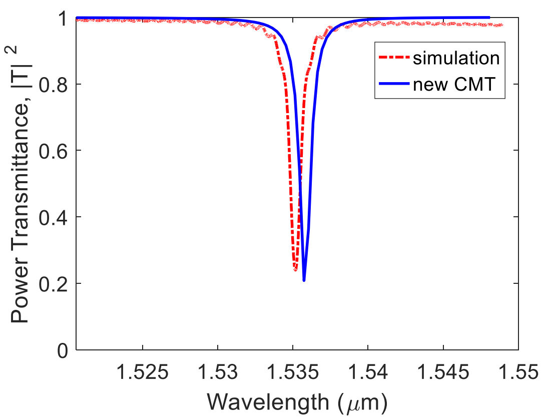

Figure 4: Power transmission coefficient calculated by the proposed temporal CMT (solid blue) and FDTD simulation (dashed red).

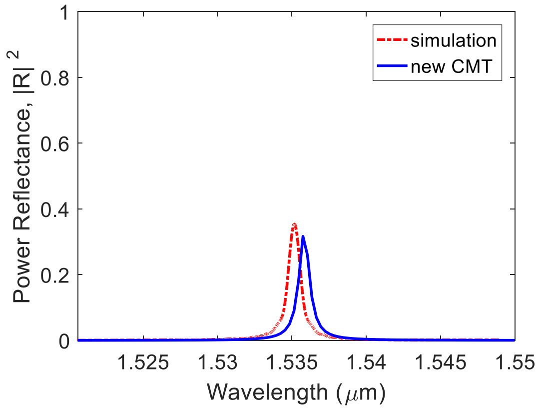

Figure 5: Power reflection coefficient calculated by the proposed temporal CMT (solid blue) and FDTD simulation (dashed red).

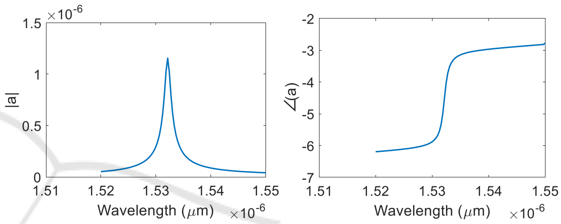

Figure 6: Amplitude (left) and phase (right) of the complex mode of the resonator.

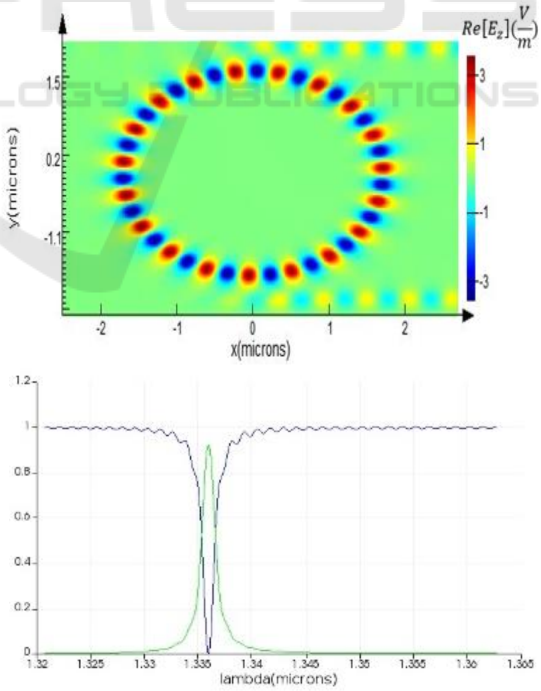

Figure 7: Electric field distribution of the TE mode of the ring resonator near resonance (upper) and transmission of the through (blue) and drop (green) ports simulated in Lumerical (bottom).

3.2 Add-drop Filter with Ring Resonator

An add-drop filter and its TE mode electric field distribution (calculated with Lumerical) are shown in figure 7. Radius of the ring, width of the ring and waveguide, and edge to edge distance between ring and each waveguide are $1.7\mu m$, $0.2\mu m$ and $0.2\mu m$ respectively. Refractive index of the guided regions and background are 3 and 1 respectively.

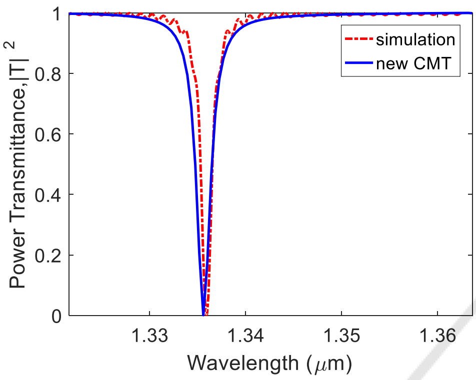

Figure 8: Power transmission coefficient calculated by the proposed temporal CMT (solid blue) and FDTD simulation (dashed red).

Since this is a traveling wave resonator that has two degenerate cw and ccw modes, result of section 2.2 is implemented to calculate transmission and reflection coefficients and results are plotted in figure 8. In this case, there is better agreement between the proposed temporal CMT and FDTD simulation, compared to the previous example. Here the quality factor of the ring is much higher than the extrinsic quality factor and effect of intrinsic loss can be neglected. In the following, amplitude of the complex cw and ccw modes of the resonator are calculated by the proposed temporal CMT as shown in figure 9. Due to the traveling wave nature of the mode, cw mode is excited mainly and ccw mode amplitude is negligible.

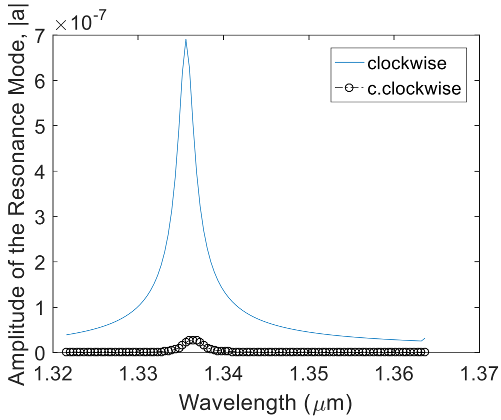

Figure 9: Amplitude of the complex cw (solid blue) and ccw (line-circle black) mode of the resonator, calculated by the proposed temporal CMT.

4 CONCLUSION

In this paper we rigorously derived temporal CMT for optical cavity-waveguide structures and obtained closed-form expressions for the coupling coefficients. Our formulation is general and can be applied to any kind of resonator, without any prior knowledge. We demonstrated its validity for structures with standing wave and traveling wave resonators, and results were verified against FDTD simulations.

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