A New Technique for Education Process Optimization via the Dual Control Approach

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Abstract: In this paper, examples arising from a problem in education process control are considered. The provision of the number of information items in a specified discipline for education to optimize a suitable performance index is treated as a "dual control" process. According to the theory of dual control, the control signal has two purposes which might be in conflict with each other: 1) to help learn about any unknown parameters and/or state of the system (estimation); 2) to contro1. In view of this, one can see that the open-loop feedback control is, from the estimation point of view, passive, since it does not take into account that learning is possible in the future. In contrast to this, a dual control is active, not only for the control purpose but also for the estimation purpose because the performance depends also on the "quality" of the estimates. Therefore, the dua1 control can be called actively adaptive since it regulates the speed and amount of learning as required by the performance index. To determine the sequence of optimal decisions, dynamic programming is used. It will be noted that the optimal policy can be compared with the non-optimal policy of optimizing stage by stage. To illustrate a new technique for education process optimization via the dual control approach, numerical examples are given.

1 INTRODUCTION

In all control problems there are certain degrees of uncertainty with respect to the process to be controlled. The structure of the process and/or the parameters of the process may vary in an unknown way. To obtain good process information it is necessary to perturb the process. Normally, the information about the process will increase with the level of perturbation. On the other hand the specifications of the closed loop system are such that the output normally should vary as little as possible. There is thus a conflict between information gathering and control quality. This problem was introduced and discussed by A. A. Feldbaum in a sequence of four seminal papers from 1960 and 1961, see (Feldbaum, 1960-61). Feldbaum's main idea is that in controlling the unknown process it is necessary that the controller has dual goals. First the controller must control the process as well as possible. Second, the controller must inject a probing signal or perturbation to get more information about the process. By gaining more process information better control can be achieved in future time. The compromise between probing and control or in Feldbaum's terminology investigating and directing leads to the concept of *dual control*. Feldbaum showed that a functional equation gives the solution to the dual control problem. The derivation is based on dynamic programming and the resulting functional equation is often called the *Bellman equation*. Alper and Smith (1967) use Feldbaum's idea to provide places in a sector of education to satisfy an unknown social demand.

This paper is an endeavour to clarify some concepts of underlying decision processes relating to education process control by utilizing the theory of automatic control. The purpose of this paper then is to indicate how some of the concepts of control theory may be utilized in order to bring about better performance of the education process. One of the most important features of control theory is its great generality, enabling one to analyze diverse systems within one unified framework.

In order to illustrate the Feldbaum's main idea, a simple example relevant to education process control is given below. The example has two outstanding virtues: the performance index is not the

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usual quadratic one, and analytic solutions without approximations are possible, thus affording the opportunity of comparison with reasonable but not optimal policies.

2 PROBLEM STATEMENT

Let us assume that one has to decide on the number of information items to be made available in a specified discipline (course of lectures). Let the number of information items (discipline information quantity) made available be designated by $\boldsymbol{u}$**.** However, the number of information items, x, successfully acquired by students, may be less than u, because the above number is limited to an unknown level H (as a function of the number of education hours and a fundamental knowledge level of students).

Thus, one may write

$$X = \min(u, H). \tag{1}$$

The outcome of the decision is assessed by considering that the cost of providing an information item is the same whether it is understood or not and that the benefit derived from a student is to be given by a factor q times as great. Therefore, the total "education effect" may be written as

$$V = q X - u = q \min(u, H) - u. \tag{2}$$

Clearly $q > 1$, otherwise one should not provide any information items at all. If H were known, V is maximized by choosing $u=H$.

It is further assumed that

$$h_1 \le H \le h_2 \tag{3}$$

and the decision maker's a priori knowledge is expressed by means of a probability density $f(h)$. The question is then: what value should be chosen for u so as to maximize the expected education effect averaged over the a priori distribution of H?

There are two possibilities:

$$H < u, \text{ with probability } \int_{h_1}^{u} f(h)dh$$

and

$$H \ge u, \text{ with probability } \int_{u}^{h_2} f(h)dh. \tag{4}$$

In the first case, $X = H$, with an expected H given by

$$E\{H\} = \int_{h_1}^{u} hf(h)dh \left(\int_{h_1}^{u} f(h)dh \right)^{-1}. \tag{5}$$

In the second case, $X = u$. The expected education effect is thus

$$E\{V\} = E\{V \mid h < u\} \Pr(h < u) + E\{V \mid h \ge u\} \Pr(h \ge u)$$

$$= \int_{h_1}^{u} (qh - u) f(h)dh + (q-1)u \int_{u}^{h_2} f(h)dh. \tag{6}$$

Therefore, the optimum value of u, designated u^* is given by

$$1/q = \int_{u^*}^{h_2} f(h)dh \tag{7}$$

and the corresponding extremized effect is

$$E\{V^*\} = q \int_{h_1}^{u^*} hf(h)dh. \tag{8}$$

3 MAXIMIZATION OF THE EXPECTED EDUCATION EFFECT OVER N STAGES

The education effect of any particular stage (say, jth) of the education process may be written as

$$V_j = q \min(u_j, H) - u_j. \tag{9}$$

Thus, the problem is to maximize the expected education effect over N stages,

$$G_N = E\left\{ \sum_{j=1}^{N} V_j \right\} = E\left\{ \sum_{j=1}^{N} [q \min(u_j, H) - u_j] \right\}$$

$$= \sum_{j=1}^{N} \left[q \left(\int_{h_1}^{u_j} hf(h)dh + \int_{u_j}^{h_2} u_j f(h)dh \right) - u_j \right]$$

$$= N(q-1) \int_{h_1}^{h_2} hf(h)dh$$

$$- \sum_{j=1}^{N} \left(\int_{h_1}^{u_j} (u_j - h) f(h)dh + (q-1) \int_{u_j}^{h_2} (h - u_j) f(h)dh \right). \tag{10}$$

Now, consider a sequence of decisions maximizing (10). Let $G_N^*(h_1)$ be the optimum total future education effect when there are N stages to go, and note that it is expressed as a function of the lower limit on H. The expected cost at the first of N stages is given by (6).

If $H < u$, the expected effect of the succeeding (N–1) stages will be given by

$$(N-1)(q-1)\int_{h_1}^{u} hf(h)dh\left(\int_{h_1}^{u} f(h)dh\right)^{-1}. \quad (11)$$

If $H \geq u$, then H is not known precisely, but the lower limit will have been raised from h_1 to u. The optimum expected education effect of the succeeding N stages can thus be denoted by $G_N^*(u)$. Consequently, by analogy with the single-stage optimization

$$G_N^*(h_1) = \max_{u_N}\left([q+(N-1)(q-1)]\int_{h_1}^{u_N} hf(h)dh \right.$$
$$\left. -u_N\int_{h_1}^{u_N} f(h)dh + [u_N(q-1)+G_{N-1}^*(u_N)]\int_{u_N}^{h_2} f(h)dh \right). \quad (12)$$

The results of the optimal policy can be compared with the outcome of repeated applications of the single-stage policy. The expected education effect over N+1 stages, Q_{N+1}, may then be interpreted in this instance as the "non-dual" solution of the problem and may be derived as

$$Q_N(h_1) = [q+(N-1)(q-1)]\int_{h_1}^{u^*} hf(h)dh + \frac{1}{q}Q_{N-1}(u^*), \quad (13)$$

where u^* is given by (7).

3.1 Procedure for the Optimal Policy

The procedure for the optimal policy is then, for the appropriate $f(h)$, as follows:

1) Determine u_N^* using (12) and the current values of h_1 and h_2.

2) Either $H < u_N^*$ in which case $u^* = H$ for the remaining N–1stages or $H \geq u_N^*$ in which case the above step is repeated again with N reduced by one and h_1 replaced by u_N^*.

3.2 Modification of the Optimization Equations

If H were known to have its expected value, then it is convenient to express (12) as follows:

$$G_N^*(h_1) = R_N^*(h_1) + N(q-1)\int_{h_1}^{h_2} hf(h)dh. \quad (14)$$

Then (12) may be rewritten as

$$R_N^*(h_1) = \max_{u_N}\left(R_{N-1}^*(u_N) - \int_{h_1}^{u_N}(u_N-h)f(h)dh \right.$$
$$\left. -(q-1)\int_{u_N}^{h_2}(h-u_N)f(h)dh \right). \quad (15)$$

As for $G_N^*(h_1)$ above, we can write

$$Q_N(h_1) = S_N(h_1) + N(q-1)\int_{h_1}^{h_2} hf(h)dh, \quad (16)$$

so that (13) can be rewritten as

$$S_N(h_1) = S_{N-1}(u^*) - q\int_{u^*}^{h_2} hf(h)dh + \int_{h_1}^{h_2} hf(h)dh. \quad (17)$$

3.3 Illustrative Example 1

(*Rectangular probability density function of a priori knowledge*). If only an upper and lower bound on H is known, then a priori knowledge about H is expressed as

$$f(h;h_1,h_2) = \frac{1}{h_2-h_1}, h_1 \leq h \leq h_2. \quad (18)$$

It follows from the above that in this case we have:

$$u_N^* = h_1 + a_N(q-1)(h_2-h_1), \quad (19)$$

where

$$a_N = \frac{1+a_{N-1}}{q+a_{N-1}(q-1)}, \quad a_0 = 0; \quad (20)$$

$$G_N^*(h_1) = \frac{q-1}{2}[N(h_1+h_2) - a_N(h_2-h_1)]; \quad (21)$$

$$Q_N(h_1) = \frac{N(q-1)}{2}[h_1+h_2-b_N(h_2-h_1)], \quad (22)$$

where

$$b_N = \frac{1}{Nq^{2N-1}}\left(\frac{q^{2N}-1}{q^2-1}\right). \quad (23)$$

The optimum policy when the number of stages to go is very large is, surprisingly enough, given by

$$u_\infty^* = \lim_{N\to\infty} u_N^* = h_1 + (q-1)\left(\sqrt{\frac{q+3}{q-1}}-1\right)(h_2-h_1) \quad (24)$$

With regard to a two-stage problem when for example, h_1=10, q=1.2, h_2=40, then G_2^* = 5.976 and Q_2 = 5.764, an improvement of 3.6 percent for the optimum policy as compared to merely treating the problem as a non-dual, repeated single-stage problem.

3.4 Illustrative Example 2

(*Exponential probability density function of a priori knowledge*). It is assumed that a priori knowledge is such that H is given as

$$f(h \mid h_1, \sigma) = \frac{1}{\sigma}\exp\left(-\frac{h-h_1}{\sigma}\right), \quad h_1 \le h \le \infty. \quad (25)$$

In this situation,

$$u_N^* = h_1 + \sigma\omega_N, \quad (26)$$

where

$$\omega_N = \ln(q + \omega_{N-1}), \quad \omega_0 = 0; \quad (27)$$

$$G_N^*(h_1) = N(q-1)(h_1+\sigma) + \sigma\omega_N; \quad (28)$$

$$Q_N(h_1) = N(q-1)(h_1+\sigma) + \frac{q^N - 1}{q^{N-1}(q-1)}\sigma \ln q. \quad (29)$$

Then

$$u_\infty^* = h_1 + \sigma v, \quad (30)$$

where

$$v = \ln(q + v). \quad (31)$$

The comparison for the dual solution versus the non-dual solution for N=2, h_1=0, q = e, σ = 1 shows that G_2^* = 2.123 and Q_2 = 2.068, an improvement of 2.56 percent.

3.5 Illustrative Example 3

(*Polynomial probability density function of a priori knowledge*). It is assumed that a priori knowledge is such that H is given as

$$f(h \mid h_1, m) = (m-1)h_1^{m-1}h^{-m}, \quad m \ge 2, \quad h_1 \le h \le \infty. \quad (32)$$

In this case,

$$u_N^* = h_1\rho_N, \quad (33)$$

where

$$\rho_N = \left[q + (m-1)\rho_{N-1}\right]^{1/(m-1)}, \quad \rho_0 = 0; \quad (34)$$

$$G_N^*(h_1) = h_1\left(\frac{m-1}{m-2}\right)\left[q + (N-1)(q-1) - \rho_N\right]; \quad (35)$$

$$Q_N(h_1) = h_1\left(\frac{m-1}{m-2}\right)\left[q + (N-1)(q-1) - q^{1/(m-1)}r_N\right] \quad (36)$$

where

$$r_N = -\frac{1}{q}\left[1 - q - q^{1/(m-1)}r_{N-1}\right], \quad r_1 = 1. \quad (37)$$

The optimum policy for the infinite-stage case is to set

$$u_\infty^* = h_1 w, \quad (38)$$

where w is greater than one and a root of

$$w^{m-1} = q + (m-1)(w-1). \quad (39)$$

For a two-stage process and q=1.2, h_1=10, m=3, we have that G_2^*=4.42 and Q_2=4.36, an improvement of 1.35 percent for the dual solution over the non-dual.

4 PREDICTIVE INFERENCES FOR EDUCATION PROCESS CONTROL

Various solutions have been proposed for the prediction problems, that is, the problems of making inferences on a random sample $\{Y_i;\ i=1, \ldots, m\}$ given independent observations $\{X_j;\ j=1, \ldots, n\}$ drawn from the same distribution. The Y_i's and the X_j's are commonly featured as "future outcomes" and "past outcomes" respectively. Inferences usually bear on some reduction Z of the Y_i's − possibly a minimal sufficient statistic − and consist of either frequentist prediction intervals or likelihood or predictive distribution of Z, depending on different authors. The following result is not new, but the example illustrates the procedure.

4.1 Illustrative Example 1

(*Predictive exponential distribution*). Suppose X_1, …, X_n, Y are independent random variables each having density function

$$f(s \mid \sigma) = (1/\sigma)\exp(-s/\sigma), \quad s > 0, \quad \sigma > 0. \quad (40)$$

The joint density is given by

$$f(x_1, \ldots, x_n, y \mid \sigma) = \frac{1}{\sigma^{n+1}}\exp\left(-\frac{\sum_{i=1}^{n} x_i}{\sigma} - \frac{y}{\sigma}\right) \quad (41)$$

and the factorization theorem gives

$$T = \sum_{i=1}^{n} X_i + Y \quad (42)$$

sufficient for σ. The conditional density of Y given $T = t$ is

$$g(y \mid t) = \begin{cases} \dfrac{n}{t}\left(1 - \dfrac{y}{t}\right)^{n-1}, & \text{for} \quad 0 < y < t, \\ 0, & \text{otherwise.} \end{cases} \quad (43)$$

This is obtained by finding the joint distribution of $\sum X_i$ and Y and transforming to $t = \sum x_i + y$ and y. Then the two-sided predictive interval for Y will be (a, b) where a and b satisfy

$$\int_a^b g(y \mid t) dy = \gamma \tag{44}$$

or

$$\left(1 - \frac{a}{t}\right)^n - \left(1 - \frac{b}{t}\right)^n = \gamma. \tag{45}$$

For example, if we take an interval of the form $(0, b)$, which will be the interval of shortest width, we have

$$(0,\ b) = \{0,\ t\ [1 - (1-\gamma)^{1/n}]\}. \tag{46}$$

Using the fact that $t = \sum x_i + y$,

$$y < t\ [1 - (1-\gamma)^{1/n}] \tag{47}$$

implies

$$y < [(1-\gamma)^{-1/n} - 1] \sum_{i=1}^{n} x_i. \tag{48}$$

Thus,

$$(0,\ b) = \left\{0,\ [(1-\gamma)^{-1/n} - 1] \sum_{i=1}^{n} X_i \right\}. \tag{49}$$

This is, of course, the same result given by the standard approach of using the fact that the pivotal statistic $nY / \sum X_i$ is distributed $F(2, 2n)$.

5 SAMPLING DISTRIBUTIONS FOR EDUCATION PROCESS CONTROL

The sampling truncated distributions have found many applications, including education process control. It is known that a sampling distribution for truncated law may be derived using, namely, the method based on characteristic functions (Bain and Weeks, 1964), the method based on generating functions (Charalambides, 1974), or the combinatorial method (Cacoullos, 1961). In this section, a much simpler technique than the above ones is proposed. It allows one to obtain the results for truncated laws more easily (Nechval *et al.*, 2002, 2008).

Suppose an experiment yields data sample $X^n = (X_1, \ldots, X_n)$ relevant to the value of a parameter $\boldsymbol{\theta}$ (in general, vector). Let $L_X(x^n|\boldsymbol{\theta})$ denote the probability or probability density of X^n when the parameter assumes the value $\boldsymbol{\theta}$. Considered as a function of $\boldsymbol{\theta}$ for given $X^n = x^n$, $L_X(x^n|\boldsymbol{\theta})$ is the likelihood function. If the data sample X^n can be summarized by a sufficient statistic $\mathbf{S}$, one can write $L_S(\mathbf{s}|\boldsymbol{\theta}) \propto L_X(x^n|\boldsymbol{\theta})$. Further, for any non-negative function $\omega(\mathbf{s})$, $\omega(\mathbf{s})L_S(\mathbf{s}|\boldsymbol{\theta})$ is also a likelihood function equivalent to $L_X(x^n|\boldsymbol{\theta})$. Suppose we recognize a function $\omega(\mathbf{s})$ such that $\omega(\mathbf{s})L_S(\mathbf{s}|\boldsymbol{\theta})$, regarded as a function of $\mathbf{s}$ for a given $\boldsymbol{\theta}$, is a density function. It can be shown that this is the sampling density of $\mathbf{S}$ if the family of recognized densities is complete.

The technique for finding sampling distributions of truncated laws is based on the use of the unbiasedness equivalence principle (UEP) and consists in the following. If

$$L_X(x^n|\boldsymbol{\theta},\vartheta) = [w(\boldsymbol{\theta},\vartheta)]^n L_X(x^n|\boldsymbol{\theta}), \tag{50}$$

represents the likelihood function for the truncated law, where $w(\boldsymbol{\theta},\vartheta)$ is some function of a parameter $(\boldsymbol{\theta},\vartheta)$ associated with truncation, ϑ is a known truncation point (in general, vector), then a sampling density for the truncated law is determined by

$$g_\vartheta(\mathbf{s} \mid \boldsymbol{\theta}) = \widehat{w}(\mathbf{s})\left[w(\boldsymbol{\theta},\vartheta)\right]^n g(\mathbf{s} \mid \boldsymbol{\theta}), \quad \mathbf{s} \in \mathrm{S}_\vartheta, \tag{51}$$

where

$$\widehat{w}(\mathbf{s})\left[w(\boldsymbol{\theta},\vartheta)\right]^n g(\mathbf{s} \mid \boldsymbol{\theta})$$

$$= \varphi(\mathbf{s}) L_S(\mathbf{s}|\boldsymbol{\theta},\vartheta) \propto L_X(x^n|\boldsymbol{\theta},\vartheta), \tag{52}$$

$g(\mathbf{s}|\boldsymbol{\theta})$ is a sampling density of a sufficient statistic $\mathbf{s}(X^n)$ (for a family of densities $\{f(x|\boldsymbol{\theta})\}$) determined on the basis of $L_X(X^n|\boldsymbol{\theta})$, $\widehat{w}(\mathbf{S})$ is *an unbiased estimator* of $1/[w(\boldsymbol{\theta},\vartheta)]^n$ with respect to $g(\mathbf{s}|\boldsymbol{\theta})$, $\mathbf{s} \in \mathrm{S}$ (a sample space of a non-truncated sufficient statistic $\mathbf{S}$), $\varphi(\mathbf{S})$ is a function of $\mathbf{S}$ for a given $\boldsymbol{\theta}$, which is equivalent to unbiased estimator $\widehat{w}(\mathbf{S})$ of $1/[w(\boldsymbol{\theta},\vartheta)]^n$, i.e.,

$$\varphi(\mathbf{S}) \propto \widehat{w}(\mathbf{S}) \tag{53}$$

or

$$\varphi(\mathbf{S}) = \widehat{w}(\mathbf{S})\left[w(\boldsymbol{\theta},\vartheta)\right]^n g_\vartheta(\mathbf{S} \mid \boldsymbol{\theta}) / L_{\mathrm{S}}(\mathbf{S} \mid \boldsymbol{\theta},\vartheta), \tag{54}$$

$g_\vartheta(\mathbf{s}|\boldsymbol{\theta})$ is the sampling density of a sufficient statistic $\mathbf{S}$ (for a family of densities $\{f_\vartheta(x|\boldsymbol{\theta})\}$) when the truncation parameter ϑ is known, S_ϑ is a sample space of a truncated sufficient statistic $\mathbf{S}$.

To illustrate the above technique, we present the following illustrative examples.

5.1 Illustrative Example 1

(*Sampling distribution for the left-truncated Poisson law*). Let the Poisson probability function be denoted by

$$f(x\,|\,\theta)=\frac{\theta^x}{x!}e^{-\theta},\quad x=0,\,1,\,2,\,\ldots\ . \tag{55}$$

The probability function of the restricted random variable, which is truncated away from some $\vartheta\geq 0$, is then

$$f_\vartheta(x\,|\,\theta)=w(\theta,\vartheta)f(x\,|\,\theta),\quad x=\vartheta+1,\ \vartheta+2,\ \ldots\ , \tag{56}$$

where

$$w(\theta,\vartheta)=\left(\sum_{j=\vartheta+1}^{\infty}\frac{\theta^j}{j\ !}e^{-\theta}\right)^{-1}=\left(1-\sum_{j=0}^{\vartheta}\frac{\theta^j}{j\ !}e^{-\theta}\right)^{-1}. \tag{57}$$

Consider a sample of n independent observations X_1, X_2, …, X_n, each with probability function $f_\vartheta(x|\theta)$, where the likelihood function is defined as

$$L_X(x^n\,|\,\theta,\vartheta)=\prod_{i=1}^{n}f_\vartheta(x_i\,|\,\theta)$$

$$=\left[w(\theta,\vartheta)\right]^n L_X(x^n\,|\,\theta)=\left[w(\theta,\vartheta)\right]^n\prod_{i=1}^{n}f(x_i\,|\,\theta)$$

$$=\left[w(\theta,\vartheta)\right]^n e^{-n\theta}\frac{\theta^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i\,!}, \tag{58}$$

and let

$$S=\sum_{i=1}^{n}X_i,\quad s=n(\vartheta+1),\ n(\vartheta+1)+1,\ \ldots\ . \tag{59}$$

It is well known that

$$S=\sum_{i=1}^{n}X_i,\quad s=0,\,1,\,\ldots\ . \tag{60}$$

is a complete sufficient statistic for the family $\{f(x|\theta)\}$. A result of Tukey (1949) states that sufficiency is preserved under truncation away from any Borel set in the range of X.

For the sake of simplicity but without loss of generality, consider the case $\vartheta=0$. It follows from (51) that

$$g_\vartheta(s\,|\,\theta)=\widehat{w}(s)\left[w(\theta,\vartheta)\right]^n g(s\,|\,\theta)$$

$$=\frac{\theta^s n\ !}{(e^\theta-1)^n s\ !}C_s^n,\quad s=n,\,n+1,\,\ldots\ , \tag{61}$$

where

$$g(s\,|\,\theta)=\frac{(n\theta)^s}{s\ !}e^{-n\theta},\quad s=0,\,1,\,\ldots\,, \tag{62}$$

$$[\mathrm{w}(\theta,\vartheta)]^n=\frac{1}{(1-e^{-\theta})^n}, \tag{63}$$

$$\widehat{w}(s)=\frac{n\ !}{n^s}C_s^n, \tag{64}$$

C_s^n denotes the Stirling number of the second kind (Jordan, 1950) defined by

$$C_s^n=\begin{cases}\dfrac{1}{n\ !}\displaystyle\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}j^s, & s=n,\,n+1,\,\ldots\,,\\ 0, & s<n,\end{cases} \tag{65}$$

$$E\{\widehat{w}(s)\}=\sum_{s=0}^{\infty}\widehat{w}(s)g(s\,|\,\theta)=\sum_{s=0}^{\infty}\frac{n\ !}{n^s}C_s^n\frac{(n\theta)^s}{s\ !}e^{-n\theta}$$

$$=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}e^{-(n-j)\theta}\sum_{s=0}^{\infty}\frac{(j\theta)^s}{s\ !}e^{-j\theta}=(1-e^{-\theta})^n, \tag{66}$$

This is the same result that of Tate and Goen (1958). Their proof was based on characteristic functions.

5.2 Illustrative Example 2

(*Sampling distribution for the right-truncated exponential law*). Let the probability density function of the right-truncated exponential distribution be denoted by

$$f_\vartheta(x\,|\,\theta)=w(\theta,\vartheta)f(x\,|\,\theta),\quad 0\leq x\leq\vartheta, \tag{67}$$

where

$$w(\theta,\vartheta)=\frac{1}{1-e^{-\vartheta/\theta}}, \tag{68}$$

$$f(x\,|\,\theta)=(1/\theta)e^{-x/\theta},\quad x\in[0,\infty). \tag{69}$$

Consider a sample of n independent observations X_1, X_2, …, X_n, each with density $f_\vartheta(x|\theta)$, where the likelihood function is determined as

$$L_X(x^n \mid \theta, \vartheta) = \prod_{i=1}^{n} f_\vartheta(x_i \mid \theta) = \left[w(\theta, \vartheta)\right]^n L_X(x^n \mid \theta)$$

$$= \left[w(\theta, \vartheta)\right]^n \prod_{i=1}^{n} f(x_i \mid \theta) = \left[w(\theta, \vartheta)\right]^n \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} x_i/\theta}. \tag{70}$$

It is well known that

$$S = \sum_{i=1}^{n} X_i, \quad s \in [0, \infty), \tag{71}$$

is a complete sufficient statistic for the family $\{f(x|\theta)\}$. It follows from (51) that

$$g_\vartheta(s \mid \theta) = \widehat{w}(s)\left[w(\theta, \vartheta)\right]^n g(s \mid \theta)$$

$$= \frac{e^{-s/\theta}}{\Gamma(n)\theta^n (1 - e^{-\vartheta/\theta})^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} [(s - (n-j)\vartheta)_+]^{n-1},$$

$$s \in [0, n\vartheta], \quad n \ge 1, \tag{72}$$

where a_+=max(0,a),

$$g(s \mid \theta) = \frac{s^{n-1}}{\Gamma(n)\theta^n} e^{-s/\theta}, \quad s \in [0, \infty), \tag{73}$$

$$[w(\theta, \vartheta)]^n = \frac{1}{(1 - e^{-\vartheta/\theta})^n}, \tag{74}$$

$$\widehat{w}(\mathrm{s}) = \frac{1}{s^{n-1}} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} [(s - (n-j)\vartheta)_+]^{n-1}, \tag{75}$$

$$E\{\widehat{w}(s)\} = \int_0^\infty \widehat{w}(s) g(s \mid \theta) ds$$

$$= \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} e^{-(n-j)\vartheta/\theta} \int_0^\infty \frac{[(s - (n-j)\vartheta)_+]^{n-1}}{\Gamma(n)\theta^n}$$

$$\times e^{-(s-(n-j)\vartheta)_+/\theta} d(s - (n-j)\vartheta)_+ = (1 - e^{-\vartheta/\theta})^n. \tag{76}$$

This is the same result that of Bain and Weeks (1964). Their proof was based on characteristic functions.

5.3 Illustrative Example 3

(*Sampling distribution for the doubly truncated exponential law*). Consider an exponential distribution (69) that is doubly truncated at a lower truncation point (ϑ_1) and an upper truncation point (ϑ_2). The probability density function of the doubly truncated exponential distribution is defined as

$$f_{\boldsymbol{\vartheta}}(x \mid \theta) = w(\theta, \boldsymbol{\vartheta}) f(x \mid \theta), \quad \vartheta_1 \le x \le \vartheta_2, \tag{77}$$

where $\boldsymbol{\vartheta}$=(ϑ_1, ϑ_2),

$$w(\theta, \boldsymbol{\vartheta}) = \frac{1}{e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta}}. \tag{78}$$

Consider a sample of n independent observations X_1, X_2, …, X_n, each with density $f_{\boldsymbol{\vartheta}}(x|\theta)$, where the likelihood function is determined as

$$L_X(x^n \mid \theta, \boldsymbol{\vartheta}) = \prod_{i=1}^{n} f_{\boldsymbol{\vartheta}}(x_i \mid \theta) = \left[w(\theta, \boldsymbol{\vartheta})\right]^n L_X(x^n \mid \theta)$$

$$= \left[w(\theta, \boldsymbol{\vartheta})\right]^n \prod_{i=1}^{n} f(x_i \mid \theta) = \left[w(\theta, \boldsymbol{\vartheta})\right]^n \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} x_i/\theta}. \tag{79}$$

It is well known that

$$S = \sum_{i=1}^{n} X_i, \quad s \in [0, \infty), \tag{80}$$

is a complete sufficient statistic for the family $\{f(x|\theta)\}$. It follows from (51) that

$$g_{\boldsymbol{\vartheta}}(s \mid \theta) = \widehat{w}(s)\left[w(\theta, \boldsymbol{\vartheta})\right]^n g(s \mid \theta)$$

$$= \frac{e^{-s/\theta}}{\Gamma(n)\theta^n (e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n}$$

$$\times \sum_{j=1}^{n} \binom{n}{j} (-1)^{n-j} [(s - n\vartheta_1 - (n-j)(\vartheta_2 - \vartheta_1)_+]^{n-1},$$

$$s \in [n\vartheta_1, n\vartheta_2], \quad n \ge 1, \tag{81}$$

where a_+=max(0,a), $g(s|\theta)$ is given by (73),

$$[w(\theta, \boldsymbol{\vartheta})]^n = \frac{1}{(e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n}, \tag{82}$$

$$\widehat{w}(\mathrm{s}) = \frac{1}{s^{n-1}} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j}$$

$$\times [(s - n\vartheta_1 - (n-j)(\vartheta_2 - \vartheta_1))_+]^{n-1}, \qquad (83)$$

$$E\{\widehat{w}(s)\} = \int_0^\infty \widehat{w}(s) g(s \mid \theta) ds = (e^{-\vartheta_1/\theta} - e^{-\vartheta_2/\theta})^n. \quad (84)$$

6 CONCLUSIONS

This work has presented the new technique for education process optimization via the dual control approach. The main feature of this technique is the fact that it is actively adaptive, i.e., the control plans the future learning of the system parameters as needed by the overall performance. This contro1 is obtained by using the dynamic programming equation in which the dual effect of the control appears explicitly. The technique yields a closed-loop control that takes into account not only the past observations but also the future observation program and the associated statistics. A detailed description of the technique is given and illustrative examples are presented.

Although the examples discussed in this paper are highly simplified and have orders of magnitude simpler than the complex situation faced by the education decision-maker, it does indicate the way to some very interesting points. The optimum procedure is to consider the situation as a dual control problem where information and action are interrelated.

The authors hope that this work will stimulate further investigation using the approach on specific applications to see whether obtained results with it are feasible for realistic applications.

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