A Set of Optimal Looks on a Symmetric Target

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Abstract: In this paper, we identify a set of multiple looks from symmetry that optimize the expected probability of detection in a mine hunting operation or in a search & rescue mission. We assume that the target exhibits mirror symmetry, i.e., that the left hand side of a target is the mirror image of the right hand side of the same target. In addition, it is assumed that the cross section is maximal at the interface between the left hand side and the right hand side and decreases monotonically as we move away from the interface. The optimal strategy consists of choosing aspect angles to inspect a target to ensure that the probability of detection is maximal. This is generally an NP-hard problem in the sense that to find the optimal angles in n dimensions normally consumes a lot of computational power. Fortunately, in this problem, we are use a novel combination of variational calculus and symmetry principles to determine analytically the locally optimal angles. The solutions will help the operators plan for an effective strategy in a mine hunting operation or in a search and rescue mission. Such a strategy is robust as most targets of interest possess approximate mirror symmetry along one or more axes. For example, a human body or a canoe or a mine when cut in half yield approximately such symmetry.

1 INTRODUCTION

In this paper, we examine a problem where a searcher can observe a target from multiple look angles. This problem arises naturally in the context of search and detection, or mine countermeasure operations. We show that the angular dependence affects the overall probability of detection significantly even though this dependence is often overlooked in the open literature.

Indeed, the formula for the probability of detecting a target in a random search is widely used, yet it assumes no angular dependence (Koopman 1999). The importance of the look angle in perception of a target is not just intuitively evident, but has also been demonstrated both theoretically and experimentally. The ideas could be found for example in (Wettergren and Baylog 2010) & (Zerr, Bovio and Stage 2000) and many more such as (Ji and Liao 2005) or (Runkle et al. 1999). As the paper unfolds, it will be seen that our approach is different from the current literature in that we identify a set of all optimal angles.

For completeness, we define the look angle as shown in Figure 1. Each look angle is associated with a look. For brevity, we call the look angle simply the angle. It is measured counter clockwise from the positive horizontal axis. The zero degree (zero radian) angle corresponds to the look on the long side of the target while the ninety degree ($\pi / 2$ radian) angle corresponds to the look on the short side of the target. The look angle is periodic with period equal to 180 degrees (π radians).

For a general class of single look angular probability of no detection, as shown in Figure 2, we provide a strategy to determine the locally optimal probability of detection based on n observations at various angles, or simply n looks. That is, if an observation is made once at an angle x then the corresponding probability of no detection $g(x)$ is the corresponding value shown in Figure 2. For illustration purposes, we assume that $g(x) = \sin(x)^2$. Note that $g(x)$ is symmetric around zero degrees (zero radians).

There are many real life targets than can be approximated with this type of symmetry including canoes, ships, submarines, mines and human bodies. In this paper we call it the mirror symmetry; that is, the left hand side of a target is the mirror image of

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the right hand side of that target. The difficulty of detecting such targets depends for example on the cross sections of the targets that are visible to the sensor. The probability of (no) detection is assumed to be proportional to the cross section of a cylindrical target.

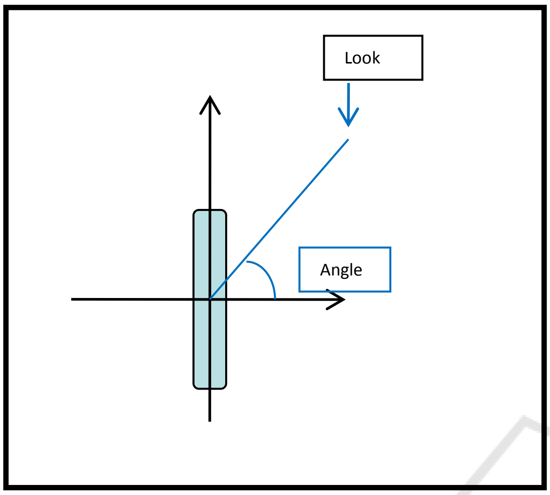

Figure 1: Look angle x .

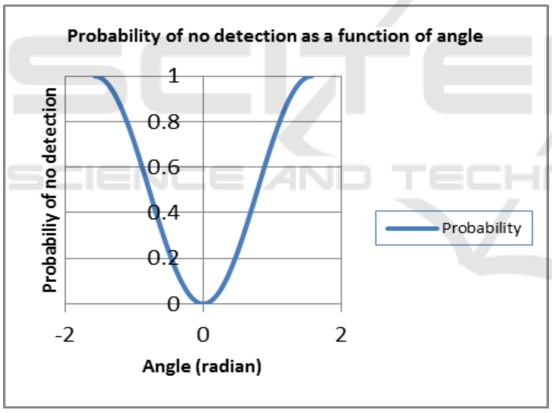

Figure 2: Probability of no detection as a function of angle.

This problem is intractable using calculus since each look is independent of one another yielding a multi-dimensional problem. The explicit expression for the overall probability of detection is hopelessly complicated even when the single probability of detection is simple and the number of looks is small.

(Press et al. 1999): "Don't be fooled by the apparent notational similarity of (multi dimension $\vec{F}\left(\vec{\mu}\right)=\vec{0}$) and (one dimension $F\left(\mu\right)=0$) (where are the roots to be determined). Simultaneous solution of equations in dimensions is much more difficult than finding roots in the one-dimensional case. The principal difference between one and many dimensions is that, in one dimension, it is possible to bracket or "trap" a root between bracketing values, and then hunt it down like a rabbit. In multidimensions, you can never be sure that the root is there at all until you have found it."

Furthermore, even if we can bracket the roots then the techniques available in (Press et al. 1999) only provide numerical solutions which vary from one model of angular probability to another and hence require extensive computations prior to a search.

(Torns and Zilinska 2007): "In general, the problem of finding the exact values $\vec{x}$ that minimize a given objective function $f\left(\vec{x}\right)$ is computationally difficult (NP-hard); …Crudely speaking, NP hardness means that it is not possible to have an algorithm that solves all optimization problems in a reasonable time."

The problem that we consider has all the features and difficulties that are described above. We resolve these difficulties by making use of the symmetry of the single angular probability of detection function. This novel symmetry argument yields a number of simple and easy to use formulae for the optimal angles. In addition, these formulae hold true for the general class of single probability of detection curves shown in Figure 2. That is, the probability of no detection is an even and decreasing function of angle based on the definition of angle shown in Figure 1. $g\left(x\right)$ can be convex, can be concave, and can be neither convex nor concave.

Note that (Waterhouse, 1983): "That is, as soon as the symmetry of a problem is brought out, people are inclined to say that *by symmetry* the extreme value must occur when the variables are equal. But a bit of thought shows that there is no simple symmetry argument to this effect. Indeed, there cannot be, because such a symmetry conclusion is sometimes false (Bouniakovsky 1854)". To avoid this pitfall, we will find all locally optimal roots.

2 SEARCH EXPERIMENT

In this Section, we provide a simple demonstration to qualitatively illustrate the dependence of the probability of detection as a function of angle. This experiment was also reported in (Nguyen and Mirshak, 2016). We put a pen on a Christmas tree and we take pictures of the tree (including the pen) as we rotate the observation angle by approximately 30 degrees each time by rotating the tree.

Figure 3: Pen observed at -90 degrees.

This pen has approximately mirror symmetry and is approximately six inches in length. The tree is about one meter in height. The distance between the camera and the tree is approximately 1.5 meters. A cat is shown to give an idea of the scale. We used a Canon Power Shot A530 digital camera to take the pictures. It is difficult to identify the pen, from Figure 4 and Figure 8, when the angle is ± 60 degrees. It is nearly impossible, from Figure 3 and Figure 9, when the angle is ± 90 degrees. However, it is easily identified when the angle is zero degrees (zero radians) or ± 30 degree. Zero degrees (zero radians) correspond to the look perpendicular to the long side of the pen.

Figure 4: Pen observed at -60 degrees.

Figure 5: Pen observed at -30 degrees.

Figure 6: Pen observed at zero degrees.

Figure 7: Pen observed at 30 degrees.

Figure 8: Pen observed at 60 degrees.

Figure 9: Pen observed at 90 degrees.

3 MODELLING THE LOOK ANGLE

As shown in Section 2, the probability of detection of a target depends on the look angle in search and rescue operations. The effectiveness of such an operation depends on the performance of the sensor. There are two types: the probability of detection as a function of range and the probability of detection as a function of look angle. The probability of detection as a function of range is nearly a constant; hence we focus only on the angular dependence. For more details on the range dependence, we refer the reader to (Nguyen et al. 2008).

The probability of detection as a function of range is primarily a characteristic of the sensor, while the probability of detection as a function of angle is primarily a characteristic of the target. All ranges and angles are measured on the two dimensional plane formed by the sensor beam and the direction of motion of the searcher carrying the sensor. Most targets of interest have approximate mirror symmetry; that is the left hand side of a target is the mirror image of the right hand side of the same target. Human bodies, canoes, ships and mines belong to this type of symmetry. Therefore, to build a robust search strategy, we assume that the target has (approximately) the mirror symmetry. The look angle is defined as the counter clockwise angle between the sensor beam and the short axis of symmetry of a cylindrical (positive horizontal axis) target as shown in Figure 1. A look angle of zero degrees corresponds to the observation of the long side of the target. A look angle of ninety degrees corresponds to the observation of the short side of the target.

The corresponding angular probability of no detection curve $(g(x))$ in Figure 2 shows that the detectability of a target reaches a maximum when its look angle is perpendicular to the sensor beam and this angular probability decreases symmetrically with respect to that perpendicular case where we use $g(x)=\sin(x)^2$ for illustration purposes. Such an expression for $g(x)$ is similar to a specific case of target angular dependence, (Gilani et al. 2015). In addition, $1-g(x)=1-\sin(x)^2$ is approximately equal to the normalized cross section of a cylindrical target. It is very clear from Figure 2 that the probability of detection is substantially degraded if the look angle differs from zero degrees (zero radians).

The following assumptions are imposed on the function $g(x)$:

1. $g(x)$ is periodic with period equal to π ;
2. $g(x)$ is an even function i.e. $g(x)=g(-x)$;
3. $g(x)$ is minimal at $x=0$ and
4. $g(x)$ is increasing between zero and $\pi/2$.

4 LOCAL OPTIMAL CONDITIONS

In this Section, we determine the set of all angles that optimize the probability of detection. We assume that a target is observed n times possibly at n distinct angles. The proof of Lemma 4A is described in (Nguyen and Bourque, 2012a & 2012b) which make use of variational calculus (Gelfand and Fomin 1963). A different approach was made in (Bourque and Nguyen, 2011) based on inequalities of a quadratic equation.

Let μ_i be the look angle of the ith observation $(i=0,...,n-1)$. We note that the probability of detection (at least one detection), $P(\vec{\mu})$ can be written as:

$$P(\vec{\mu}) = 1 - G(\vec{\mu}) \tag{1}$$

where $G(\vec{\mu})$ is the probability of no detection defined as:

$$G(\vec{\mu}) = \int_{-\pi/2}^{\pi/2} \frac{dx}{\pi} \cdot g(x+\mu_0) \cdot ... \cdot g(x+\mu_{n-1}) \tag{2}$$

Eqn (2) assumes that the orientation of the target is randomly uniform. That is, the probability density function is equal to $1/\pi$. The set of n looks yields no detection when each look yields no detection. Therefore, the probability of no detection based on n looks is the product of the probability of no detection of each look: $g(x+\mu_0) \cdot ... \cdot g(x+\mu_{n-1})$. When this product is integrated over all angles x and weighted by the density distribution $1/\pi$, we obtain the expected probability of no detection based on n looks: $G(\vec{\mu})$.

Lemma 4A. $G(\vec{\mu})$ is locally optimal, i.e., $\vec{\nabla} G(\vec{\mu}) = \vec{0}$ if and only if $f_i(x) = f_i(-x)$ for $i = 1,...,n-1$ where

$$\begin{aligned}
\partial_0 G(\vec{\mu}) &= \int_{-\pi/2}^{\pi/2} \frac{dx}{\pi} \cdot g'(x) \cdot g(x+\mu_1) \cdot ... \cdot g(x+\mu_{n-1}) \\
&= \int_0^{\pi/2} \frac{dx}{\pi} \cdot g'(x) \cdot (f_0(x) - f_0(-x)) \\
\partial_1 G(\vec{\mu}) &= \int_{-\pi/2}^{\pi/2} \frac{dx}{\pi} \cdot \begin{Bmatrix} g(x-\mu_1) \cdot g'(x) \cdot g(x+\mu_2-\mu_1) \cdot \\ ... \cdot g(x+\mu_{n-1}-\mu_1) \end{Bmatrix} \\
&= \int_0^{\pi/2} \frac{dx}{\pi} \cdot g'(x) \cdot (f_1(x) - f_1(-x)) \\
&... \\
\partial_{n-1} G(\vec{\mu}) &= \int_{-\pi/2}^{\pi/2} \frac{dx}{\pi} \cdot \begin{Bmatrix} g(x-\mu_{n-1}) \cdot \\ ... \cdot g(x+\mu_{n-2}-\mu_{n-1}) \cdot g'(x) \end{Bmatrix} \\
&= \int_0^{\pi/2} \frac{dx}{\pi} \cdot g'(x) \cdot (f_{n-1}(x) - f_{n-1}(-x))
\end{aligned} \tag{3}$$

and

$$f_i(x) = \frac{1}{g(x)} \cdot \prod_{j=0}^{n-1} g(x+\mu_j-\mu_i) \tag{4}$$

For example,

$$\begin{aligned}
& f_0(x) = g(x+\mu_1) \cdot ... \cdot g(x+\mu_{n-1}) \\
& = g(x-\mu_1) \cdot ... \cdot g(x-\mu_{n-1}) \\
& = g(-x+\mu_1) \cdot ... \cdot g(-x+\mu_{n-1}) \\
& = f_0(-x)
\end{aligned} \tag{5}$$

Lemma 4A implies that

$$\begin{aligned}
& g(x+\mu_1) \cdot ... \cdot g(x+\mu_{n-1}) \\
& = g(x-\mu_1) \cdot ... \cdot g(x-\mu_{n-1})
\end{aligned} \tag{6}$$

Lemma 4B. The above holds when $\{\mu_1,..,\mu_{n-1}\} = \{-\mu_1,...,-\mu_{n-1}\} \pmod{\pi}$ for $i,j=1,...,n-1$.

Proof of Lemma 4B. We can show this by induction. We have shown above that it is true for $n-1=1$. Assume the contrary, i.e. there is no pair $\mu_i = -\mu_j \mod \pi$ and $g(0)=0$, we can choose $x = -\mu_1$ so that the LHS of Eqn (6) is zero while the RHS is not zero. This is impossible. Hence, we can infer that $\mu_1 = -\mu_1 \mod \pi$ without loss of generality. Therefore,

$$\begin{aligned}
& g(x+\mu_2) \cdot ... \cdot g(x+\mu_{n-1}) \\
& = g(x-\mu_2) \cdot ... \cdot g(x-\mu_{n-1})
\end{aligned} \tag{7}$$

The above implies that by induction:

$$\{\mu_2,...,\mu_{n-1}\} = \{-\mu_2,...,-\mu_{n-1}\} \mod \pi \tag{8}$$

Therefore,

$$\{\mu_1,...,\mu_{n-1}\} = \{-\mu_1,...,-\mu_{n-1}\} \mod \pi \tag{9}$$

Lemma 4C. Assume that $\mu_i = i \cdot \mu \mod \pi$ for $i=0,...,n-1$ then $\mu = k \cdot \pi / n$ for $k=0,...,n$ or $\mu = k' \cdot \pi/(n-1)$ for $k'=0,...,n-1$.

Beside the roots in Lemma 4C, there are other roots satisfying Eqn (8) as shown below.

First, we can set any root μ_i $(i=0,...,n-1)$ to be zero. Second, the critical point must obey the following symmetry $2 \cdot \vec{\mu} = -2 \cdot \vec{\mu} \pmod{2\pi}$. The two criteria signify that there are two types of critical points as shown below.

Theorem 4A. Using the two criteria, we obtain two types of critical points.

Type 1 critical points consist of roots that appear together from $\{0, \pi/m, \ldots, (m-1)\cdot\pi/m\}$ possibly more than once as well as $\{0, \pi/(2\cdot m), \ldots, (2\cdot m-1)\cdot\pi/(2\cdot m)\}$ also possibly more than once where $1 \le m, 2\cdot m \le n$.

Type 2 critical points consist of roots that differ by $\mu = m\cdot\pi/n$ where $m = 1, \ldots, n-1$. That is, $\vec{\mu} = \{0, \mu, \ldots, (n-1)\cdot\mu\}$.

Proof. We can infer from the second symmetry, the periodicity and the monotonicity of $g(x)$ that $\vec{\mu}$ is composed of multiple sub cycles consisting of $\{0, \pi/m, \ldots, (m-1)\cdot\pi/m\}$ for $1 \le m \le n$.

It is simple to show that the two types of critical points satisfy the two criteria above. To show completeness of the two types, we make use of contradiction. Specifically, we assume that $\vec{\mu}$ is comprised of either type in addition to at least another sub cycle $\{0, \pi/m', \ldots, (m'-1)\cdot\pi/m'\}$ where $1 \le m' \le n$ such that $\pi/m' \ne \pi/m, \pi/(2\cdot m) \pmod{\pi}$ and $\pi/m' \ne \mu \pmod{\pi}$ and show that $\vec{\mu}$ does not satisfy the two criteria. We will rotate all the roots of $\vec{\mu}$ such that $\pi/m' \to \pi/m' - \pi/m' = 0$.

For type 1 critical points,

$$
\begin{aligned}
&2\cdot\{0, \pi/m, \ldots, (m-1)\cdot\pi/m\}^{a} \\
&\to 2\cdot\{0-\pi/m', \pi/m-\pi/m', \ldots, (m-1)\cdot\pi/m-\pi/m'\}^{a} \\
&\ne -2\cdot\{0-\pi/m', \pi/m-\pi/m', \ldots, (m-1)\cdot\pi/m-\pi/m'\}^{a} \\
&\pmod{\pi}
\end{aligned}
$$

and

$$
\begin{aligned}
&2\cdot\{0, \pi/(2\cdot m), \ldots, (2\cdot m-1)\cdot\pi/(2\cdot m)\}^{b} \\
&\to 2\cdot\{0-\pi/m', \pi/(2\cdot m)-\pi/m', \ldots, (2\cdot m-1)\cdot\pi/(2\cdot m)-\pi/m'\}^{b} \\
&\ne -2\cdot\{0-\pi/m', \pi/(2\cdot m)-\pi/m', \ldots, (2\cdot m-1)\cdot\pi/(2\cdot m)-\pi/m'\}^{b} \\
&\pmod{\pi}
\end{aligned}
$$

where a and b are the redundancies of the corresponding roots. It is shown above that the transformed roots no longer satisfy the second criteria. That is, $2\cdot\vec{\mu} \ne -2\cdot\vec{\mu} \pmod{2\pi}$. The proof is similar for type 2 critical points.

Example 4B. For clarity, we provide below all the critical points for six looks $(n = 6)$:

a. $\{0, 0, 0, 0, 0, 0\}$;
b. $\{0, 0, 0, 0, 0, \pi/2\}$;
c. $\{0, 0, 0, 0, \pi/2, \pi/2\}$;
d. $\{0, 0, 0, \pi/2, \pi/2, \pi/2\}$;
e. $\{0, \pi/2\}$ and $\{0, \pi/4, \pi/2, 3\cdot\pi/4\}$;
f. $\{0, \pi/3, 2\cdot\pi/3\}^2$ where the superscript 2 means that the set $\{0, \pi/3, 2\cdot\pi/3\}$ is repeated twice and
g. $\{0, \pi/6, 2\cdot\pi/6, 3\cdot\pi/6, 4\cdot\pi/6, 5\cdot\pi/6\}$.

Technically there are also other critical points such as $\{0, 0, \pi/2, \pi/2, \pi/2, \pi/2\}$. However, by the periodicity of $g(x)$ we can shift these roots by $\pi/2$ without changing the probability of detection i.e. $\{-\pi/2, -\pi/2, 0, 0, 0, 0\}$ yields the same probability of detection as the one from $\{0, 0, \pi/2, \pi/2, \pi/2, \pi/2\}$. Furthermore, the evenness of $g(x)$ allows us to infer that $\{\pi/2, \pi/2, 0, 0, 0, 0\}$ yield the same probability of detection as the one from $\{0, 0, \pi/2, \pi/2, \pi/2, \pi/2\}$.

5 GLOBALLY OPTIMAL ROOTS

It turns out that the critical point consisting of $\{0, \pi/n, \ldots, (n-1)\cdot\pi/n\}$ yields the globally maximum probability of detection. We name this critical point the equidistant roots.

Theorem 5A. If $g(x)$ is logarithmically concave then the equidistant roots yield the globally maximum detection probability.

Proof of Theorem 5A. In both types of critical points, the root that is equal to zero occurs at least twice except for the equidistant roots. We choose $\delta > 0$ and infinitesimal then modify the two zero roots one by $+\delta$ and the other by $-\delta$. This will generate a higher detection probability than the one with two zero roots. To show this, we define $\vec{\mu}' = (\delta, -\delta, *)$ where the $*$ represents all the remaining roots of a critical point $\vec{\mu}$. We now determine the expected probability of detection of $\vec{\mu}'$:

$$G(\vec{\mu}') = \int_{-\pi/2}^{\pi/2} \frac{dx}{\pi}\cdot g(x+\delta)\cdot g(x-\delta)\cdot(**) \tag{10}$$

where $(**)$ represents the product of gs with the remaining roots of the critical point $\vec{\mu}$. Since δ is infinitesimal, the first order expansion in δ can be written as:

$$G\left(\vec{\mu}'\right) = \int_{-\pi/2}^{\pi/2} \frac{dx}{\pi} \cdot \begin{Bmatrix} g\left(x\right)^2 - \delta^2 \cdot \\ \left(g'\left(x\right)^2 - g\left(x\right)\cdot g''\left(x\right)\right) \end{Bmatrix} \cdot (**) + O\left(\delta^3\right) \quad (11)$$

Logarithmic concavity means that $g'\left(x\right)^2 - g\left(x\right)\cdot g''\left(x\right) \geq 0$. In the non-trivial case where $g'\left(x\right)^2 - g\left(x\right)\cdot g''\left(x\right) \neq 0$, this implies that:

$$G\left(\vec{\mu}'\right) < G\left(\vec{\mu}\right) \quad (12)$$

Therefore, a critical point $\vec{\mu}$ with at least two zero roots cannot yield the globally maximum probability of detection. This eliminates all the critical points except for the equidistant roots. Hence, the equidistant roots must be the globally maximum probability of detection.

We observe that there are many functions for a symmetric target that are logarithmically concave such as:

a. $g\left(x\right) = \left(\sin^2\left(x\right)\right)^\varepsilon$ where $\varepsilon > 0$;

b. $g\left(x\right) = \left(\left(\frac{x}{\pi/2}\right)^2\right)^\varepsilon$ where $\varepsilon > 0$;

c. the normal density distribution and

d. the exponential density distribution.

For the practical purpose of planning a search & detection operation, the types of $g\left(x\right)$ listed above are representative of most targets of interest. This is sufficient to guarantee the global optimality of the equidistant roots.

We provide here examples when assuming $g\left(x\right) = \sin\left(x\right)^2$. For convenience, we break the two types of critical points into the following four categories. For $i = 0, ..., n-1$:

1. $\{0\}^a$ and $\{\pi/2\}^b$ where $a + b = n$;
2. $\{0, \pi/m, ..., \left(m-1\right)\cdot\pi/m\}^a$ and $\{0, \pi/\left(2m\right), ..., \left(2m-1\right)\pi/\left(2m\right)\}^b$ such as $\{0, \pi/2\}^a$ and $\{0, \pi/4, 2\pi/4, 3\pi/4\}^b$ where $m < n$ and $a\cdot m + b\cdot 2\cdot m = n$;
3. $\{0, \pi/m, ..., \left(m-1\right)\pi/m\}^p$ where $m\cdot p = n$ and
4. $\{0, \pi/n, ..., \left(n-1\right)\pi/n\}$.

The results below make use of the following identity (Gradshteyn and Ryzhik 1980):

$$\sin\left(x\right)\cdot\sin\left(x + \frac{\pi}{v}\right)\cdot ... \sin\left(x + \left(v-1\right)\frac{\pi}{v}\right) = \frac{1}{2^{v-1}}\cdot\sin\left(v\cdot x\right) \quad (13)$$

where v is a positive integer. Using simple calculus, we get $G\left(\vec{\mu}\right)$ as follows:

1. $1/\pi\cdot B\left(a + 1/2, b + 1/2\right)$;
2. $1/\left(4^{a+2b}\pi\right)\cdot B\left(a + b + 1/2, b + 1/2\right)$ for the case $\{0, \pi/2\}^a$ and $\{0, \pi/4, 2\pi/4, 3\pi/4\}^b$;
3. $1/\left(2\cdot 4^{n-1}\right)$ and
4. $1/\left(\pi\cdot 4^{n-p}\right)B\left(p + 1/2, 1/2\right)$.

where $B\left(x, y\right)$ is the B function (Zwillinger, 1996). We observe that all of the four results above can be rewritten using factorial of integers. However, for concision we express the results using the B function.

For illustration, we assume $n = 6$. Table 1 displays the set of all optimal roots for four looks and the corresponding probabilities of no detection.

Clearly, we see that $G\left(0,0,0,0,0,0\right) = 231/1024 = 0.2256$ is the largest among the five optimal roots. While $G\left(0, \pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6\right) = 1/2048 = 0.0004883$ is the smallest implying that it provides the greatest probability of detection.

It is also worth observing that the probability of detection for one look is equal to $1 - 1/\left(2\cdot 4^0\right) = 1/2$ while the one for two looks with roots $\left(0, \pi/2\right)$ is equal to $1 - 1/\left(2\cdot 4\right) = 7/8$. Therefore, the probability of detection almost doubles when we go from one look to two (optimal) looks. Figure 10 shows two search patterns: one yielding one look and the other two looks.

Table 1: Exhaustive set of optimal roots and their corresponding probabilities of no detection.

$\vec{\mu}$	$G\left(\vec{\mu}\right)$
$\left(0,0,0,0,0,0\right)$	$231/1024$
$\left(0,0,0,0,0,\pi/2\right)$	$21/1024$
$\left(0,0,0,0,\pi/2,\pi/2\right)$	$7/1024$
$\left(0,0,0,\pi/2,\pi/2,\pi/2\right)$	$5/1024$
$\left(0,\pi/2\right)\cup\left(0,\pi/4,\pi/2,3\pi/4\right)$	$1/1024$
$\left(0,\pi/3,2\pi/3\right)^2$	$3/2048$
$\left(0,\pi/6,\pi/3,\pi/2,2\pi/3,5\pi/6\right)$	$1/2048$

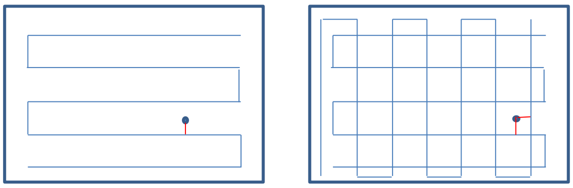

Figure 10: One (lhs) and two (rhs) optimal look search patterns.

For comparison, we also compute the probability of no detection for two points that are not locally optimal. Namely $G(0,\pi/2,\pi/4,\pi/8,\pi/16,\pi/32)=0.02384$ and $G(0,0,\pi/2,\pi/2,\pi/4,\pi/4)=0.01074$. Clearly they lie between $G(0,0,0,0,0,0)=0.2256$ and $G(0,\pi/6,\pi/3,\pi/2,2\pi/3,5\pi/6)=0.0004883$.

6 DISCUSSION

In this paper, we show the significance of the look angle dependency. The probability of detection can improve substantially when we increase the number of looks in addition to choosing the optimal looks.

As observed in Section 5, the probability of detection almost doubles when we go from one look to two optimal looks.

We have derived the optimality condition using variational calculus that allows determining the optimal roots in a general way. That is, the single look no detection probability obeys a broad class of functions that requires only symmetry and monotonicity that is similar to $g(x)=\sin(x)^2$. In addition, the results apply to general n dimension which is normally NP hard even when we seek for numerical solutions.

In the near future, we will provide a stronger proof showing the global optimality of the equidistant angles and the effect of repeated looks.

Future work might also include the development of general search patterns that would make use of the optimal angles. There is already some evidence in the open literature such as (Bays et al. 2011) and (Nguyen et al. 2008) which assumes angular dependencies and which we will build upon to develop new concepts of search and rescue operations.

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