Boundedness of the Riesz Potential in Generalized Morrey Spaces

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Abstract: The purpose of this paper is to prove the necessary and sufficient condition for the boundedness of Riesz operators on homogeneous generalized Morrey spaces. Further, we will make use the Q-Ahlfors regularity condition in the proof instead of usual doubling conditions.

1 INTRODUCTION

In this paper, we shall discuss about the boundedness of a Riesz potential integral operator. The boundedness of operator $I_\alpha$ on the several homogeneous metric measure spaces has been proved by some researchers (Eridani and Gunawan, 2009; Eridani, Kokilashvili and Meskhy, 2009; Nakai, 2000; Petree, 1969). Such boundedness results have been obtained in the several kinds of Morrey spaces thanks to the doubling condition obeyed by the measure of homogeneous metric measure spaces on the Euclidian spaces (Adams, 1975; Chiarenza et al. 1987; Petree, 1969). The Euclidian spaces combined with Lebesgue measure is the most trivial example of the boundedness result of $I_\alpha$ on homogeneous spaces. The generalized Morrey spaces was introduced later by (Nakai, 2000) who also proved the boundedness of $I_\alpha$ in those spaces. Following from this progress, Eridani and Gunawan obtained proof for the boundedness of the fractional integral operator $I_\alpha$ on the generalized Morrey spaces (Eridani & Gunawan, 2009). The further results in the same line were obtained by Sobolev, Spanne, Adams, Chiarenza dan Frasca, Nakai and Gunawan and Eridani related to the boundedness of $I_\alpha$ on generalized Morrey spaces on Euclidian spaces equipped with Euclid norm $|\cdot|$ (Adams, 1975; Chiarenza et al. 1987; Eridani and Gunawan, 2009; Nakai, 2000). Furthermore, the result obtained by Utoyo has described the generalized necessary and sufficient condition for the boundedness of $I_\alpha$ on classic and generalized Morrey spaces (Utoyo et al. 2012).

The boundedness of $I_\alpha$ in the results was obtained using doubling condition obeyed by measure on generalized Morrey spaces. This type of spaces is called homogeneous spaces, the metric measure spaces on which the measure obeys the doubling condition. As the generalization of homogeneous properties of spaces, Ahlfors defined the regularity condition $C_0 r^\beta \leq \mu(B) \leq C_1 r^\beta$ where $C_0$ and $C_1$ are some positive constants.

In this paper, we will prove the necessary and sufficient conditions for the boundedness of $I_\alpha$ on the generalized Morrey spaces similar to the previous results using Ahlfors regularity condition. All the results in this article can be considered as the alternative for the corresponding homogeneous results.

2 LITERATURE REVIEW

Our result of the boundedness result of $I_\alpha$ on the homogeneous generalized Morrey spaces generalizes the following theorem about the boundedness property of fractional integral operator on the homogeneous classic Morrey space. The theorem stated as the following.

Theorem 2.1. Let $X$ be a homogeneous metric measure space, $\nu$ be a measure on $X$, $1 < p < \infty$, $1 < \alpha < \beta$ and $C_0 r^\beta \leq \mu(B(x,r)) \leq C_1 r^\beta$. Then $I_\alpha$ is bounded from $\mathcal{L}^p(X,\mu)$ to $\mathcal{L}^q(X,\nu)$ if and only if there is a constant $C > 0$ such that for every ball $B$ on $X$, $\nu(B) \leq C \mu(B)^{(\frac{1-p}{q}-\frac{1}{\beta})}$. 

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The modification of the preceding theorem, replacing the condition \( v(B) \leq C \mu(B) q(1/p - \alpha/\beta) \), is stated as the following.

**Theorem 2.2.** Let \( 1 < p < q < \infty \) and \( 0 < \alpha < \beta - \frac{q}{p} \). The operator \( I_\alpha \) is bounded from \( L^p(X, \mu) \) to \( L^q(X, v) \) if and only if \( v(B) \leq C r^{\beta - \alpha} \frac{p}{p-1} \) with \( p' = \frac{p}{p-1} \) and \( C r^{\alpha} \mu(\mathbb{R}^n) \leq \mu(B) \leq C_1 r^{\alpha} \).

As the generalization of the above theorems, in this article, we will prove the necessary and sufficient conditions for the boundedness of \( I_\alpha \) on the homogeneous generalized Morrey space. The generalized Morrey space is denoted by \( L^p_\phi = L^p_\phi(\mathbb{R}^n) \) defined as the set of functions \( f \in L^p \) such that

\[
\left\| f; L^p_\phi \right\| = \sup_{B} \frac{1}{B(a,r)} \frac{1}{\phi(r)} \left( \frac{1}{|B|} \int_B |f(y)|^p dy \right)^{\frac{1}{p}} < \infty
\]

where \( \phi: (0, \infty) \to (0, \infty) \) is a function satisfying \( \phi(B(a,r)) = \phi(r) \) and \( 1 \leq p < \infty \). The generalized Morrey space \( L^p_\phi \) is the strong generalization of classic Morrey spaces \( L^{p,\lambda} \). By choosing \( \phi(r) = r^{\frac{\lambda-n}{\lambda}} \) where \( 0 \leq \lambda < n \), then the corresponding generalized Morrey space reduces to classic Morrey spaces \( L^{p,\lambda} \) and hence is so for Lebesgue spaces \( L^p \).

Analysis of boundedness of \( I_\alpha \) on the generalized Morrey spaces requires two condition for function \( \phi \), that is

1. \( \phi \) is said to satisfy doubling condition, denoted by \( \phi \in (DCF) \) if there is a constant \( C > 1 \) such that for every \( r > 0 \) and \( t > 0 \), if \( \frac{r}{2} \leq t \leq 2r \) then \( \frac{1}{r} \leq \frac{\phi(t)}{\phi(r)} \leq C \).
2. \( \phi \) is said to satisfy integral condition (integration condition) and denoted by \( \phi \in (ICF) \) if there is a constant \( C > 1 \) such that for every \( r > 0 \), \( \int_r^{2r} \frac{\phi(t)}{t} dt \leq C \phi(r) \).

The boundedness results for fractional integral operator \( I_\alpha \) on generalized Morrey spaces has been proven by (Nakai, E.) in the following theorem.

**Theorem 2.3.** (Kokilashvili and Meskhi, 2005) If \( 1 < p < q < \infty, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta} \), \( \phi \in (DCF), \) \( t^\alpha \phi(t) \in (ICF), \) with functions \( \psi: (0, \infty) \to (0, \infty) \) satisfy: there is a constant \( C > 0 \) such that for every \( r > 0 \), \( r^\alpha \phi(r) \leq C \psi(r) \) then \( I_\alpha \) is bounded from \( L^p_\phi \) to \( L^q_\psi \).

This result shows that \( I_\alpha \) is bounded from \( L^p_\phi \) to \( L^q_\psi \). Furthermore, the statement in the above theorems is the implication statement, in sense that it only says about the sufficient condition of the boundedness of the operator. For that reason, in determining of complete theory about the boundedness of the fractional integral operator \( I_\alpha \) on the generalized Morrey spaces. In this article, we will construct the necessary conditions for the boundedness of the operator as accompagné to the theorem above. Using the theorem from Adams-Zhiarenza-Frasca, Gunawan and Eridani, which states that (Eridani and Gunawan, 2009) shows that \( I_\alpha \) is bounded from \( L^p_\phi \) to \( L^{p,q}_\phi \). Their result is stated by the following theorem.

**Theorem 2.4.** (Eridani and Gunawan, 2009) Let \( \phi \in (DCF), \phi^p \in (ICF), 1 < p < \frac{n}{\alpha} \) and there is a constant \( C > 1 \) such that for every \( t > 0, \phi(t) \leq Ct^\beta \) where \( -\frac{n}{p} < \beta < -\alpha \), then \( I_\alpha \) is bounded from \( L^p_\phi \) to \( L^{q,p}_\psi \) where \( q = \frac{\beta p}{\alpha + \beta} \).

As the preceding results, the boundedness theorem of \( I_\alpha \) on the generalized Morrey spaces stated above is the implication statement. Then, also will be developed in this article to be, the boundedness from \( L^p(X, \mu) \) to \( L^{p,q}_\psi(X, v) \) and \( L^p(X, \mu) \) to \( L^{p,q}_\psi(X, v) \) with biimplication form on the metric measure space.

3 RESULTS

The first result in our paper is the boundedness property of fractional integral operator similar to that of Theorem 2.1, and 2.2. The difference is that the measures used in the spaces are made to be different cause maximal operator to be unusable to prove the boundedness properties of the operator. Also, the condition of the boundedness of the fractional integral operator \( I_\alpha \) in our result uses Ahlfors regularity condition instead of the traditional doubling condition. The following is the definition of generalized Morrey spaces equipped with
measures $\mu$ and $v$ which is allowed to be different in the later boundedness results.

**Definition 3.1.** Let $v$ be a measure on $X$, $1 \leq p < \infty$, and function $\phi:(0,\infty) \to (0,\infty)$. The generalized Morrey space $\mathcal{L}^{p,\phi}(X, v, \mu)$ is defined as the set of functions $f \in \mathcal{L}^p(X, v, \mu)$ such that the following condition holds

$$\|f\| = \sup_B \frac{1}{\phi(\mu(B))} \left( \frac{1}{\mu(B)} \int_B |f(y)|^p v(y) \right)^{\frac{1}{p}} < \infty,$$

with the supremum is evaluated over every ball $B(a,r)$ on $X$.

**Remark 3.2.** If $v = \mu$, then $\mathcal{L}^{p,\phi}(X, v, \mu) = \mathcal{L}^{p,\phi}(X, \mu)$.

In the above equation, the measure $\mu$ is almost assumed to satisfy the following two conditions:
1. $\phi(r)$ is almost decreasing function, that is, there is a constant $C > 0$ such that for every $t \leq r$, $\phi(r) \leq C \phi(t)$.
2. $r^p \phi(r)^p$ is almost increasing function, that is, there is a constant $C > 0$ such that for every $t \leq r$, $r^p \phi(t)^p \leq C r^p \phi(r)^p$.

The above conditions ensure that the functions $\phi$ and $\psi$, appearing in the boundedness property, do not too rapidly blow up to infinity nor rapidly decay to zero respectively. The following theorem states about the condition that must be satisfied by the functions $\phi$ and $\psi$, and also measure $\mu$ appearing in the spaces, in order to ensure the boundedness property of $I_\alpha$ form the spaces $\mathcal{L}^{p,\phi}(X, \mu)$ to $\mathcal{L}^{p,\phi}(X, v, \mu)$.

**Theorem 3.3.** Let $(X, \delta, \mu)$ be a homogeneous metric space, $1 < p < q < \infty$ and $a \in (0, \frac{p}{q})$. If $\phi \in (ADF), \phi(t) \in (AIP)$, and $\mu(B)^\frac{1}{p-1} \phi(\mu(B)) \leq (\psi(\mu(B)))$, then $\mu$ satisfies the Q-Ahlfors regularity condition, and

$$\int_r^\infty \mu(B(a,t)) \frac{\phi(a,t)}{t^\frac{a}{p}} \left( \frac{\mu(B(a,t))}{\mu(B)} \right)^\frac{1}{q} \, dt \leq C \mu(B(a,t))^\frac{a}{q} \mu(B(a,t))(a,r)$$

then, $I_\alpha$ is bounded from $\mathcal{L}^{p,\phi}(X, \mu)$ to $\mathcal{L}^{p,\phi}(X, v)$.

**Proof. Necessity.** Suppose that $I_\alpha$ is bounded from $\mathcal{L}^{p,\phi}(X, \mu)$ to $\mathcal{L}^{p,\phi}(X, v)$ such that

$$\|I_\alpha f : \mathcal{L}^{p,\phi}(X, v)\| \leq C \|f : \mathcal{L}^{p,\phi}(X, \mu)\|.$$

Then,

$$\frac{1}{\psi(\mu(B))} \left( \frac{1}{\mu(B)} \int_X |I_\alpha f|^q v \right)^{\frac{1}{q}} \leq C \phi(\mu(B))^{-\frac{a}{p}} \mu(B)^\frac{1}{p} \left( \frac{1}{\mu(B)} \int_X |X_B f|^p \, d\mu \right)^{\frac{1}{p}}.$$

If $f := X_B$ were $a \in X$ and $r > 0$ thus,

Since $p' = \frac{p}{p-1}, \mu(B)^\frac{1}{p-1} \phi(\mu(B)) \leq (\psi(\mu(B)))$ and $C r^\alpha \leq \mu(B) \leq C r^\alpha$.

$$v(\mu(B)) = \mu(B)^\frac{a}{p} r^{-\alpha},$$

$$v(\mu(B))^\frac{1}{q} \leq C r^\alpha r^{-\alpha},$$

$$v(\mu(B))^\frac{1}{q} \leq C r^\alpha r^{-\alpha}.$$

**Sufficiency.** Let ball $B$ be an arbitrary ball on $X$ that is $B : B(a,r) \subset X$. Assume that $B(a,r)$ and $f \in \mathcal{L}^{p,\phi}(\mu)$. then we write

$$f = f_1 + f_2 := f_{X_B} + f_{X_B^c},$$

$$\|f_1 : \mathcal{L}^p(\mu)\| = \left( \int_B |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}.$$
\[
\begin{align*}
\mu(B)^{\frac{1}{\beta}}\phi(\mu(B)) &= \frac{1}{\phi(\mu(B))} \frac{1}{\mu(B)} \left( \int_B |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \\
&\leq \mu(B)^{\frac{1}{\beta}}\phi(\mu(B)) \|f: L^p,\phi(\mu)\|.
\end{align*}
\]

If \( f_1 \in L^p(X,\mu) \) then, according to Hardy-Littlewood-Sobolev inequality, we obtain

\[
\frac{1}{\psi(\mu(B))} \left( \frac{1}{\mu(B)} \int_B |f_1|^q \, d\mu(x) \right)^{\frac{1}{q}} \leq \mu(B)^{\frac{1}{\beta}}\phi(\mu(B)) \|f_1: L^p(\mu)\|
\]

\[
\leq C \psi(\mu(B))^{-1} \mu(B)^{\frac{1}{\beta}}\phi(\mu(B)) \|f_1: L^p(\mu)\|.
\]

Next, we estimate \( I_{\alpha} f_2 \). According to definition of \( I_{\alpha} \), we have

\[
|I_{\alpha} f_2(x)| \leq \int_{\{2 \delta \}} \frac{|f(y)|}{\delta(x,y)^{p-\alpha}} \, d\mu(y)
\]

\[
\leq \int_{\delta(x,y)^{2\alpha}} \frac{|f(y)|}{\delta(x,y)^{p-\alpha}} \, d\mu(y)
\]

\[
= \sum_{k=0}^{\infty} \int_{2^{k+1} \delta(x,y) \leq 2^{k+1} \delta} \frac{|f(y)|}{\delta(x,y)^{p-\alpha}} \, d\mu(y)
\]

\[
= \sum_{k=0}^{\infty} \left( \frac{2^{k+1}}{2^{k+1}} \right)^{\frac{1}{\beta}} \int_{2^{k+1} \delta} \frac{|f(y)|}{d\mu(y)}.
\]

Then, using Holder’s inequality, we obtain

\[
|I_{\alpha} f_2|^\frac{1}{p} \leq \sum_{k=0}^{\infty} \left( \frac{2^{k+1}}{2^{k+1}} \right)^{\frac{1}{\beta}} \int_{2^{k+1} \delta} \frac{|f(y)|}{d\mu(y)}.
\]

Hence, according to the hypothesis of the theorem, we obtain

\[
v(B) \leq C \left( \frac{\delta-a}{\delta} \right)^q \quad \text{and} \quad \mu(B)^{\frac{1}{\beta}} \phi(\mu(B)) \leq C \psi(\mu(B)).
\]

Following the above results, the next corollary is the simple implication of Theorem 3.3.

**Theorem 3.4.** Let \( X, \delta, \mu \) be a homogeneous metric spaces, \( 1 < p < q < \infty, \alpha \in (0, \frac{p}{p}), \) and satisfies Q-Ahlfors regularity condition. if \( \phi \in (ADF), \phi(t) \in \]
(AIF), and $v(B) \leq C r^{(\beta - \frac{\alpha}{p} - \frac{q}{p})}$, for some constant $C > 0$, that is $\mu$ satisfies the Q-Ahlfors regularity condition, and

$$\int_r^\infty \mu(B(a,t))^{\frac{\alpha - \beta}{q}} \mu(B(a,t))^{\frac{\phi(a,t)}{t}} dt \leq C \mu(B(a,t))^{\frac{\alpha - \beta}{q}} \mu(B(a,t))^{\phi(a,r)}.$$  

Then, $I_\alpha$ is bounded from $L^p,\Phi(X,\mu)$ to $L^q,\Psi(X,v)$ if and only if

$$\mu(B)^{\frac{1}{q} - \frac{1}{p}} \phi(B) \leq (\psi \mu(B))$$

When $Q = \beta$, the above theorem implies the following corollary.

**Corollary 3.5.** Let $(X, \delta, \mu)$ be a homogeneous metric space, $1 < p < q < \infty$ and $\alpha \in \left(0, \frac{1}{p}\right)$. If $\phi \in (ADF), \phi(t) \in (AIF)$, and $\mu(B)^{\frac{1-q}{p}} \phi(B) \leq (\psi \mu(B))$, then $\mu$ satisfies the $\beta$- Ahlfors regularity condition, and

$$\int_r^\infty \mu(B(a,t))^{\frac{\alpha - \beta}{q}} \mu(B(a,t))^{\frac{\phi(a,t)}{t}} dt \leq C \mu(B(a,t))^{\frac{\alpha - \beta}{q}} \mu(B(a,t))^{\phi(a,r)}.$$  

Then, $I_\alpha$ is bounded from $L^p,\Phi(X,\mu)$ to $L^q,\Psi(X,v)$ if and only if

$$v(B) \leq C \mu(B)^{\phi(\frac{1}{q} - \frac{1}{p})}.$$  

**Corollary 3.6.** Let $(X, \delta, \mu)$ be a homogeneous metric space, $1 < p < q < \infty$ and $\alpha \in \left(0, \frac{1}{p}\right)$. If $\phi \in (ADF), \phi(t) \in (AIF)$, and, that is $\mu$ satisfies the $\beta$- Ahlfors regularity condition, and $v(B) \leq C \mu(B)^{\phi(\frac{1}{q} - \frac{1}{p})}$

$$\int_r^\infty \mu(B(a,t))^{\frac{\alpha - \beta}{q}} \mu(B(a,t))^{\frac{\phi(a,t)}{t}} dt \leq C \mu(B(a,t))^{\frac{\alpha - \beta}{q}} \mu(B(a,t))^{\phi(a,r)}.$$  

Then, $I_\alpha$ is bounded from $L^p,\Phi(X,\mu)$ to $L^q,\Psi(X,v)$ if and only if

$$\mu(B)^{\frac{1}{q} - \frac{1}{p}} \phi(B) \leq (\psi \mu(B)).$$

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**REFERENCES**


