The Owen and the Owen–Banzhaf Values Applied to the Study of the Madrid Assembly and the Andalusian Parliament in Legislature 2015–2019

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Abstract: This work focuses on the Owen value and the Owen–Banzhaf value, two classical concepts of solution defined on games with structure of coalition blocks. We provide a computation procedure for these solutions based on a method of double-level work obtained from the multilinear extension of the original game. Moreover, two applications to several possible political situations in the Madrid Assembly and the Andalusian Parliament (legislatures 2015–2019) are also given.

1 INTRODUCTION

Introduced in (Aumann and Drèze, 1974), the notion of *game with a coalition structure* gave a new impulsion to the development of value theory. These authors extended the Shapley value to this new framework in such a manner that the game really splits into subgames played by the unions isolatedly from each other, and every player receives the payoff allocated to him by the Shapley value in the subgame he is playing within his union.

A second approach was used in (Owen, 1977), when introducing the first *coalitional value*, called now the *Owen value*. The Owen value is the result of a *two–step procedure*: first, the unions play a *quotient game* among themselves, and each one receives a payoff which, in turn, is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by the Shapley value.

The same procedure is applied in (Owen, 1982) to the Banzhaf value and it is obtained the *modified Banzhaf value* or *Owen–Banzhaf value*. In this case the payoffs at both levels (unions in the quotient game and players within each union) are given by the Banzhaf value.

In (Alonso and Fiestras, 2002), the authors suggested to modify the two–step allocation scheme and use the Banzhaf value for sharing in the quotient game and the Shapley value within unions. This gave rise to the *symmetric coalitional Banzhaf value* or *Alonso–Fiestras value*. On the other hand, in (Amer et al., 2002) was considered a sort of "counterpart" of the Alonso–Fiestras value where the Shapley value is used in the quotient game and the Banzhaf value within unions.

The *multilinear extension* of a cooperative game was introduced in (Owen, 1972) and then it was applied to the calculus of the Shapley value. The computing technique based on the multilinear extension has been applied to many values: to the Banzhaf value in (Owen, 1975); to the Owen value in (Owen and Winter, 1992); to the Owen–Banzhaf value in (Carreras and Magana, 1994); to the quotient game in (Carreras and Magana, 1997); to the binomial semivalues and multinomial probabilistic indices in (Puente, 2000); to the coalitional semivalues in (Amer and Giménez, 2003); to the α–decisiveness and Banzhaf α–indices in (Carreras, 2004); to the Alonso–Fiestras value in (Alonso et al., 2005); to the symmetric coalitional binomial semivalues in (Carreras and Puente, 2011); to all semivalues in (Carreras and Giménez, 2011); and to coalitional multinomial probabilistic values in (Carreras and Puente, 2013).

The present paper focuses on giving a new computational procedure for the Owen and the Owen–Banzhaf value by means of the multilinear extension of the game.

The organization of the paper is as follows. In Section 2, some preliminaries are provided. Section 3 is devoted to the computation of the Owen and the Owen–Banzhaf values in terms of the multilinear ex-

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tension. Section 4 contains two applications of these values to the analysis of the Madrid Assembly and the Andalusian Parliament (legislatures 2015–2019).

2 PRELIMINARIES

2.1 Cooperative Games and Values

Let N be a finite set of *players* and 2^N be the set of its *coalitions* (subsets of N). A *cooperative game* on N is a function $v : 2^N \to \mathbb{R}$, that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$ and *simple* if, moreover, $v(S) = 0$ or 1 for every $S \subseteq N$. A player $i \in N$ is a *dummy* in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$, and *null* in v if, moreover, $v(\{i\}) = 0$. Two players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. Given a nonempty coalition $T \subseteq N$, the restriction to T of a given game v on N is the game $v_{|T}$ on T that we will call a *subgame* of v and is defined by $v_{|T}(S) = v(S)$ for all $S \subseteq T$.

Endowed with the natural operations for real–valued functions, *i.e.* $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set of all cooperative games on N is a vector space $\mathcal{G}_N$. For every nonempty coalition $T \subseteq N$, the *unanimity game* u_T is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$, so that $\dim(\mathcal{G}_N) = 2^n - 1$ if $n = |N|$. Each game $v \in \mathcal{G}_N$ can then be uniquely written as a linear combination of unanimity games, and its components are the *Harsanyi dividends* (Harsanyi, 1959):

$$v = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T u_T, \quad \text{where}$$
$$\alpha_T = \alpha_T(v) = \sum_{S \subseteq T} (-1)^{t-s} v(S)$$

and, as usual, $t = |T|$ and $s = |S|$.

By a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \to \mathbb{R}^N$, that assigns to every game v a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

Particularly, the Shapley value (Shapley, 1953) φ, and the Banzhaf value (Owen, 1975) β, are defined by

$$\varphi_i[v] = \sum_{S \subseteq N \backslash \{i\}} 1/n \binom{n-1}{s} [v(S \cup \{i\}) - v(S)]$$

and

$$\beta_i[v] = \sum_{S \subseteq N \backslash \{i\}} \frac{1}{2^{n-1}} [v(S \cup \{i\}) - v(S)]$$

for all $i \in N$ and all $v \in \mathcal{G}_N$.

As it is well known, the Shapley value is the unique value that satisfies:
(i) *additivity*: $\varphi[v + v'] = \varphi[v] + \varphi[v']$, for all $v, v' \in \mathcal{G}_N$;
(ii) *anonymity*: $\varphi_{\theta i}[\theta v] = \varphi_i[v]$ for all permutation θ on N, $i \in N$, and $v \in \mathcal{G}_N$;
(iii) *dummy player property*: if $i \in N$ is a dummy in game v, then $\psi_i[v] = v(\{i\})$.
(iv) *efficiency*: $\sum_{i \in N} \varphi_i[v] = v(N)$

The Banzhaf value follows a similar scheme, satisfying the *total power property* (Owen, 1975)

$$\sum_{i \in N} \beta_i[v] = \sum_{i \in N} \sum_{S \subseteq N \backslash \{i\}} \frac{1}{2^{n-1}} [v(S \cup \{i\}) - v(S)]$$

for all $v \in \mathcal{G}_N$, instead of additivity.

Notice that these two values are defined for each N. In fact, these values are defined on cardinalities rather than on specific player sets. When necessary, we shall write $\varphi^{(n)}$ and β^n for the Shapley and Banzhaf values on cardinality n. In both cases, $\varphi^{(n)}$ and β^n induce values $\varphi^{(t)}$ and β^t for all cardinalities $t < n$.

The *multilinear extension* (Owen, 1972) of a game $v \in \mathcal{G}_N$ is the real–valued function defined on $\mathbb{R}^N$ by

$$f_v(X_N) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \in N \backslash S} (1 - x_j) v(S).$$

where X_N denotes the set of variables x_i for $i \in N$.

As it is well known, both the Shapley and Banzhaf values of any game v can be easily obtained from its multilinear extension. Indeed, $\varphi[v]$ can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0, 1]^N$ (Owen, 1972), while the partial derivatives of that multilinear extension evaluated at point $(1/2, 1/2, \ldots, 1/2)$ give $\beta[v]$ (Owen, 1975). This latter procedure extends well to any p–binomial semivalue evaluating the derivatives at point $(p, p, \ldots, p)$, as we can see in (Puente, 2000), (Freixas and Puente, 2002), or (Amer and Giménez, 2003).

2.2 Games with Coalition Structure and Coalitional Values

Given $N = \{1, 2, \ldots, n\}$, we will denote by $B(N)$ the set of all partitions of N. Each $B \in B(N)$ is called a *coalition structure* in N, and a *union* each member of B. The so–called *trivial coalition structures* are $B^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ (individual coalitions) and $B^N = \{N\}$ (grand coalition). A *cooperative game with a coalition structure* is a pair $[v; B]$, where $v \in \mathcal{G}_N$ and $B \in B(N)$ for a given N. Each partition B gives a pattern of cooperation among players. We denote by $\mathcal{G}_N^{cs} = \mathcal{G}_N \times B(N)$ the set of all cooperative games with a coalition structure and player set N.

If $[v;B] \in \mathcal{G}_N^{cs}$ and $B = \{B_1, B_2, \ldots, B_m\}$, the *quotient game* v^B is the cooperative game played by the unions or, rather, by the *quotient set* $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^B(R) = v(\bigcup_{r \in R} B_r) \quad \text{for all } R \subseteq M.$$

By a *coalitional value* on $\mathcal{G}_N^{cs}$ we will mean a map $g : \mathcal{G}_N^{cs} \to \mathbb{R}^N$, which assigns to every pair $[v;B]$ a vector $g[v;B]$ with components $g_i[v;B]$ for each $i \in N$.

If f is a value on $\mathcal{G}_N$ and g is a coalitional value on $\mathcal{G}_N^{cs}$, it is said that g is a *coalitional value of* f iff $g[v;B^n] = f[v]$ for all $v \in \mathcal{G}_N$.

The *Owen value* (Owen, 1977) is the coalitional value Φ defined by

$$\Phi_i[v;B] = \sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq B_k \setminus \{i\}} \frac{1}{mb_k} \frac{1}{\binom{m-1}{r}} \frac{1}{\binom{b_k-1}{t}} [v(Q \cup T \cup \{i\}) - v(Q \cup T)]$$

for all $i \in N$ and $[v;B] \in \mathcal{G}_N^{cs}$, where $B_k \in B$ is the union such that $i \in B_k$ and $Q = \bigcup_{r \in R} B_r$.

It was axiomatically characterized in (Owen, 1977) as the only coalitional value that satisfies the following properties: the natural extensions to this framework of

− *efficiency*
− *additivity*
− *the dummy player property*

and also

− *symmetry within unions*: if $i, j \in B_k$ are symmetric in v then

$$\Phi_i[v;B] = \Phi_j[v;B]$$

− *symmetry in the quotient game*: if $B_r, B_s \in B$ are symmetric in $[v;B]$ then

$$\sum_{i \in B_r} \Phi_i[v;B] = \sum_{j \in B_s} \Phi_j[v;B].$$

The Owen value is a *coalitional value of the Shapley value* φ in the sense that $\Phi[v;B^n] = \varphi[v]$ for all $v \in \mathcal{G}_N$. Besides, $\Phi[v;B^N] = \varphi[v]$. Finally, as Φ is defined for any N, the following property makes sense and is also satisfied:

− *quotient game property*: for all $[v;B] \in \mathcal{G}_N^{cs}$,

$$\sum_{i \in B_k} \Phi_i[v;B] = \Phi_k[v^B;B^m] \quad \text{for all } B_k \in B.$$

The Owen value can be viewed as a two–step allocation rule. First, each union B_k receives its payoff in the quotient game according to the Shapley value; then, each B_k splits this amount among its players by applying the Shapley value to a game played in B_k as follows: the worth of each subcoalition T of B_k is the Shapley value that T would get in a "pseudoquotient game" played by T and the remaining unions on the assumption that $B_k \backslash T$ leaves the game, *i.e.* the quotient game after replacing B_k with T. This is the way to bargain within the union: each subcoalition T claims the payoff it would obtain when dealing with the other unions in absence of its partners in B_k.

The Owen–Banzhaf value Γ (Owen, 1982) follows a similar scheme. The resulting formula parallels that of the Owen value given above with the sole change of coefficient $1/mb_k\binom{m-1}{r}\binom{b_k-1}{t}$ by $2^{1-m}2^{1-b_k}$. This value, which is a *coalitional value of the Banzhaf value* β, does not satisfy efficiency, but neither symmetry in the quotient game nor the quotient game property. The bargaining interpretation is the same as in the case of the Owen value by replacing everywhere the Shapley value with the Banzhaf value.

3 A NEW COMPUTATIONAL PROCEDURE

In this section we give a new computational procedure to calculate the Owen and the Owen–Banzhaf values in terms of the MLE of the game. First of all we need the following lemma.

Lemma 3.1. *Let* $[v;B] \in \mathcal{G}_N^{cs}$, $B = \{B_1, B_2, \ldots, B_m\}$ *a coalition structure in N. Then, the allocations given by* Φ *and* Γ *to players belonging to a union* B_j *can be obtained as a linear combination of the allocations to unanimity games* u_T, *where* $T = V \cup W$, $V \subseteq B_j$ *and* $W \in 2^{B \backslash B_j}$.

Proof. Each game $v \in \mathcal{G}_N$ can be uniquely written as linear combination of unanimity games

$$v = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T u_T,$$

where $\alpha_T = \alpha_T(v) = \sum_{S \subseteq T} (-1)^{t-s} v(S)$.

For all $i \in B_j$, by linearity, $\Phi_i[v;B] = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T \Phi_i[u_T]$ and it suffices consider unanimity games u_T with

$$T = V \cup A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_p},\ V \subseteq B_j,$$
$$\{i_1, i_2, ..., i_p\} \subseteq M \setminus \{j\},\ \emptyset \neq A_{i_q} \subseteq B_{i_q},\ q = 1, ..., p.$$

According to the definition of the Owen value it is easy to check that the allocations to players in B_j only depend on the allocations in the unanimity games defined on inside coalitions in B_j and entire unions outside B_j. That is,

$$\Phi_i[u_T;B] = \Phi_i[u_{V \cup A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_p}};B] = \Phi_i[u_{V \cup B_{i_1} \cup B_{i_2} \cup \ldots \cup B_{i_p}};B].$$

Analogously for the Owen–Banzhaf value. $\square$

Notice that the number of unanimity games of this form is $(2^{b_j}-1)2^m$ with $b_j = |B_j|$ and $m = |M|$.

Proposition 3.2. *Let $B = \{B_1, B_2, \ldots, B_m\}$ be a coalition structure in N. Fixed a union B_j,*

(i) The allocation to a player i belonging to B_j in a unanimity game u_T, $T = V \cup B_{i_1} \cup \cdots \cup B_{i_h}$, $V \subseteq B_j$ and $\{i_1, ..., i_h\} \subseteq M \setminus \{j\}$, given by the Owen value Φ is

$$\Phi_i[u_T;B] = \begin{cases} \frac{1}{h+1}\frac{1}{v} & i \in T \\ 0 & i \notin T \end{cases}$$

where $\frac{1}{h+1}$ and $\frac{1}{v}$ are the induced coefficients of $\varphi^{(h+1)}$ and $\varphi^{(v)}$, respectively.

(ii) The allocation to a player i belonging to B_j in a unanimity game u_T, $T = V \cup B_{i_1} \cup \cdots \cup B_{i_h}$, $V \subseteq B_j$ and $\{i_1, ..., i_h\} \subseteq M \setminus \{j\}$, given by the Owen–Banzhaf value Γ is

$$\Gamma_i[u_T;B] = \begin{cases} 2^{-h}2^{-(v-1)} = 2^{-(h+v-1)} & i \in T \\ 0 & i \notin T \end{cases}$$

where $\frac{1}{2^h}$ and $\frac{1}{2^{v-1}}$ are the induced coefficients of $\beta^{(h+1)}$ and $\beta^{(v)}$, respectively.

Proof. (i) For $i \in T$, we have

$$\Phi_i[u_T;B] = \sum_{R \subseteq M \setminus \{j\}} 1/m \binom{m-1}{r} \sum_{S \subseteq B_j \setminus \{i\}} 1/b_j \binom{b_j-1}{s} [u_T(Q \cup S \cup \{i\}) - u_T(Q \cup S)]$$

where $Q = \bigcup_{r \in R} B_r$, $b_j = |B_j|$, and $s = |S|$.

Only $u_T(Q \cup S \cup \{i\}) - u_T(Q \cup S)$ does not vanish for coalitions R such that $\{i_1, ..., i_h\} \subseteq R \subseteq M \setminus \{j\}$ and for coalitions S such that $V \setminus \{i\} \subseteq S \subseteq B_j \setminus \{i\}$. Then,

$$\Phi_i[u_T;B] = \sum_{r=h}^{m-1} \binom{m-1-h}{r-h} 1/m \binom{m-1}{r} \sum_{s=v-1}^{b_j-1} \binom{b_j-v}{s-v+1} 1/b_j \binom{b_j-1}{s} = \frac{1}{h+1}\frac{1}{v}$$

In case of $i \notin T$, all marginal contributions $u_T(Q \cup S \cup \{i\}) - u_T(Q \cup S)$ vanish.

(ii) For $i \in T$, we have

$$\Gamma_i[u_T;B] = \sum_{R \subseteq M \setminus \{j\}} 2^{-(m-1)} \sum_{S \subseteq B_j \setminus \{i\}} 2^{-(b_j-1)} [u_T(Q \cup S \cup \{i\}) - u_T(Q \cup S)]$$

where $Q = \bigcup_{r \in R} B_r$, $b_j = |B_j|$, and $s = |S|$.

Analogously to the previous case, only $u_T(Q \cup S \cup \{i\}) - u_T(Q \cup S)$ does not vanish for coalitions R such that $\{i_1, ..., i_h\} \subseteq R \subseteq M \setminus \{j\}$ and for coalitions S such that $V \setminus \{i\} \subseteq S \subseteq B_j \setminus \{i\}$. Then,

$$\Gamma_i[u_T;B] = \sum_{r=h}^{m-1} \binom{m-1-h}{r-h} 2^{-(m-1)} \sum_{s=v-1}^{b_j-1} \binom{b_j-v}{s-v+1} 2^{-(b_j-1)} = 2^{-(h+v-1)}$$

In case of $i \notin T$, all marginal contributions $u_T(Q \cup S \cup \{i\}) - u_T(Q \cup S)$ vanish. $\square$

Example 3.3. *On the set $N = \{1,2,3,4,5,6,7\}$, let $B = \{\{1,2,3\},\{4,5\},\{6\},\{7\}\}$ be a coalition structure.*

We will obtain the allocations to players $i \in B_1$ according to Φ for the unanimity games $u_{\{1,2,4,6,7\}}$ and $u_{\{1,2,4,5,6,7\}}$. They are

$$\Phi_i[u_{\{1,2,4,6,7\}};B] = \frac{1}{4}\cdot\frac{1}{2}, \textit{ for } i = 1,2 \textit{ and}$$
$$\Phi_3[u_{\{1,2,4,6\}};B] = 0,$$

In a similar way and according to Lemma 3.1, for $u_{\{1,2,4,5,6,7\}}$ we obtain

$$\Phi_i[u_{\{1,2,4,5,6,7\}};B] = \frac{1}{4}\cdot\frac{1}{2}, \textit{ for } i = 1,2 \textit{ and}$$
$$\Phi_3[u_{\{1,2,4,5,6,7\}};B] = 0,$$

Notice that the allocations in both games are the same because coalitions $\{1,2,4,6,7\}$ and $\{1,2,4,5,6,7\}$ intersect the same unions B_2, B_3 and B_4.

The computing technique based on the multilinear extension has been applied to many coalitional values: to the Owen value in (Owen and Winter, 1992); to the Owen–Banzhaf value in (Carreras and Magana, 1994); to the quotient game in (Carreras and Magana, 1997); to the coalitional semivalues in (Amer and Giménez, 2003); to the Alonso–Fiestras value in (Alonso et al., 2005); to the symmetric coalitional binomial semivalues in (Carreras and Puente, 2011); and to the coalitional multinomial probabilistic values in (Carreras and Puente, 2013). In next theorems we present a new method to compute the Owen and the Owen–Banhaf values by means of the multilinear extension of the game.

Theorem 3.4. *Given in N a game with coalition structure, $[v;B] \in \mathcal{G}_N^{cs}$, $B = \{B_1, B_2, \ldots, B_m\}$ coalition structure in N, the following steps lead to the Owen value of any player $i \in B_j$ in $[v;B]$.*

1. *Obtain the multilinear extension $f(x_1, x_2, \ldots, x_n)$ of game v.*

2. *For every $r \neq j$ and all $h \in B_r$, replace the variable x_h with y_r. This yields a new function of x_k for $k \in B_j$ and y_r for $r \in M \setminus \{j\}$.*
3. *In this new function, reduce to 1 all higher exponents, i.e. replace with y_r each y_r^q such that $q > 1$. This gives a new multilinear function denoted as $g_j((x_k)_{k \in B_j}, (y_r)_{r \in M \setminus \{j\}})$ –the modified multilinear extension of union B_j–*
4. *After some calculus, the obtained modified multilinear extension reduces to*
$$g_j((x_k)_{k \in B_j}, (y_r)_{r \in M \setminus \{j\}}) = \sum_{V \subseteq B_j} \sum_{W \subseteq M \setminus \{j\}} \lambda_{V \cup W} \prod_{k \in V} x_k \prod_{r \in W} y_r$$
5. *Multiply each product $\prod_{k \in V} x_k$ by $\frac{1}{v}$ and each product $\prod_{r \in W} y_r$ by $\frac{1}{w+1}$ obtaining a new multilinear function called $\overline{g}_j$.*
6. *Obtain the partial derivative of $\overline{g}_j$ with respect to x_i evaluated at point $(1, \ldots, 1)$ and*
$$\Phi_i[v;B] = \frac{\partial \overline{g}_j}{\partial x_i}(1_{B_j}, 1_{M \setminus \{j\}}).$$

Proof. Steps 1–3 have been already used in (Owen and Winter, 1992), (Carreras and Magana, 1994), (Puente, 2000), (Freixas and Puente, 2002), (Alonso et al., 2005), (Carreras and Puente, 2011) and (Carreras and Puente, 2013) to obtain the modified multilinear extension of union B_j. Step 4 shows the modified multilinear extension as a linear combination of multilinear extensions of unanimity games. Step 5 weights each unanimity game according to Proposition 3.2 so that step 6 gives as usual the marginal contribution of player i and his allocation $\Phi_i[v;B]$ is obtained. □

Theorem 3.5. *Given in N a game with coalition structure, $[v;B] \in \mathcal{G}_N^{cs}$, $B = \{B_1, B_2, \ldots, B_m\}$ coalition structure in N, the following steps lead to the Owen–Banzhaf value of any player $i \in B_j$ in $[v;B]$.*

1. *Obtain the multilinear extension $f(x_1, x_2, \ldots, x_n)$ of game v.*
2. *For every $r \neq j$ and all $h \in B_r$, replace the variable x_h with y_r. This yields a new function of x_k for $k \in B_j$ and y_r for $r \in M \setminus \{j\}$.*
3. *In this new function, reduce to 1 all higher exponents, i.e. replace with y_r each y_r^q such that $q > 1$. This gives a new multilinear function denoted as $g_j((x_k)_{k \in B_j}, (y_r)_{r \in M \setminus \{j\}})$ –the modified multilinear extension of union B_j–*
4. *After some calculus, the obtained modified multilinear extension reduces to*
$$g_j((x_k)_{k \in B_j}, (y_r)_{r \in M \setminus \{j\}}) = \sum_{V \subseteq B_j} \sum_{W \subseteq M \setminus \{j\}} \lambda_{V \cup W} \prod_{k \in V} x_k \prod_{r \in W} y_r$$
5. *Multiply each product $\prod_{k \in V} x_k$ by $2^{-(v-1)}$ and each product $\prod_{r \in W} y_r$ by 2^{-w} obtaining a new multilinear function called $\overline{g}_j$.*
6. *Obtain the partial derivative of $\overline{g}_j$ with respect to x_i evaluated at point $(1, \ldots, 1)$ and*
$$\Gamma_i[v;B] = \frac{\partial \overline{g}_j}{\partial x_i}(1_{B_j}, 1_{M \setminus \{j\}}).$$

Proof. Analogously to the previous theorem. □

Example 3.6. *Let us consider the 4–person weighted majority game $v \equiv [65; 48, 37, 27, 17]$ and the coalition structure $B = \{\{1,4\}, \{2\}, \{3\}\}$. We will compute $\Phi[v;B]$.*

The set of minimal winning coalitions is
$$W^m(v) = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3,4\}\},$$
and the multilinear extension of v is given by
$$f(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3x_4 - x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4.$$
Notice that players 2 and 3 became null in the quotient game and, by the quotient game property, $\Phi_2[v;B] = \Phi_3[v;B] = 0$ and it is not necessary to compute the corresponding modified multilinear extensions g_2 and g_3.

Steps 1–4 in Theorem 3.4 give the modified multilinear extension g_1 of union B_1:
$$g_1(x_1, x_4, y_2, y_3) = x_1y_2 + x_1y_3 + x_1x_4 + y_2y_3x_4 - x_1y_2y_3 - x_1y_2x_4 - x_1y_3x_4,$$
Step 5 leads to $\overline{g}_1$.
$$\overline{g}_1(x_1, x_4, y_2, y_3) = \frac{1}{2}x_1y_2 + \frac{1}{2}x_1y_3 + \frac{1}{2}x_1x_4 + \frac{1}{3}y_2y_3x_4 - \frac{1}{3}x_1y_2y_3 - \frac{1}{2}\frac{1}{2}x_1y_2x_4 - \frac{1}{2}\frac{1}{2}x_1y_3x_4$$
Step 6 yields
$$\Phi_1[v;B] = \frac{2}{3}$$
$$\Phi_4[v;B] = \frac{1}{3}.$$

4 TWO APPLICATIONS TO THE POLITICAL ANALYSIS

Example 4.1. *We consider here the Madrid Assembly in legislature 2015–2019.*

Four parties elected members to the Madrid Assembly (129 seats) in the elections held on 24 May 2015. The seat distribution of the parties are as follows.

Table 1: Classical measures of power in the Madrid Assembly, legislature 2015–2019.

	(a)			(b)			(c)		
	(–)	(R)	(L)	(–)	(R)	(L)	(–)	(R)	(L)
1. PP	0.5000	0.6666	0.0000	0.7500	0.7500	0.0000	0.7500	0.7500	0.0000
2. PSOE	0.1666	0.0000	0.3333	0.2500	0.0000	0.2500	0.2500	0.0000	0.3333
3. Podemos	0.1666	0.0000	0.3333	0.2500	0.0000	0.2500	0.2500	0.0000	0.3333
4. C's	0.1666	0.3333	0.3333	0.2500	0.2500	0.2500	0.2500	0.2500	0.3333

1: *PP (Partido Popular), conservative party: 48 seats.*

2: *PSOE (Partido Socialista Obrero Español), moderate left–wing party: 37 seats.*

3: *Podemos, radical left–wing party: 27 seats*

4: *C's (Ciudadanos), Spanish nationalist liberal party: 17 seats.*

Under the standard absolute majority rule, and assuming voting discipline within parties, the structure of this parliamentary body can be represented by the weighted majority game

$$v \equiv [65; 48, 37, 27, 17].$$

Therefore, the strategic situation given by means of the set of minimal wining coalitions

$$W^m(v) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\},$$

shows that players 2, 3 and 4 are symmetric in v, and the multilinear extension of v is

$$f(x_1,x_2,x_3,x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3x_4 - x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4.$$

A main feature of the Madrid Assembly issued from the elections was the absence of a party enjoying absolute majority, so a coalition government was expected to form. We will not try to give here a full description of the complexity of the Madrid politics. We wish only to state that the politically most likely coalitions to form, and the corresponding coalition structures to the analysis of which we will limit ourselves, were clearly the following:

- *PP + C's, the "right"–wing majority alliance:* $B_R = \{\{1,4\},\{2\},\{3\}\}$.
- *PSOE + Podemos + C's, the "left"–wing majority alliance:* $B_L = \{\{1\},\{2,3,4\}\}$.

We would like to analyze these two situations. Of course, our main interest will center on the strategic possibilities of party 4 (C's), whose position is crucial in the two–alternative scenario we are considering.

A classical approach to study the problem would consist in using either (a) the Shapley value and the Owen value, (b) the Banzhaf value and the Owen–Banzhaf value, or (c) the Banzhaf value and the symmetric coalitional Banzhaf value, in order to evaluate the strategic possibilities of each party under both hypotheses. The results are given in Table 1, where (–) means no coalition formation, (R) means that PP + C's forms, and (L) means that PSOE + Podemos + C's forms.

According to (a), C's gets the same profit in both alliances. The same conclusion is obtained according to (b). Instead, according to (c), C's would strictly prefer joining PSOE and Podemos instead of PP. Moreover, by symmetry, the power of C's when there is not a coalition formation coincides with the power of PSOE. According to (a), when the "right"-wing alliance is formed, the outside parties are reduced to a null position and the power of C's increases regarding to the initial power in v. The same happens when the "left"–wing alliance is formed

As we have seen, in the present Legislature, studied here, in order to form a government coalition the role of C's was crucial. Thus, C's was faced to the dilemma of choosing among either a a "left"–wing majority coalition with PSOE and Podemos or a "right"–wing majority coalition with PP, which was finally formed in 2015.

Example 4.2. *We consider here the Andalusian Parliament (legislature 2015–2019).*

Five parties elected members to the Andalucia Parliament (109 seats) in the elections held on 22 March 2015. The seat distribution of the parties are as follows.

1: *PSOE (Partido Socialista Obrero Español), moderate left–wing party: 47 seats.*

2: *PP (Partido Popular), conservative party: 33 seats.*

3: *Podemos, radical left–wing party: 15 seats*

4: *C's (Ciudadanos), Spanish nationalist liberal party: 9 seats.*

5: *IULV–CA, Coalition of eurocommunist parties, federated to Izquierda Unida, and ecologist*

Table 2: Classical measures of power in the Andalusian Parliament, legislature 2015–2019.

	(a)				(b)			
	(–)	(R)	(L)	(N)	(–)	(R)	(L)	(N)
1. PSOE	0.5000	0.3333	0.6666	0.6666	0.7500	0.5000	0.7500	0.7500
2. PP	0.1666	0.1666	0.0000	0.0000	0.2500	0.2500	0.0000	0.0000
3. Podemos	0.1666	0.3333	0.3333	0.0000	0.2500	0.5000	0.2500	0.0000
4. C's	0.1666	0.1666	0.0000	0.3333	0.2500	0.2500	0.0000	0.2500
5. IULV–CA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

groups : 5 seats

Under the standard absolute majority rule, and assuming voting discipline within parties, the structure of this parliamentary body can be represented by the weighted majority game

$$v \equiv [55; 47, 33, 15, 9, 5].$$

Therefore, the strategic situation is given by

$$W^m(v) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\},$$

so that players 2, 3 and 4 are symmetric in v, player 5 is null and the multilinear extension of v is

$$f(x_1,x_2,x_3,x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3x_4 - x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4.$$

As in the same case as the Madrid Assembly, a coalition government was expected to form. We will analyze the following alliances:

- *PP + C's, the "right" alliance:* $B_R = \{\{2,4\},\{1\},\{3\},\{5\}\}$.
- *PSOE + Podemos, the "left"–wing majority alliance:* $B_L = \{\{1,3\},\{2\},\{4\},\{5\}\}$.
- *PSOE + C's, the "neutral"–wing majority alliance:* $B_N = \{\{1,4\},\{2\},\{3\},\{5\}\}$

To study the problem we will use either (a) the Shapley value and the Owen value and (b) the Banzhaf value and the Owen–Banzhaf value, in order to evaluate the strategic possibilities of each party under the three hypotheses. The results are given in Table 2, where (–) means no coalition formation, (R) means that PP + C's forms, and (L) means that PSOE + Podemos forms and (N) that PSOE + C's forms.

In general, we can conclude that the formation of a two-person coalition block is favorable for its members and, especially, for the one that was obtained a fewer number of seats.

5 CONCLUSIONS

We have obtained the allocations to the players according to solution concepts modified by coalition structures following a double-level procedure based on the multilinear extension of the game. The two levels are (i) the modification of the multilinear extension according to the quotient game and (ii) the weighting of each product in the modified multilinear extension according to the solution concept that we want to compute.

This procedure has been effective for the computation of allocations according to the Owen value and the Owen–Banzhaf value by means of simple steps. In this way, the calculus in Section 4 of several situations in two territorial Spanish Parliaments has been easy to compute.

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