Quantum Bit Error Rate Analysis of the Polarization based BB84 Protocol in the Presence of Channel Errors

Ágoston Schranz and Eszter Udvary
Department of Broadband Infocommunications and Electromagnetic Theory,
Budapest University of Technology and Economics,

Keywords: Quantum Key Distribution (QKD), BB84, Polarization Switching, Polarization Rotation, Quantum Bit Error Rate (QBER).

Abstract: In the BB84 quantum key distribution (QKD) protocol, the communicating parties do a quantum bit error rate (QBER) test to determine whether there is an eavesdropper trying to gain information about the secret key. However, the QBER is not only influenced by the eavesdropper's strategies, but also by the imperfections of the physical devices and the channel through which the quantum states propagate. We developed a simple channel model with error parameters describing the channel and the potential polarization switching in the transmitter, to see how those effects influence the QBER in a polarization-qubit BB84 implementation. Certain well-defined probabilistic channel models are compared to see which is responsible for the highest error probability.

1 INTRODUCTION

The first quantum key distribution protocol, commonly referred to as BB84, was published in 1984 by Charles Bennett and Gilles Brassard (Bennett and Brassard, 1984). During the key exchange, the transmitter (Alice) sends a qubit randomly prepared in one of two conjugate bases, and the receiver (Bob) chooses one of the bases also randomly, to measure the quantum state. Afterwards, the two parties disclose their basis choices on a public channel, keeping only those measured values where they chose the same and discarding everything else. A random subsequence of the raw key is then compared to calculate the quantum bit error rate. This part of the protocol makes it possible to detect eavesdropping, and the key distribution is aborted if the QBER exceeds a predefined threshold value.

The original BB84 paper already introduced an eavesdropper (Eve) performing the so called intercept-and-resend (I-R) strategy, measuring Alice's qubits randomly and sending the measured state towards Bob. If Eve performs it for every qubit, she gains an average information of 0.5 bits per key bit, but this attack introduces an average QBER of 0.25, even if the transmitter and the channel are both ideal. Performing it for a smaller portion of qubits reduces the QBER, but also decreases the average information gained about the key bits. This is due to the no-cloning theorem, which states, that no arbitrary unknown quantum state can be perfectly copied by the same device (Wootters and Zurek, 1982). We must note, that there exist more refined attacks than the I-R, resulting in lower QBERs, most notably the phase-covariant cloning machine with a QBER of 0.14644 (Bruß et al., 2000). In this paper, we deal with the most common implementation of qubits in BB84, that of linearly polarized single photons.

However, physical channels are not perfect in terms of quantum state transmission, which may introduce errors in the probabilistic measurements at the receiving end. Such imperfections might originate even from the transmitter. As an example, vertical-cavity surface-emitting lasers (VCSELs) are known to exhibit a phenomenon called polarization switching (PS), which causes the output light that is originally polarized along one of two orthogonal directions (polarization eigemodes) to rapidly switch to the orthogonal polarization (San Miguel et al., 1995; Martín-Regalado et al., 1997). If Bob chose the same basis as Alice, such a switch would introduce a certain quantum bit error assuming that the channel is free from any further imperfections, and there is no eavesdropper present. Polarization switching is depending on the laser's current, and with careful considerations, it can be eliminated for any specific device, or as shown in our previous work, even utilized for polarization modulation in the BB84 protocol (Schranz and Udvary, 2018). Regard-

Schranz, Á. and Udvary, E.
Quantum Bit Error Rate Analysis of the Polarization based BB84 Protocol in the Presence of Channel Errors.
DOI: 10.5220/0007384101810189
In *Proceedings of the 7th International Conference on Photonics, Optics and Laser Technology (PHOTOPTICS 2019)*, pages 181-189
ISBN: 978-989-758-364-3

less, in this paper, we maintain the possibility of a PS at the transmitter, always happening between two orthogonal states.

The more important issue is that of qubit errors resulting from the fact that the channel (be it an optical fiber or free-space) can alter the quantum states in such a way that even measurements in the correct basis will yield erroneous results. These can be small disturbances which may only present themselves with very low probability, but nevertheless, their effects are best not to be neglected.

Furthermore, in this paper we are not dealing with the fact that due to a number of reasons (absorption, coupling losses, the non-unit quantum efficiency of single photon detectors, the Poissonian photon statistics of semiconductor lasers used as a substitute for true single photon sources, etc.), some of the states sent by Alice will not be detected by Bob. We also assume that all such losses are polarization independent, affecting every quantum state ($|0\rangle$, $|1\rangle$, $|+\rangle$ and $|-\rangle$ representing linear polarizations with angles 0°, 90°, +45° and −45°, respectively) in the same way. If the QBER is calculated using only those time bins when both Alice and Bob used the same basis and Bob did receive a photon, then excluding the effects of losses does not reduce the generality of the error rate analysis.

In Section 2, we define the parameters necessary for our analysis and set up a polarization rotation error model for the channel. In Section 3, we derive the QBER rates for all possible combinations of errors caused by eavesdropping, polarization switches and polarization rotation. Section 4 deals with obtaining the error parameters of several channel models defined by their polarization angle distribution. Finally, in Section 5 we show how this analysis might be useful, when one wants to determine the amount of QBER originating from eavesdropping. QBER is also referred to as erroneous measurement probability throughout the paper.

2 GENERAL DEFINITIONS AND CHANNEL MODELLING

We introduce the polarization switching rate (PSR) parameter r as the average ratio of qubits for which a PS happens in Alice's transmitter, and channel error rates e_1 and e_2 for channel sections 1 and 2, respectively. The latter are defined as the *probability that measuring polarization in the correct basis yields an erroneous result.* For simplicity, we assume that all errors in all channel sections are independent of the input state and the basis it was sent in (every polarization state is affected likewise), and that PSR is also basis and state independent.

Two simple conclusions regarding an eavesdropping-free system arise from our definitions.

1. Given a transmitter with PSR r and an error-free channel, the probability of correct and bad measurement results in the correct basis is $C_{0,r} = 1 - r$ and $E_{0,r} = r$, respectively.
2. Given a PS-free transmitter and a channel with an error rate e, the probability of correct and bad measurement results in the correct basis is $C_{e,0} = 1 - e$ and $E_{e,0} = e$, respectively. This results directly from our definition of e.

The total error rate calculations are always concerning the raw keys, not taking into account the discarded results due to basis choice differences. This ultimately does not change the calculations, because polarization switches and channel errors are independent from (and uncorrelated with) Bob's basis choices.

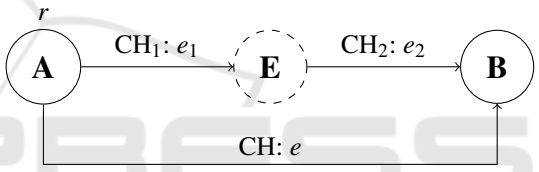


Figure 1: Diagram of the channel model in the presence and in the absence of eavesdropping. A: Alice, E: Eve, B: Bob. The presence of Eve cuts the channel CH into two sections, CH_1 and CH_2. The respective PSR and error parameters are denoted above every element.

2.1 The Channel Error Model

In the previous sections we described the channel only by its general error probability parameter e, but we did not specify the origin and nature of those errors. This leads to problems in cases where eavesdropping and channel errors are both present, because our definition of e didn't include how it influences measurements when the transmitting and receiving bases are different. Before calculating these general error rates, we need to have more information about how the channel introduces errors. Our basic assumption is, that this is done by rotating the polarization by an angle ϑ.

Malus' law states that if a polarizer is irradiated by linearly polarized light angled at ϑ relative to the polarizing axis, a proportion of $\cos^2(\vartheta)$ of the light is transmitted, while a proportion of $\sin^2(\vartheta)$ will be blocked or reflected. This can be translated to the single photon level as the following: if a linearly polarized photon is sent and measured in the same basis, but its polarization angle is rotated along the way by an

angle ϑ, the probability of measuring the state to be orthogonal to the original state is equal to $\sin^2(\vartheta)$.

2.1.1 Fixed Angle Polarization Rotation

First assume a channel, that introduces a systematic error by rotating the polarization of all incoming states by a fixed value of α. Although fixed polarization rotations or those following a discrete probability distribution might be unphysical, analyzing their properties provides a good understanding for practical cases with continuous rotation angle distributions.

For every qubit sent, the rotation angle, therefore the measurement error probability is constant, meaning that e can be calculated by Malus' law as

$$e = \sin^2(\alpha). \tag{1}$$

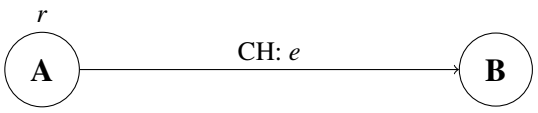
2.1.2 Random Polarization Rotation

In a practical case, the error probability depends on the rotation angle θ, which itself is a random variable characterized by a probability density function (PDF) $p_\theta(\vartheta)$. The error probability $\sin^2(\theta)$ is a function of the angle, becoming a random variable itself. Thus we need to average the errors of all possible rotations weighed by the PDF of the rotation angle itself. This way e can be calculated as the expectation value of the error probability, $\mathbb{E}[\sin^2(\theta)]$.

3 ERROR RATES FOR ALL COMBINATIONS OF ERROR SOURCES

In Section 2. we have defined two parameters e and r as error rates for situations without eavesdropping, if either only the channel, or only the transmitter may be responsible for faulty measurements. In this section, we include the possibility of an eavesdropper Eve, splitting the channel into two sections. We assume Eve to use the simple intercept-and-resend attack for every single qubit, to maximize the amount of information gained. During the following subsections, all possible combinations of error sources are taken into account and analyzed individually, with a small channel model depicting the actual parameters. The resulting correct and erroneous measurement probabilities at Bob's side are calculated as well, following the notation of $C_{e,r}$ / $E_{e,r}$ if there is no eavesdropping, and $C_{e_1,e_2,r}$ / $E_{e_1,e_2,r}$ to denote the presence of Eve.

3.1 Polarization Switching and Channel Errors in the Absence of Eavesdropping



We have seen the error rates if only either polarization switching or channel polarization rotation is present. However, combining both error sources, the situation gets more complicated. In this case, polarization switches and channel errors may counteract each other, leading to correct measurements. This can be understood easily: a polarization switch would introduce a certain measurement error, but a polarization rotation makes it possible that Eve measures the original state. Measurement errors happen in two cases: when polarization does not switch but the channel introduces an error, or when the polarization switches and the channel does not introduce an error, leading to a total error rate of

$$E_{e,r} = (1-r)e + (1-e)r = e + r - 2re. \tag{2}$$

Conversely, correct measurements happen when either none of the two problems arise, or the two effects cancel each other out.

$$C_{e,r} = (1-r)(1-e) + re \tag{3}$$
$$= 1 - e - r + 2re = 1 - E_{e,r} \tag{4}$$

It is easy to see that these equations are symmetric with respect to e and r. An interesting consequence of the error cancelling is that the correct measurement rate $C_{e,r}$ is near one if both parameters are very low or both parameters are very high, obtaining the maximum value if $e = r = 0$ or $e = r = 1$ ($C_{0,0} = C_{1,1} = 1$). In turn, almost all measurements are faulty if one of the parameters is very high with the other being very low, reaching zero if $e = 1, r = 0$ or $e = 0, r = 1$. Plotting $E_{e,r}$ and $C_{e,r}$ as a function of the parameters yields saddle-like surfaces (Fig. 2).

3.2 Eavesdropping, Polarization Switching and Channel Errors

Introducing eavesdropping to the analysis requires some easily justifiable restraints. Therefore, the basis choices of Eve and Bob are taken to be independent from each other. Moreover, for every qubit, we suppose that Bob chooses the correct basis (otherwise the results are later discarded, not presenting themselves in the error calculations).

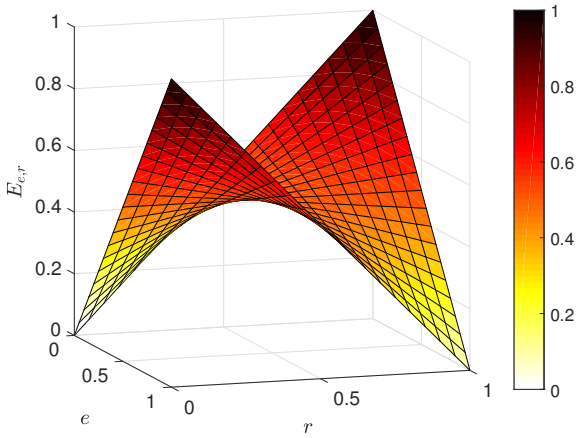
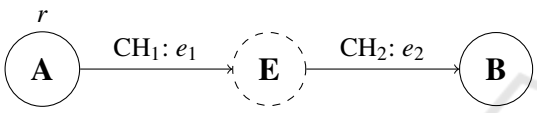


Figure 2: Erroneous measurement probability in the correct basis as a function of PSR r and channel error parameter e, in the absence of eavesdropping.



First, we analyze events that happen before Eve's measurement, letting $e_2 = 0$. Table 1 represents the probabilities by which the eavesdropper would obtain correct or wrong measurement results choosing the correct or wrong bases. Since Eve is the first to measure the qubits, if she chooses the correct basis, these values are the same as Bob's in Section 3.1, replacing e with e_1. Also, if Eve chose correctly, Bob will always measure the same result as Eve. On the other hand, if Eve chose the wrong basis, our parameter definition for e is not enough to correctly describe the probabilities of measured values; a more detailed knowledge about the channel would be necessary. This lack of knowledge is represented by p. However, the state Eve resends and Bob receives will certainly be one of the wrong basis states, and his outcome would be completely random (0.5). This happens regardless of the value of p, and the nature of the first channel section; Eve's measurement erases the previous polarization rotation, therefore p is irrelevant.

Table 1: Eve's measurement probabilities, when both polarization switches and the first channel section may cause errors.

Eve's meas. res.	Eve's basis choice	
	Correct	**Wrong**
Correct	$e_1 r + (1-e_1)(1-r)$	$0.5 + p$
Wrong	$e_1(1-r) + (1-e_1)r$	$0.5 - p$

Now set $r = e_1 = 0$ and analyze the section between Eve and Bob. Table 2 lists Eve's measurement probabilities for this case. Since there is no error between Alice and the eavesdropper, if Eve chose the correct basis, Bob would only see errors caused by the second channel section. The correct and wrong measurement probabilities are thus, by definition, $1 - e_2$ and e_2. On the other hand, if Eve uses the wrong basis and sends Bob a state that is rotated by 45 degrees with respect to his basis states, it is impossible to give a general description of what will happen, based only on our definition of e_2, which only accounts for the error probability when the sent state was in the final measurement basis. The exact nature of the channel is necessary for a complete analysis.

Table 2: Eve's measurement probabilities, when the only source of error is the second channel section.

Eve's meas. res.	Eve's basis choice	
	Correct	**Wrong**
Correct	1	0.5
Wrong	0	0.5

We can introduce the following assumption to reduce the complexity of the problem. Let's restrict the possible polarization rotations performed by the channel to those, which can be described by a probability density function symmetric around zero, an even function $f_\theta^{\text{even}}(\vartheta)$. This restriction will be maintained for the rest of the calculations for simplicity.

Assume a simple case when Alice sends a bit 1 in the rectilinear basis, $|1\rangle$, which is randomly rotated in CH_1 by either an angle $+\vartheta$ or $-\vartheta$. Eve measures the state in the diagonal basis. For an undisturbed $|1\rangle$ state, her measurement results could be 0 or 1, both with $p_1 = p_0 = \cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}$. If the state was rotated by $+\vartheta$, the probabilities would change to $p_0 = \cos^2\left(\frac{\pi}{4} - \vartheta\right)$ and $p_1 = \cos^2\left(\frac{\pi}{4} + \vartheta\right)$. On average, the total error probability in this case is

$$p_{\text{error}}^{|1\rangle} = \frac{1}{2}\left[p_{\text{error}}^{|1\rangle}|_{(\theta=+\vartheta)} + p_{\text{error}}^{|1\rangle}|_{(\theta=-\vartheta)}\right] \quad (5)$$

$$= \frac{1}{2}\left[\cos^2\left(\frac{\pi}{4} - \vartheta\right) + \cos^2\left(\frac{\pi}{4} + \vartheta\right)\right]. \quad (6)$$

Since we know that

$$\cos^2\left(\frac{\pi}{4} + \vartheta\right) = \frac{1}{2} - \frac{\sin(2\vartheta)}{2}, \quad (7)$$

this probability reduces to

$$p_{\text{error}}^{|1\rangle} = \frac{1}{2}\left(\frac{1}{2} - \frac{\sin(2\vartheta)}{2} + \frac{1}{2} - \frac{\sin(-2\vartheta)}{2}\right) \quad (8)$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{\sin(2\vartheta)}{2} + \frac{1}{2} + \frac{\sin(2\vartheta)}{2}\right) = \frac{1}{2} \quad (9)$$

The same end result applies for every other original state $|0\rangle$, $|+\rangle$ and $|-\rangle$. For the latter two, ϑ should

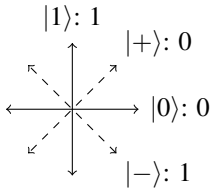


Figure 3: Bit mapping in the BB84 protocol. Polarization states $|0\rangle$ and $|+\rangle$ represent zeros, $|1\rangle$ and $|-\rangle$ represent ones in their respective bases.

be replaced by $-\vartheta$, owing to the bit-to-state mapping of BB84 (Fig. 3). All original states are equiprobable, and all of them have an error probability of $\frac{1}{2}$, therefore the total error probability in case when Eve measures in the wrong basis is $\frac{1}{2}$, regardless of the second channel section's errors.

This conclusion can be extended to any case, where the rotation angle θ is a zero-mean random variable with an even PDF $f_\theta^{\text{even}}(\vartheta)$. At this point, we can turn the weighted sum in Eq. 5 into an expectation value calculation (10).

$$p_{\text{error}}^{|1\rangle} = \int_{-\infty}^{\infty} \cos^2\left(\frac{\pi}{4}+\vartheta\right) \cdot f_\theta^{\text{even}}(\vartheta)\, d\vartheta \tag{10}$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2}+\frac{\sin(2\vartheta)}{2}\right) \cdot f_\theta^{\text{even}}(\vartheta)\, d\vartheta \tag{11}$$

$$= \frac{1}{2} \tag{12}$$

Usint the linearity of the integral, the first part of the sum in (11) evaluates to $1/2$ since the area under the curve of any PDF is unit. The second part is an integral of an odd function with symmetric limits, which – further assuming that it exists – yields zero.

The most general case is when all three errors may happen during a single qubit's transmission. Summarizing the individual analysis of the errors happening before and after the eavesdropper, we can calculate the QBER. If Eve choose the wrong basis, Bob would measure any value with probability 0.5, regardless of any errors happening due to the second channel section. If Eve choose, however, the correct basis, correct measurement at Bob's can occur in the following situations:

- Eve measured the correct value and the second section caused no error at the receiver. This happens with a probability of $[e_1 r + (1-e_1)(1-r)](1-e_2)$.
- Eve measured the incorrect value, but the second section's rotation caused Bob to measure the original value, ultimately. The probability is $[e_1(1-r)+(1-e_1)r]e_2$.

This can be read as the following: if an even number of errors "happen" during the transmission, they will cancel each other's effects and lead to a correct measurement value, while an odd number of errors lead to an erroneous value.

Averaging all situations, we can arrive at the following general formulae for the correct and wrong measurement probabilities, the latter representing the QBER.

$$C_{e_1,e_2,r} = 0.75 - \frac{e_1+e_2+r}{2} + (e_1 r + e_2 r + e_1 e_2) - 2\cdot e_1 e_2 r \tag{13}$$

$$E_{e_1,e_2,r} = 0.25 + \frac{e_1+e_2+r}{2} - (e_1 r + e_2 r + e_1 e_2) + 2\cdot e_1 e_2 r \tag{14}$$

3.3 Summary

The formulae (concluded for the QBER in Table 3) are consistent with each other, meaning that it could be easily shown that all specific results (when one or more error sources are not present) can be obtained if we insert zeros in the most general equations for the error parameters not present in the individual cases. Also, switching any two parameter values would mean no difference, since all formulae are symmetric with respect to all of e_1, e_2 and r. Note that these results only apply for the instance of simple intercept-and-resend attacks, and it would take a different approach to derive similar formulae for other types of eavesdropping strategies using the same error parameter definitions.

4 CHANNELS WITH WELL-DEFINED PROBABILITY DISTRIBUTIONS FOR POLARIZATION ROTATION

In the previous sections we developed a framework for error calculations, mostly independent from the channel's exact probability distribution for the polarization angle rotation θ, represented by the probability density function $f_\theta(\vartheta)$. The only restriction limited the generality of this framework for distributions with zero-mean symmetric distributions. In this section, we discuss several channel models with well defined error mechanisms, with an emphasis on the calculation of the e parameter, defined as the expectation value of $\sin^2(\theta)$, being a function of the distribution parameters. We use the notation σ^2 (σ) for the variance (standard deviation) of the probability distribution, and in Sec. 4.4 reparametrize the distributions with σ for the simplicity of comparison.

Table 3: Conclusive table of erroneous measurement probabilities for all combinative cases of eavesdropping, polarization switching and channel errors.

$\mathbf{e_1/e_2/r}$	**Without eavesdropping**	**With eavesdropping**
$0/0/0$	0	0.25
$e_1/0/0$	e	$0.25+0.5\cdot e_1$
$0/e_2/0$		$0.25+0.5\cdot e_2$
$e_1/e_2/0$		$0.25+0.5\cdot(e_1+e_2)-e_1e_2$
$0/0/r$	r	$0.25+0.5\cdot r$
$e_1/0/r$	$e+r-2\cdot er$	$0.25+0.5\cdot(e_1+r)-e_1r$
$0/e_2/r$		$0.25+0.5\cdot(e_2+r)-e_2r$
$e_1/e_2/r$		$0.25+0.5\cdot(e_1+e_2+r)-(e_1r+e_2r+e_1e_2)+2\cdot e_1e_2r$

4.1 Symmetric Two-point Rotation Angle Distribution

A discrete two-point distribution $\mathcal{T}(a,b,q)$ is a generalization of the Bernoulli distribution, where the two obtainable values are a and b, with probabilities q and $1-q$, respectively. Symmetry is achieved when the two outcomes are equiprobable ($q=0.5$). Assuming a channel with $\theta\sim\mathcal{T}(-\alpha,\alpha,0.5)$, rotating the polarization angle of the incoming states randomly by either $+\alpha$ or $-\alpha$, the PDF can be written in terms of the Dirac delta function $\delta(\vartheta)$:

$$f_\theta^{\text{tp}}(\vartheta)=0.5\left[\delta(\vartheta+\alpha)+\delta(\vartheta-\alpha)\right], \tag{15}$$

Since $\sin^2(\cdot)$ is an even function, this channel's error parameter is equivalent to that of a fixed angle polarization rotating channel with an angle $+\alpha$ or $-\alpha$, confirmed by the calculations as well:

$$e_{\text{tp}}=\mathbb{E}[\sin^2(\theta)]=\int_{-\infty}^{\infty}f_\theta^{\text{tp}}(\vartheta)\cdot\sin^2(\vartheta)\,d\vartheta \tag{16}$$

$$=0.5\int_{-\infty}^{\infty}\sin^2(\vartheta)\cdot\delta(\vartheta+\alpha)\,d\vartheta \tag{17}$$

$$+0.5\int_{-\infty}^{\infty}\sin^2(\vartheta)\cdot\delta(\vartheta-\alpha)\,d\vartheta$$

$$=0.5\left(\sin^2(\alpha)+\sin^2(-\alpha)\right) \tag{18}$$

$$=\sin^2(\alpha). \tag{19}$$

The error parameter of the symmetric two-point distribution is periodic with a period of π, oscillating around 0.5, between minima with value 0 at points $\alpha=\pi/2+k\pi, k\in\mathbb{Z}$ and maxima with value 1 at points $\alpha=k\pi, k\in\mathbb{Z}$. The periodicity is a property specific only to this specific distribution in the set of all zero-mean symmetric distributions. Since all continuous distributions in the set will spread increasingly as the variance grows, their error parameters will tend to 0.5 as $\sigma\to\infty$.

4.2 Uniform Rotation Angle Distribution

Now assume a channel that rotates the polarization angle of each qubit by a random angle θ following a continuous uniform distribution $\mathcal{U}(-\alpha,\alpha)$ within limits $-\alpha$ and α. This distribution is parametrized by a mean value of zero and standard deviation of $\sigma=\alpha/\sqrt{3}$. The PDF of the rotation angle is then given by

$$f_\theta^{\text{uni}}(\vartheta)=\begin{cases}\frac{1}{2\alpha}, & \text{if } \vartheta\in[-\alpha;\alpha]\\ 0, & \text{otherwise.}\end{cases} \tag{20}$$

The expectation value of the error probability can be analytically calculated, resulting in

$$e_{\text{uni}}=\mathbb{E}[\sin^2(\theta)] \tag{21}$$

$$=\int_{-\infty}^{\infty}f_\theta^{\text{uni}}(\vartheta)\cdot\sin^2(\vartheta)\,d\vartheta \tag{22}$$

$$=\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\sin^2(\vartheta)\,d\vartheta \tag{23}$$

$$=\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\frac{1-\cos(2\vartheta)}{2}\,d\vartheta \tag{24}$$

$$=\frac{1}{2}-\frac{\sin(\alpha)\cos(\alpha)}{2\alpha}. \tag{25}$$

Note that e_{uni} is undefined for the limiting case of $\alpha=0$, but $\lim_{\alpha\to0}e_{\text{uni}}=0$, agreeing with the expectations that a degenerate distribution (a certain event) with zero mean and variance causes no measurement errors. As expected, for increasing variance (the case of large α), e_{uni} approaches 0.5, oscillating around this value.

4.3 Normal Rotation Angle Distribution

For our third model, we chose an example of higher practical value: a normal distribution with mean zero and variance σ^2, denoted by $\mathcal{N}(0,\sigma^2)$. This has been reported to be the approximate distribution for the angle of polarization rotation caused by turbulence during free-space propagation, as a result of the central-limit theorem (Zhang et al., 2014; Zhang et al., 2016; Zhang et al., 2018). The approximation can be applied when the turbulence strength is low enough, so that the magnitude of the depolarized field component is small compared to the magnitude of the original, linearly polarized electric field ($M = |E_y| / |E_0|$). In this situation, the rotation angle can be approximated by the ratio of depolarized field components ($\theta \approx M$) (Strohbehn and Clifford, 1967).

The PDF of this distribution has the form of

$$f_\theta^{\text{norm}}(\vartheta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\vartheta^2}{2\sigma^2}}. \tag{26}$$

The calculation of the expectation value can be done analytically once again, resulting in

$$\begin{aligned} e_{\text{norm}} &= \mathbb{E}[\sin^2(\theta)] && (27)\\ &= \int_{-\infty}^{\infty} f_\theta^{\text{norm}}(\vartheta)\cdot\sin^2(\vartheta)\,d\vartheta && (28)\\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\sin^2(\vartheta)\cdot e^{-\frac{\vartheta^2}{2\sigma^2}}\,d\vartheta && (29)\\ &= \frac{1}{2}(1-e^{-2\sigma^2}) && (30) \end{aligned}$$

The error parameter of the normal distribution is thus a monotonically increasing function of σ, obtaining a value of 0 for $\sigma = 0$ and approaching 0.5 as $\sigma \to +\infty$, but its value never exceeds 0.5, as opposed to the uniform distribution's e_{uni}.

4.4 Comparison of Different Models

To obtain a more appropriate comparison between them, we parametrized the three previously mentioned distributions to have the same mean and variance (0 and σ^2): a two-point distribution $\mathcal{T}(-\sigma,\sigma,0.5)$, a uniform distribution $\mathcal{U}(-\sqrt{3}\sigma,\sqrt{3}\sigma)$ and a normal distribution $\mathcal{N}(0,\sigma^2)$. The parameter of the uniform distribution is reparametrized as

$$e_{\text{uni}} = \frac{1}{2} - \frac{\sin(\sqrt{3}\sigma)\cos(\sqrt{3}\sigma)}{2\sqrt{3}\sigma}. \tag{31}$$

In Fig. 4, we can see the error parameters of all three distributions as the function of standard deviation σ. The three curves are distinctly different

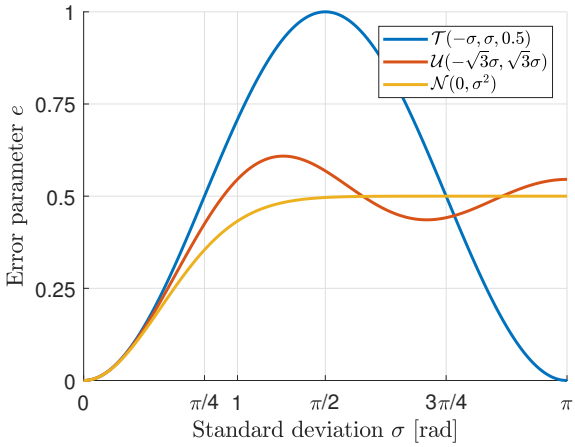


Figure 4: Error parameter e as a function of standard deviation σ in case of a two-point, a uniform and a normal rotation angle distribution.

when shown for a wide range of standard deviation, $\sigma \in [0,\pi]$, However, zooming in on the section with the most physical significance, when σ is small, would reveal that the e values are very similar across all these distributions. Quantitatively, for a given standard deviation $\sigma < \sigma_0^{\text{A,B}}$ – except for $\sigma = 0$, where e is identically zero) –, the two-point distribution has a higher error parameter than both the uniform and the normal, while the uniform distribution has a higher e compared to the normal. Given two distributions A and B, the limit deviation $\sigma_0^{\text{A,B}}$ is the highest standard deviation, for which

$$e_{\text{B}}(\sigma) \le e_{\text{A}}(\sigma), \forall\sigma : 0 < \sigma \le \sigma_0^{\text{A,B}}. \tag{32}$$

The respective approximate limit deviations for the three distributions are the following: $\sigma_0^{\text{tp,uni}} \approx 2.409$, $\sigma_0^{\text{tp,norm}} \approx 2.356$, $\sigma_0^{\text{uni,norm}} \approx 1.816$. All these values represent high standard deviations with respect to polarization rotation, far from the small-angle approximation for which the theoretical normal distribution caused by turbulence was derived. We examined the differences for $0 < \sigma < 0.2$ rad, which is still a wider range than for which the approximations hold, and found that they are almost negligible (Fig. 5). The absolute (defined as $e_{\text{tp}} - e_{\text{norm}}$ and $e_{\text{uni}} - e_{\text{norm}}$) and relative differences (normalized by e_{norm} in both cases) for small σ are monotonically increasing with growing standard deviation. For $\sigma = 0.2$, the higher absolute difference is $\sim 1.028 \cdot 10^{-3}$, while the higher relative difference is $\sim 2.67 \cdot 10^{-2}$.

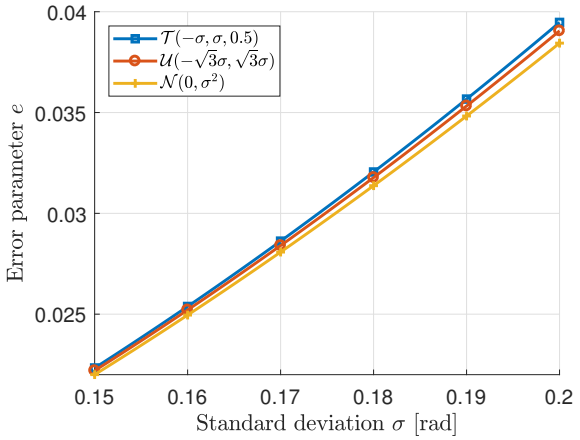


Figure 5: Close-up of the error parameter e as a function of standard deviation σ for the three different distributions for small values of σ.

5 PRACTICAL APPLICATIONS IN QKD SYSTEMS

The framework and the results derived in previous sections can be used in practical QKD systems to analyze what portion of the QBER is a result of channel errors and/or polarization switching in the transmitter, and what is the portion that cannot be described by these factors, presumably originating from eavesdropping.

The QKD transmitter is assumed to be inaccessible by Eve, and we can observe its behaviour to almost full extent, therefore we presume to possess accurate information about its polarization switching rate, the r parameter. In a perfect channel, the measured QBER ratio $E_{0,0,r}$ can be easily corrected by subtracting $0.5 \cdot r$, to see what percentage of the errors originate from other sources, mostly eavesdropping.

The advantage of the framework is that there is no need to have an accurate description of the channel in terms of the probability distribution of polarization angle rotation. After obtaining a measurement about the standard deviation the channel's polarization angle rotation and calculating the worst-case error parameter (which looks to be that of the two-point distribution, but it needs to be proven), we can correct our measured QBER. The corrected value will tell us a rough number of how much information the potential eavesdropper has gained, making it easier to find the optimal rejection threshold value. Not that since error variance measurements describe the whole channel, there remains the question of how the error parameters of the two channel sections relate to that of the full, eavesdropping-free channel, a problem that depends on Eve's exact location, presumed to be unknown by Alice and Bob.

In a practical situation, where protection against eavesdropping is the main goal, it is safer to choose the model with the highest possible error parameter to derive the final rejection threshold of QBER, above which the presence of an eavesdropper is presumed. The damages resulting from aborting the process even in the absence of eavesdropping due to an overestimation of error would be very rare if σ is small, because of the low differences between the error parameters of the analyzed distributions. Additionally, a false abortion is still better than to underestimate the error, and let an eavesdropped process continue, allowing Eve to gain significant information about the key.

6 CONCLUSION

We have seen that by introducing a simple model we can describe the effects of device and channel imperfections on the QBER of the polarization-qubit based BB84 QKD protocol. This can help determine the two communicating parties the approximate portion of QBER originating from eavesdropping, thus the information possibly gained by the unauthorized eavesdropper. QBER formulae have been derived for all possible combinations of errors described in the model, assuming that Eve uses a simple intercept-and-resend attack strategy. Different channel models were analyzed and their respective error parameters were calculated and compared, with a high emphasis on the case of small polarization angle rotations.

Further examination of the possible PDFs describing the channel's polarization angle rotation is necessary, to prove whether the two-point distribution is indeed the one with the highest error parameter for a given standard deviation, in the case of small deviations.

ACKNOWLEDGEMENTS

This research was supported by the National Research Development and Innovation Office of Hungary within the Quantum Technology National Excellence Program (Project No. 2017-1.2.1-NKP-2017-00001).

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