Modeling the Esscher Premium Principle for a System of Elliptically Distributed Risks

Tomer Shushi
Department of Business Administration, Guilford Glazer Faculty of Business and Management, Ben-Gurion University of the Negev, Beer-Sheva, Israel

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Abstract: The Esscher premium principle provides an important framework for allocating a certain loaded premium for some claim (risk) in order to manage the risks of insurance companies. In this paper, we show how to model the celebrated Esscher premium principle for a system of elliptically distributed dependent risks, where each risk is greater or equal than its value-at-risk. Furthermore, we present calculations of the proposed multivariate risk measure, investigate its properties and formulas, and show how special elliptical models can be implemented in the theory.

1 INTRODUCTION

Recently, there is a growing interest in multivariate risk measures. The motivation behind considering a multivariate risk measure is that it provides more accurate measurements of risks that are mutually dependent on each other. There are several attempts to obtain such multivariate measures (Jouini et al., 2004; Molchanov and Cascos, 2016; Cousin and Di Bernardino, 2014; Feinstein and Rudloff, 2017; Landsman et al., 2016; Shushi, 2018). For instance, Landsman et al. (2016) introduced the multivariate tail conditional expectation (MTCE) with the following form

$$MTCE_q(X) = E(X | X > VaR_q(X)).$$

Here $X = (X_1, X_2, \ldots, X_n)^T$ is $n \times 1$ vector of risks that are mutually depending on each other, and $VaR_q(X) = (VaR_{q_1}(X_1), VaR_{q_2}(X_2), \ldots, VaR_{q_n}(X_n))^T$ is $n \times 1$ vector, where $VaR_{q_i}(X_i)$ is the value at risk of $X_i$ under the $q_i$-th quantile, $q_i \in (0, 1)$. In this notation, for two $n$-variate random vectors $X$ and $Y$, $X \geq Y$ means that $X_i \geq_{st} Y_i, i = 1, \ldots, n$. The multivariate tail covariance measure was also introduced in the literature by (Landsman et al., 2018), and obtained for the class of elliptical distributions. The Esscher premium principle is a widely used measure in risk measurement and portfolio theory, which allows to quantify insurance premiums (Kamps, 1998; Van Heerwaarden et al., 1989; Landsman, 2004; Shushi, 2017; Chi et al., 2017). In the theory of risks there exist vast number of different models to calculate insurance premiums (Goovaerts et al., 1984; Wang and Dhaene, 1998; Déniz et al., 2000).

The Esscher premium was first introduced in the seminal paper of Buhlmann (Bühlmann, 1980). In his paper, Buhlmann claimed that actuaries think about premiums as a measure of risks, which are considered random. Unlike actuarial premiums, economical premiums depend also on market conditions which can be characterized by another random risk. In this paper we focus on actuarial premiums, and thus we are not taking into account any market conditions.

Let $X$ be a random risk. Then, the Esscher premium of $X$ takes the following form

$$\pi_\lambda(X) = \frac{E(Xe^{\lambda X})}{M_X(\lambda)}, \quad (1)$$

where $\lambda > 0$, $M_X(\lambda) = E(e^{\lambda X})$ is the moment generating function (MGF) of $X$, and $E(Xe^{\lambda X}) < \infty$.

The Wang’s premium (Wang et al., 2002) introduced as an exponential tilting of some risk, $X$, induced by another risk, $Y$,

$$\pi_h(X,Y) = \frac{E(Xe^{\lambda Y})}{M_Y(\lambda)}, \quad (2)$$

and the Esscher premium is the special case of $\pi_h(X,X) = \pi_\lambda(X)$. Furthermore, (2) has actuarial
sense behind the quantitative measure. For a portfolio consisting \( n \times 1 \) risks, \( X \), the measure \( \pi_{\lambda}(X_i,S) \) quantifies the amount of risk of \( X_i \) to the aggregate risks, which is the sum of the risks, i.e., \( S = X_1 + X_2 + \ldots + X_n \).

In Shushi (2018), a multivariate conditional version of the Esscher premium has been introduced which takes into account only the tail of the multivariate distribution of the vector of risks. In this paper, we generalize the Esscher premium into a conditional framework such that each risk \( X_i \) is greater than its value-at-risk measure and consider the most general case which is a portfolio that consists of \( n \)-variate dependent risks.

The multivariate conditional Esscher premium (MCEP) takes the following form

\[
\pi_{\lambda_k}(X) = \frac{E(X_i e^{\lambda_k} S | X > \text{VaR}_\mu(X_i))}{E(e^{\lambda_k} S | X > \text{VaR}_\mu(X_i))}, \lambda_k > 0, \quad (3)
\]

\( \lambda = (\lambda_1, \ldots, \lambda_n)^T, i = 1, 2, \ldots, n. \)

The motivation behind this multivariate risk measure is that it provides a conservative premium principle, in the sense that it quantifies the premium under the assumption that the \( i \)-th loss is greater than its value-at-risk. \( X_i > \text{VaR}_\mu(X_i), i = 1, 2, \ldots, \) and therefore the MCEP measure is greater than an equal to the Esscher premium:

\[
\pi_{\lambda_k}(X) \leq \pi_{\lambda_k}(X),
\]

where \( \pi_{\lambda_k}(X) = (\pi_{\lambda_k}(X_1), \pi_{\lambda_k}(X_2), \ldots, \pi_{\lambda_k}(X_n))^T. \)

We define the conditional analog to the Wang’s premium, as follows:

\[
\pi_{\lambda_k}(X_i, S) = \frac{E(X_i e^{\lambda_k} S | S > \sum_{i=1}^n \text{VaR}_\mu(X_i))}{E(e^{\lambda_k} S | S > \sum_{i=1}^n \text{VaR}_\mu(X_i))}.
\]

In the next Section, we give a concise definition of the family of elliptical distributions, and in Section 3 we analyze the proposed MCEP measure by providing its main properties and their implications. In Section 4 we compute the MCEP for a system of mutually dependent elliptically distributed risks, and in Section 5 we give examples. Section 6 offers a discussion to the paper.

2 THE CLASS OF ELLIPTICAL DISTRIBUTIONS

The class of elliptical distributions consists many important distributions such as the normal, Student-t, logistic, and Laplace distributions. In fact, it is a natural generalization of the normal distribution (Camba-Méndez et al., 1981). This class has attempting properties which will be shown in the sequel.

Let \( X \) be \( n \times 1 \) random vector following elliptical distribution, \( X \sim E_n(\mu, \Sigma, g_n) \). Then, the pdf of \( X \) is

\[
f_X(x) = \frac{c_n}{\sqrt{\left| \Sigma \right|}} g_{n} \left( \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right), x \in R,
\]

where \( c_n \) is the normalizing constant, \( g(u), u \geq 0, \) is called the density generator of \( X, \mu \) is an \( n \times 1 \) location vector, and \( \Sigma \) is an \( n \times n \) scale matrix.

The characteristic function of \( X \) takes the following form

\[
\varphi_X(t) = \exp(it^T \mu) \psi_{\mu}(\frac{1}{2} it^T \Sigma t),
\]

some function \( \psi_{\mu}(u) : [0, \infty) \to R \), called the characteristic generator.

The marginal distributions of the elliptical distribution are also elliptical with the same characteristic generator. For a random vector \( X \) such that

\[
X = (X_1, X_2)^T \sim E_n \left( \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, g_n \right),
\]

where \( X_1 \) and \( X_2 \) are \( m \times n \) and \( n - m \) random vectors, the characteristic function of \( X, \varphi_X(t) \), takes the form

\[
\varphi_X(t) = \exp(i \left( t_1^T \mu_1 + t_2^T \mu_2 \right)) \cdot \varphi_{X_1} \left( \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}^T \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)^{1/2},
\]

\( t_1 \in R^n, t_2 \in R^{n-m} \).

Then, for the marginal \( X_1 \) of \( X \) we take \( t_2 = 0 \) where \( 0 \) is vector of \( n-m \) zeros,

\[
\varphi_{X_1}(t_1) = \exp(i t_1^T \mu_1) \cdot \psi_{\mu_1} \left( \frac{1}{2} t_1^T \Sigma_{11} t_1 \right)
\]

\( t_1 \in R^n \).

As can be clearly seen, the above equation takes the same form as (5) with vector of locations \( \mu_1 \), scale matrix \( \Sigma_{11} \), and characteristic generator \( \psi(u) \). Therefore, \( X_1 \sim E_m(\mu_1, \Sigma_{11}, g_m) \).

For \( m \times n \) matrix \( B \) with rank \( m \leq n \) and \( m \times 1 \) vector \( c \), the transformation \( BX + c \) is \( m \)-variate elliptical random vector, i.e., \( BX + c \) is distributed \( E_m(\mu' = B\mu + c, \Sigma' = B\Sigma B^T, g_m) \). This can be shown by the form of elliptical characteristic function. From (5) it follows that

\[
\varphi_{BX+c}(t) = e^{it^T \varphi_X(Bt)}
\]

\[
= \exp(it^T \left( B\mu + c \right)) \cdot \psi_{\mu'} \left( \frac{1}{2} (Bt)^T \Sigma (Bt) \right)
\]

\[
= \exp(it^T \mu') \cdot \psi_{\mu'} \left( \frac{1}{2} \Sigma t \right)
\]

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From this property, we immediately establish that the marginal distribution is also elliptical as has been shown previously.

Let matrix \( B \) be
\[
B = \begin{pmatrix}
I_{m \times m} & 0_{(n-m) \times m} \\
0_{m \times (n-m)} & 0_{(n-m) \times (n-m)}
\end{pmatrix},
\]
where \( I_{m \times m} \) is the identity matrix, \( 0_{n \times m} \) is \((n-m) \times m\) matrix with zero entries, \( 0_{(n-m) \times (n-m)} \) is \((n-m) \times (n-m)\) matrix with zero entries, and \( 0_{(n-m) \times m} = 0^T_{m \times (n-m)} \). Then, random vector \( BX \) is the marginal random vector \( X_1 \). Furthermore, in the case that \( B = b \) is a \( n \times 1 \) vector, then \( b^T X \), representing weighted-sum, is distributed \( E_1(b^T \mu, b^T \Sigma b, g_1) \).

3 THE PROPERTIES OF THE MCEP MEASURE

Let us now show some important and desirable properties of the MCEP measure for the elliptical model.

**Proposition 1.** Proposition 1. Let \( X \sim E_0(\mu, \Sigma, g_0) \) be a system of \( n \) elliptically distributed risks. Then, the MCEP follows the properties:

1. **Translation Invariance:** For any random vector of risks \( X \) and any vector of constants \( \alpha \in \mathbb{R}^n \),
   \[
   \pi_{\alpha, \lambda}(X + \alpha) = \pi_{\alpha, \lambda}(X) + \alpha.
   \]
   (7)

2. **Independence of risks:** If the vector of risks \( X \) has independent components. Then
   \[
   \pi_{\alpha, \lambda}(X) = \begin{pmatrix}
   \pi_{\alpha, \lambda}(X_1) \\
   \pi_{\alpha, \lambda}(X_2) \\
   \vdots \\
   \pi_{\alpha, \lambda}(X_n)
   \end{pmatrix}.
   \]
   (8)

3. **Monotonicity:** Suppose \( Y, X \), are \( n \times 1 \) random vectors of risks and \( Y \geq X \). Then
   \[
   \pi_{\alpha, \lambda}(Y - X) \geq 0,
   \]
   (9)
   where \( 0 \) is vector of \( n \) zeros.

4. **Semi-Positive Homogeneity:** For some positive constant \( a > 0 \), The MCEP follows the following equality
   \[
   \pi_{\alpha, \lambda}(ax) = a \pi_{\alpha, \lambda}(X).
   \]
   (10)

5. **Semi-subadditivity of \( \pi_{\alpha, \lambda}(X) \) for elliptical distributions:** Consider an \((2n) \times 1\) elliptical random vector \( X \) with the partition \( X = [X_1^T, X_2^T]^T \), \( X_1 = (X_1, ..., X_n)^T \), \( X_2 = (X_{n+1}, ..., X_{2n})^T \). Then, the following inequality holds
   \[
   \pi_{\alpha, \lambda}(X_1 + X_2) \leq \pi_{\alpha, \lambda}(X_1) + \pi_{\alpha, \lambda}(X_2),
   \]
   (11)
   where \( \pi_{\alpha, \lambda}(X) = \begin{pmatrix}
   \pi_{\alpha, \lambda}(X_1)^T \\
   \pi_{\alpha, \lambda}(X_2)^T
   \end{pmatrix}^T \).

The motivation behind the semi-subadditivity property can be found in Landsman et al. (Landsman et al., 2016). In our case (11) means that combining risks provides less premium than separating them.

**Proof.**

1. The translation invariance property can be proved after some algebraic calculations. We notice that \( VaR_q(X + \alpha) = VaR_q(X) + \alpha \), so
   \[
   \pi_{\alpha, \lambda}(X + \alpha) = \frac{E \left( e^{\lambda^T (X + \alpha)} | X + \alpha > VaR_q(X + \alpha) \right)}{E \left( e^{\lambda^T X} | X + \alpha > VaR_q(X + \alpha) \right)}
   \]
   \[
   = \frac{E \left( e^{\lambda^T X} | X > VaR_q(X) \right)}{E \left( e^{\lambda^T X} | X > VaR_q(X) \right)} = \alpha + \pi_{\alpha, \lambda}(X).
   \]

2. Since we assumed that \( (X_1, X_2, ..., X_n) \) are mutually independent random risks the probability density function (pdf) of \( X \) is the multiplication for the pdf’s of the i-th component of \( X \), so
   \[
   \pi_{\alpha, \lambda}(X) = \frac{E \left( e^{\lambda^T X} | X > VaR_q(X) \right)}{E \left( e^{\lambda^T X} | X > VaR_q(X) \right)} = \prod_{i=1}^{n} \frac{E \left( e^{\lambda^T X_i} | X_i > VaR_q(X_i) \right)}{E \left( e^{\lambda^T X_i} | X_i > VaR_q(X_i) \right)}
   \]
   \[
   = \pi_{\alpha, \lambda}(X_1) \pi_{\alpha, \lambda}(X_2) \cdots \pi_{\alpha, \lambda}(X_n).
   \]

3. Notice that as \( Y \) is greater than \( (a.s.) X \), we can define a random vector in which its components get only non-negative values. \( V = Y - X \geq 0 \), where \( 0 \) is \( n \times 1 \) vector of zeros. Then, as \( V \) is non-negative random vector \( VaR_q(U) \geq 0 \), and thus \( \pi_{\alpha, \lambda}(V) \geq 0 \),
   \[
   \pi_{\alpha, \lambda}(Y - X) = E(V | V > VaR_q(V)) \geq 0.
   \]

4. Similar to the Esscher premium, the MCEP is not positive homogenous, but, the semi-positive ho-
mogeneity property holds, as follows:
\[
\pi_{\alpha,\lambda}(aX) = \frac{E(Xe^{a\lambda^T X}|aX > a\var{\text{VaR}_q}(X))}{E(e^{a\lambda^T X}|aX > a\var{\text{VaR}_q}(X))} = a\pi_{\alpha,\lambda}(X).
\]

5. The proof is similar to the proof of the semi-
subadditivity of the MTCE for elliptical dis-
tributions shown in Landsman et al. (2016). From
(McNeil et al., 2005), Theorem 6.8, we notice
that for any matrix \( B \) with \( n \times (2n) \) dimen-
sions, in the case of elliptical random vector \( X \),
\[
\text{VaR}_q(BX) \leq B\text{VaR}_q(X).
\]
In our case \( B = (I_n I_n) \). Then, since
\( \{BX > B\var{\text{VaR}_q}(X)\} = \{X > \var{\text{VaR}_q}(X)\} \) we have
\[
\pi_{\alpha,\lambda}(X_1 + X_2) \leq \pi_{\alpha,\lambda}(X_1) + \pi_{\alpha,\lambda}(X_2).
\]

4 DERIVATION OF MCEP FOR ELLIPTICAL MODELS

In risk measurement, the family of elliptical dis-
tributions is important since this family has desirable
properties which were shown in the previous Section.
This class is used to model loss distributions of some
random risks associated with this family (Landsman,
2004; Valdez and Cherinh, 2003; Xiao and Valdez,
2015). Therefore, it is natural to derive the condi-
tional Esscher premium for the family of elliptical dis-
tributions.

Before we derive the MCEP measure for ellipti-
cal models, we define a cumulative generator \( \var{\Omega}(u) \),
(Landsman and Valdez, 2003), which takes the fol-
lowing form
\[
\var{\Omega}(u) = \int_0^\infty g_n(q) dq.
\]
Furthermore, let us define a shifted cumulative gener-
ator \( \var{\Omega}_{n-1}(u) \) (Landsman et al., 2016)
\[
\var{\Omega}_{n-1}(u) = \int_u^\infty g_n(q+a) dq, a \geq 0, n > 1,
\]
under the condition that \( \var{\Omega}_{n-1}(0) < \infty \). For the
sequel, let us define the random vector of risks
\( X \sim E_n(\mu, \Sigma, g_n) \), and a standard random vector
\( Z = \Sigma^{-1/2}(X - \mu) \sim E_n(0, I, g_n) \). Furthermore, de-
fine \( \zeta_q = \Sigma^{-1/2}(\var{\text{VaR}_q}(X) - \mu) \), \( x_q = \var{\text{VaR}_q}(X) \), and
\( \zeta_{q,i-1} = (\zeta_{q,1}, \ldots, \zeta_{q, i-1}, 1, \zeta_{q,i+1}, \zeta_{q,n})^T \), and we
introduce the tail function of \((n - 1)\)-variate random
vector \( Y_i, F_{Y_i}(y) \),

\[
F_{Y_i}(y) = \int_y^\infty f_{Y_i}(u) du, u, y \in \mathbb{R}^{n-1}, du = du_1 du_2 \ldots du_n.
\]
where \( f_{Y_i}(u) \) is the elliptical pdf
\[
f_{Y_i}(y) = c_n \var{\Omega}_{n-1}(y) \left( \frac{1}{2} y^T \Sigma y + \frac{1}{2} \zeta_q^T \zeta_q \right).
\]

Lemma 1. Lemma 1. If \( M_{X,q}(\lambda) < \infty \), the conditional
tail function of \( Z \) is given by

\[
M_{X,q}(\lambda) = e^{\lambda^T \var{\mu}} \var{\Omega}(\var{\Sigma} \lambda) \frac{F_\var{\mu}(\zeta_q)}{F_X(\var{\mu})},
\]
where \( F_\var{\mu} \) is the tail function of a random vector \( \var{\mu} \) with
the pdf
\[
f_{\var{\mu}}(t) = \var{\psi}_{\var{\mu}} \left( \frac{1}{2} t^T \Sigma t \right)^{-1} e^{\lambda^T \Sigma^{-1/2} t} a_n \left( \frac{1}{2} t^T t \right).
\]

Proof. From the definition of \( M_{X,q}(\lambda) \), we have
\[
M_{X,q}(\lambda) = \int_{\var{VaR}_q(X)}^\infty \frac{e^{\lambda^T x} \cdot \Sigma^{-1/2} \cdot g_n \left( \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) dx}{F_X(\var{\mu})}.
\]
After the transformation \( z = \Sigma^{-1/2}(x - \mu) \), we have
\[
M_{X,q}(\lambda) = \int_{\zeta_q}^\infty \frac{c_n \var{\Omega}(\var{\Sigma} \lambda) \cdot g_n \left( \frac{1}{2} z^T \Sigma z \right) dz}{F_X(\var{\mu})}.
\]
Taking into account (15), we conclude that
\[
c_n \var{\Omega}(\var{\Sigma} \lambda) \cdot g_n \left( \frac{1}{2} z^T \Sigma z \right) dz = \var{\psi}_{\var{\mu}} \left( \frac{1}{2} \lambda^T \Sigma \lambda \right) F_\var{\mu}(\zeta_q),
\]
and finally,
\[ M_{\mathbf{X}, q}(\lambda) = cn e^{T \psi \varphi \lambda} \left( \frac{1}{2} \lambda T \Sigma \lambda \right) F_{\mu}(\gamma_{q}) \]

We note that the proof of Lemma 1 is based on the same method introduced in (Landsman et al., 2013) which derived the TCE and TV of the elliptical distributions, respectively.

**Theorem 1.** Theorem 1. Suppose that the conditional moment generating function of \( \mathbf{X} \), \( M_{\mathbf{X}, q}(\lambda) \), exist, and that \( E \left( X_{i} e^{T \mathbf{X}} \mathbf{X} \right) \mathbf{X} > \text{Var}(\mathbf{X})(\mathbf{X}) \) \(< \infty \) for \( i = 1, 2 \ldots \). Then, the MCEP for the multivariate elliptical distribution, \( n > 1 \), takes the form

\[ \pi_{\alpha, \lambda}(X) = \mu + \Sigma^{1/2} \chi_{q}, \lambda. \] (16)

Here \( \chi_{q, \lambda} \) is an \( n \times 1 \) vector of that depends on the \( q \) th percentile

\[ \chi_{q, \lambda} = \left( \chi_{1, q} \chi_{2, q} \cdots \chi_{n, q} \right)^{T}, \] (17)

where each component of (17) is

\[ \chi_{i, q} = \psi_{\varphi_{n-1, i}} \left( \frac{1}{2} \left( \lambda_{i} \Sigma^{1/2} \right)^{T} \lambda_{i} \Sigma^{1/2} \right) c_{n} F_{\mu}(\gamma_{q}) \psi_{\varphi_{n}} \left( \frac{1}{2} \lambda_{i} \Sigma \lambda \right) \psi_{\varphi_{n}} \left( \frac{1}{2} \lambda_{i} \Sigma \lambda \right) \]

with the pdf’s of \( \theta^{*} \in \mathbb{R}^{n} \) and \( \theta_{i}^{*} \in \mathbb{R}^{n-1}, i = 1, 2, \ldots, n, \)

\[ f_{\theta^{*}}(t) = \psi_{\varphi_{n}} \left( \frac{1}{2} \lambda^{T} \Sigma \right)^{-1} e^{T \Sigma^{1/2} \cdot t} \cdot c_{n} G_{n} \left( \frac{1}{2} \lambda^{T} \Sigma \lambda \right), t \in \mathbb{R}^{n}. \] (18)

and

\[ f_{\theta_{i}^{*}}(u) = \psi_{\varphi_{n-1, i}} \left( \frac{1}{2} \lambda_{i} \Sigma \right)^{-1} e^{(T \lambda_{i}^{T} \Sigma) \cdot u} \]

\[ c_{n-1, i} G_{n-1, i} \left( \frac{1}{2} \lambda_{i} \Sigma \right) \cdot u \in \mathbb{R}^{n-1}, \]

with the cumulative generator

\[ G_{n-1, i}(u) = \int_{0}^{\infty} g_{n}(q + 1/2 \chi_{i}) dq. \]

where \( c_{n} \) and \( c_{n-1, i} \) are the normalizing constants of (18) and (19), respectively.

**Proof.** From the definition of \( \pi_{\alpha, \lambda}(X) \), we have

\[ \pi_{\alpha, \lambda}(X) = \frac{1}{M_{\mathbf{X}, q}(\lambda)}. \]

\[ c_{n} \int_{\text{Var}(\mathbf{X})(\mathbf{X})} e^{(T \lambda \Sigma) \cdot X} \cdot |\Sigma|^{-1/2} \cdot g_{n} \left( \frac{1}{2} \varphi \lambda^{T} \Sigma \varphi \cdot (X - \mu) \right) dX \]

\[ = \frac{e^{T \mu \varphi \lambda} \cdot |\Sigma|^{-1/2} \cdot g_{n} \left( \varphi \lambda^{T} \Sigma \varphi \right) \cdot |\Sigma|^{1/2} dX}{F_{\varphi}(\gamma_{q})}. \]

Now, substituting \( \varphi \lambda = \Sigma^{1/2} \cdot (X - \mu) \), we obtain

\[ \pi_{\alpha, \lambda}(X) \]

\[ c_{n} \int_{\text{Var}(\mathbf{X})(\mathbf{X})} e^{(T \mu \varphi) \cdot X} \cdot |\Sigma|^{-1/2} \cdot g_{n} \left( \varphi \lambda^{T} \Sigma \varphi \right) \cdot |\Sigma|^{1/2} dX \]

\[ = \frac{e^{T \mu \varphi \lambda} \cdot |\Sigma|^{-1/2} \cdot g_{n} \left( \varphi \lambda^{T} \Sigma \varphi \right) \cdot |\Sigma|^{1/2} dX}{F_{\varphi}(\gamma_{q})}. \]

where \( \chi_{q, \lambda} \) is an \( n \times 1 \) vector of the form

\[ \chi_{q, \lambda} = \left( \alpha_{1, q} \alpha_{2, q} \cdots \alpha_{n, q} \right)^{T}, \]

where

\[ \alpha_{i, q} = \int_{0}^{\infty} e^{T \lambda_{i}^{T} \Sigma^{1/2} \cdot u} \cdot c_{n} G_{n} \left( \frac{1}{2} \lambda_{i}^{T} \Sigma \lambda \right) \cdot d\varphi \lambda \cdot dG_{n} \left( \frac{1}{2} \lambda_{i}^{T} \Sigma \varphi \cdot (X - \mu) \right) dX. \]

From (13), and after some algebraic calculations, we have

\[ \alpha_{i, q} \]

\[ = - \int_{\text{Var}(\mathbf{X})(\mathbf{X})} \cdot dG_{n} \left( \frac{1}{2} \lambda_{i}^{T} \Sigma \varphi \cdot (\varphi \lambda^{T} \Sigma \varphi ; \gamma_{q}) \right) \]

\[ \cdot dG_{n} \left( \frac{1}{2} \lambda_{i}^{T} \Sigma \varphi \cdot (\varphi \lambda^{T} \Sigma \varphi ; \gamma_{q}) \right) \]

where \( \varphi \lambda = (\varphi \lambda_{1}, \varphi \lambda_{2}, \ldots, \varphi \lambda_{n})^{T}, \varphi \lambda_{i} = (\varphi \lambda_{i, 1}, \varphi \lambda_{i, 2}, \ldots, \varphi \lambda_{i, n})^{T}, \) and \( \gamma_{q,i} \) is the \( i \)th
element of vector $\zeta_q$.

$$\alpha_{i,q} = \int_{\zeta_{q,-i}}^{\infty} dz_{n-1,-i} e^{(\lambda^T \Sigma^{1/2})_{i,n-1,i}} [e^{(\lambda^T \Sigma^{1/2})_{i,n-1,i}} - 1].$$

Then, after some calculations, and by using the marginality property of the elliptical distributions,

$$\alpha_{i,q} = \int_{\zeta_{q,-i}}^{\infty} dz_{n-1,-i} e^{(\lambda^T \Sigma^{1/2})_{i,n-1,i}} [e^{(\lambda^T \Sigma^{1/2})_{i,n-1,i}} - 1].$$

Finally, from the random vectors $\theta^*_i$ and $\theta^*$, we obtain

$$\alpha_{i,q} = \frac{1}{c_{n-1,i}} \psi_{\Sigma_{n-1,i}} \left( \frac{1}{2} \left( \lambda^T \Sigma^{1/2} \right) \right)_{i,n-1,i} + \frac{(\lambda^T \Sigma^{1/2})_{i,n-1,i}}{c_{n-1,i}} \psi_{\Sigma_{n-1,i}} \left( \frac{1}{2} \lambda^T \Sigma \right) \Gamma_0^{*\prime} \left( \zeta_{q,1} \right).$$

**Remark 1.** Remark. The calculation of the components $\chi_{i,q}$ can be computed explicitly for special members of the elliptical distributions (e.g., the normal, logistic and Laplace distributions), in the same way, that was obtained in (Dhaene et al., 2008).

**Corollary 1.** Corollary 1. Suppose that $X = (X_1^T, X_2^T)^T \sim E_{2n}(\mu, \Sigma, g_{2n})$, where $(X_1^T, X_2^T)^T$, $X_1, X_2 \in \mathbb{R}^n$, has uncorrelated components (i.e., $\Sigma$ is a diagonal matrix). Then, the MCEP takes the form

$$\pi_{\alpha,\lambda}(X) = \mu + \sigma_{\zeta_{i,q}}. \quad (21)$$

Here $\sigma = \text{diag}(\sqrt{\sigma_{11}}, \ldots, \sqrt{\sigma_{nn}})^T$ where $\sigma_{ii}$ is the variance of the $i$-th random variable of $X$, and $\zeta_{i,q}$ is expressed as follows:

$$\chi_{i,q} = \psi_{\Sigma_{n-1,i}} \left( \frac{1}{2} (\lambda^T \Sigma)_{i,n-1,i} \right) F_0^{*\prime}(\zeta_{q,n-1,i}) + \frac{\lambda_{n-1,i}^T}{\psi_{\Sigma_{n-1,i}} \left( \frac{1}{2} (\lambda^T \Sigma)_{i,n-1,i} \right) F_0^{*\prime}(\zeta_{q,n-1,i})} \psi_{\Sigma_{n-1,i}} \left( \frac{1}{2} (\lambda^T \Sigma)_{i,n-1,i} \right) F_0^{*\prime}(\zeta_{q,n-1,i})$$

where $1_k = (1, 1, \ldots, 1)$ is vector of $k$ ones, and $z_q = \text{VaR}_q(Z)$, $Z \sim E_1(0, 1, g_1)$.

**Proof.** As $X_1$ and $X_2$ are uncorrelated random vectors, $\Sigma$ is a diagonal matrix, so

$$\zeta_{i,q} = \Sigma^{-1/2} (\text{VaR}_q(X) - \mu) = z_q X_n.$$

This gives us the following expression of $\pi_{\alpha,\lambda}(X)$

$$\pi_{\alpha,\lambda}(X) = \mu + \sigma_{\zeta_{i,q}} \sigma_i.$$

**Lemma 2.** Lemma 2. For the random vector $(X_1, X_2)^T \sim E_2(\mu, \Sigma, g_2)$, where $X_1, X_2 \in \mathbb{R}^n$, has uncorrelated components (i.e., $\Sigma$ is a diagonal matrix). Then, the conditional Wang Esscher premium $\pi_{\alpha,\lambda}(X_1, X_2)$ is

$$\pi_{\alpha,\lambda}(X_1, X_2) = \mu + \sigma_{1} c_2 \psi_{\Sigma_1} \left( \frac{1}{2} (\lambda^T \Sigma_1)_{1,1} \right) F_0^{*\prime}(\zeta_{q,1})$$

where $\sigma_i = \sqrt{\sigma_{ii}}$.

**Proof.** From the definition of $\pi_{\alpha,\lambda}(X_1, X_2)$, we have

$$\pi_{\alpha,\lambda}(X_1, X_2) = E(X_1 e^{X_2} | X_2 > \text{VaR}_q(X_2))$$

$$= \frac{1}{\text{VaR}_q(X_2)} \int_{-\infty}^{\text{VaR}_q(X_2)} x_1 e^{x_2} \cdot |x_1|^{-1/2} g_2 \left( \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) dx$$

$$\pi_{\alpha,\lambda}(X_1, X_2) = \frac{1}{\text{VaR}_q(X_2)} \int_{-\infty}^{\text{VaR}_q(X_2)} x_1 e^{x_2} \cdot |x_1|^{-1/2} g_2 \left( \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) dx$$

Then, after some calculations, and by using the marginality property of the elliptical distributions,
\[
\pi_{\alpha,\lambda}(X_1, X_2) = \mu_1 + \frac{\sigma_{1c2}}{c_1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( \frac{1}{2} \lambda^2 \sigma_{22} z^2 \right) F_0 \left( \frac{\lambda \sigma_{11} c_1 c_2}{c_1} z_2 \right) \, dz_2
\]
and from the characteristic function of the elliptical distributions
\[
\pi_{\alpha,\lambda}(X_1, X_2) = \mu_1 + \frac{\sigma_{1c2}}{c_1} \psi_{G_{1}} \left( \frac{1}{2} \lambda^2 \sigma_{22} \right) F_{\theta} \left( \frac{\lambda \sigma_{11} c_1 c_2}{c_1} \right),
\]

\textbf{Theorem 2.} Let \( X \sim E_n(\mu, \Sigma, g_n) \) and let \( S = X_1 + X_2 + \ldots + X_n, \) so
\[
(X_1, S)^T \sim E_n \left( \sum_{j=1}^{n} \mu_j, \sum_{j=1}^{n} \sigma_{ij} \sum_{j=1}^{n} \sigma_{ij} \right),
\]
where \( \sigma_{SS} = \sum_{i,j=1}^{n} \sigma_{ij}. \) Then
\[
\pi_{\alpha,\lambda}(X_1, S) = \mu_1 + \frac{\sigma_{1c2}}{c_1} \psi_{G_{1}} \left( \frac{1}{2} \lambda^2 \sigma_{22} \right) F_{\theta} \left( \frac{\lambda \sigma_{11} c_1 c_2}{c_1} \right),
\]
where \( \xi_{S,S} = \text{Var}_R(S). \)

\textbf{Proof.} From the marginal properties of the elliptical distributions, we know that the distribution of \((X_1, S)^T\) is (23). Then, from Lemma 2, we immediately have
\[
\pi_{\alpha,\lambda}(X_1, S) = \mu_1 + \frac{\sigma_{1c2}}{c_1} \psi_{G_{1}} \left( \frac{1}{2} \lambda^2 \sigma_{22} \right) F_{\theta} \left( \frac{\lambda \sigma_{11} c_1 c_2}{c_1} \right).
\]

\section{5 EXAMPLES}

In this Section, we show several special members of the elliptical family where the MCEP can be computed. For computing the MCEP we need to compute \( \chi_{\alpha,\lambda}; \) (17).

\subsection{5.1 Normal Distribution}
Suppose that \( X \sim N_n(\mu, \Sigma). \) Then \( g_n(u) = e^{-u}, \) so \( c_{e,g_n}(\frac{1}{2}X^T \Sigma X) = \Phi_n(x) = (2\pi)^{-n/2} \exp(-\frac{1}{2}x^T X) \) and \( \Phi_n(x) \) is the \( n \)-th multivariate standard normal pdf and cdf, respectively. In this case \( \sigma_n = (2\pi)^{-n/2}, \) \( \psi_{G_n}(u) = e^{-u} = g_n(u). \) Thus
\[
\psi_{G_{n-1,j}}(\frac{1}{2}y^T y) = \exp(-\frac{1}{2} y^T y + \psi_{G_{n}}(\frac{1}{2}y^T y))
\]
so
\[
f_{\theta}^{*}(t) = f_{\theta}(t) \propto \exp \left( -\frac{1}{2} t^T \Sigma t + \lambda^T \Sigma^{1/2} t - \frac{1}{2} t^T \Sigma t \right)
\]
and
\[
f_{\theta}^{*}(u) \propto \exp \left( -\frac{1}{2} u^T \Sigma u + \lambda^T \Sigma^{1/2} u - \frac{1}{2} u^T \Sigma u \right)
\]

\subsection{5.2 Logistic Distribution}
Suppose that \( X \) has a logistic distribution. Then its pdf is
\[
f_X(x) = \frac{c_n}{\sqrt{2\pi}} \frac{\exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right)}{1 + \exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right)^2},
\]
and we write \( X \sim L_{n}(\mu, \Sigma) \) (Gupta et al., 2013). In this case the density generator is
\[
g_n(u) = \frac{\exp(-u)}{1 + \exp(-u)}
\]
and \( c_n \) is
\[
c_n = (2\pi)^{-n/2} \left[ \sum_{j=0}^{\infty} (-1)^{j-1} j^{-n/2} \right]^{-1},
\]
see (Landsman and Valdez, 2003), and the cumulative generator \( G_n(u) \) is
\[
\psi_{G_n}(u) = \int_{u}^{\infty} \frac{e^{-x}}{1 + e^{-x}} \, dx = \frac{e^{-u}}{1 + e^{-u}}.
\]

Then, \( f_{\theta}(t), f_{\theta}^{*}(t), \) and \( f_{\theta}^{*}(u) \) are, respectively,
\[
f_{\theta}(t) \propto \psi_{G_n} \left( -\frac{1}{2} t^T \Sigma t \right)^{-1} e^{\lambda^T \Sigma^{1/2} t} \frac{\exp \left( -\frac{1}{2} t^T t \right)}{1 + \exp \left( -\frac{1}{2} t^T t \right)}.
\]
\[
f_{\theta}^{*}(t) \propto \psi_{G_n} \left( -\frac{1}{2} t^T \Sigma t \right)^{-1} e^{\lambda^T \Sigma^{1/2} t} \frac{\exp \left( -\frac{1}{2} t^T t \right)}{1 + \exp \left( -\frac{1}{2} t^T t \right)}
\]
\[
f_{\theta}^{*}(u) \propto \psi_{G_n} \left( -\frac{1}{2} u^T \Sigma u \right)^{-1} e^{\lambda^T \Sigma^{1/2} u} \frac{\exp \left( -\frac{1}{2} u^T u \right)}{1 + \exp \left( -\frac{1}{2} u^T u \right)}
\]
and

\[ f_{\theta^*}(u) \propto \psi_{G \theta^*}(\frac{1}{2} u^T u)^{-1} e^{(\lambda^T \Sigma_1/2) \cdot u} \cdot \frac{\exp \left( -\frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right)}{1 + \exp \left( -\frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right)} \cdot \exp \left( -\frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right), \]

\[ u \in \mathbb{R}^{n-1}. \]

We note that while \( \psi_{G \theta^*} \) and \( \psi_{G \theta^*-1} \) can be difficult to calculate, these characteristic functions are reduced when applying them in the MCEP \( \pi_{q, \lambda}(X) \). In fact, for the \( i - th \) component of \( X_{q, \lambda} \)

\[ X_{q, \lambda} = \frac{c_n}{F_0(\zeta_q)} \int e^{(\lambda^T \Sigma_1/2) \cdot u} \cdot \frac{\exp \left( -\frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right)}{1 + \exp \left( -\frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right)} \cdot \exp \left( -\frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right) \cdot \left( -\frac{1}{2} u^T z + \lambda^T \Sigma_1/2 z \right) d\zeta_q. \]

**5.3 Laplace Distribution**

We say that \( X \) is multivariate Laplace random vector if its pdf has the form (Fang, 2017)

\[ f_X(x) = \frac{\Gamma(n/2)}{2\pi^{n/2} \Gamma(n)} \exp \left( \frac{(x-\mu)^T \Sigma^{-1} (x-\mu)^{1/2}}{2} \right) \]

and we write \( X \sim \text{La}(\mu, \Sigma) \). Then, the density generator and the characteristic generator are, respectively,

\[ g_n(u) = e^{-\sqrt{2u}} \text{ and } \psi_{g_n}(u) = \frac{1}{1 + u}. \]

In this case \( G_n(u) \) is

\[ G_n(u) = \int_u^{\infty} e^{-\sqrt{2u}} dx = \left( 1 + \sqrt{2u} \right) e^{-\sqrt{2u}}, \]

Then, \( f_{\theta^*}(t) \), \( f_{g^*}(t) \), and \( f_{\psi^*}(u) \) are, respectively,

\[ f_{\theta^*}(t) \propto \left( 1 + \frac{1}{2} t^T \Sigma t \right) e^{\lambda^T \Sigma_1/2 - \sqrt{2t^T \Sigma t}}. \]

\[ f_{g^*}(t) \propto \psi_{g_n} \left( \frac{1}{2} t^T \Sigma t \right)^{-1} e^{\lambda^T \Sigma_1/2 - \sqrt{2t^T \Sigma t}} \cdot \left( 1 + \sqrt{t^T \Sigma t} \right), t \in \mathbb{R}^d. \]

and

\[ f_{\psi^*}(u) \propto \psi_{g_n} \left( \frac{1}{2} u^T u \right)^{-1} e^{(\lambda^T \Sigma_1/2) \cdot u} \cdot \left( 1 + \frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right) \cdot \exp \left( -\frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right), \]

\[ u \in \mathbb{R}^{n-1}. \]

Notice that although \( \psi_{G \theta^*} \) and \( \psi_{G \theta^*-1} \) can be difficult to calculate they can be reduced when applying them in the MCEP \( \pi_{q, \lambda}(X) \). For the \( i - th \) component of \( X_{q, \lambda} \)

\[ X_{q, \lambda} = \frac{c_n}{F_0(\zeta_q)} \int e^{(\lambda^T \Sigma_1/2) \cdot u} \cdot \frac{\exp \left( -\frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right)}{1 + \exp \left( -\frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right)} \cdot \exp \left( -\frac{1}{2} u^T u + \frac{1}{2} \sum_{q,i} z^2 \right) \cdot \left( -\frac{1}{2} u^T z + \lambda^T \Sigma_1/2 z \right) d\zeta_q. \]

**6 DISCUSSION**

In this paper, we have shown how to model the Esscher premium principle for a system of mutually dependent risks with the underlying elliptical model, which is common in the world of risk measurement and actuarial science. Furthermore, we derived the conditional moment generating function for the family of multivariate elliptical distribution, in which the MTCE measure is a special case,

\[ MTCE_{q, \lambda}(X) = \frac{\partial}{\partial \lambda} M_{X_{q, \lambda}}(\lambda) \big|_{\lambda = 0}. \]

The MCEP measure quantifies the premium of a vector of dependent risks under the condition that an event outside a given probability level has occurred. We derived a general formula of the MCEP for the elliptical distributions

\[ \pi_{\alpha, \lambda}(X) = \mu + \Sigma^{1/2} \chi_{q, \lambda}. \]

We then derived the MCEP for aggregate risks, based on the Wang’s premium with exponential tilting,

\[ \pi_{\alpha, \lambda}(X_t, S) = \mu_t + \frac{c_1}{c_t} \psi_{g_n} \left( \frac{1}{2} \lambda^2 \sigma_{SS} \right) F_{\alpha^*}(\zeta_{s,q}) \cdot \psi_{g_n} \left( \frac{1}{2} \lambda^2 \sigma_{SS} \right) F_{\alpha^*}(\zeta_{s,q}). \]
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