Packing Circles and Irregular Polygons using Separation Lines

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Abstract: In this paper we propose a nonlinear mathematical model for the problem of packing circles, convex and non-convex irregular polygons, within a rectangular envelope to be produced, obeying containment constraints and non-overlapping constraints; the objective of the problem is to minimize the area of the rectangular envelope. We consider free rotations of the polygons and use separation lines to ensure non-overlapping. Computational tests were run using instances presented in the literature that deal with circles and polygons simultaneously; different solutions, in which the area of the rectangular envelope is smaller than or equal to the ones found in the literature were found in most cases, and the execution time is very low. This indicates that our model is computationally efficient.

1 INTRODUCTION

In production processes, such as garment manufacturing, shoe manufacturing, furniture making, among others, the optimization of layouts of items of different shapes is necessary. These items can be represented by circles or polygons. In the literature, there are many works dealing with circle packing problem (e.g., (Szabó et al., 2007) and (Castillo et al., 2008)) and polygon packing problem (see (Oliveira and Ferreira, 1993), (Toledo et al., 2013), and (Jones, 2013)), but few papers deal with the problem of packing circles and polygons simultaneously ((Kallrath, 2009; Stoyan et al., 2012; Mundim et al., 2017)).

In packing problems, items must be positioned into a certain shape, aiming to optimize an objective function. In this work, we consider the positioning of circles and polygons into a rectangle is such a manner that the produced area is minimized. That is, the objective is to minimize the multiplication of the rectangle dimensions which must be determined.

The main constraint in packing problems is the non-overlapping between items, but it is generally very complex for a computational program to determine if two items are overlapping, touching or separated. When dealing with polygons, there are tools for solving this issue ((Bennell and Oliveira, 2008)), such as the raster methods, direct trigonometry, no-fit polygon, and phi-function. When dealing with circles, the problem is commonly formulated as a problem of Euclidean geometry, either in cartesian coordinates (George et al., 1995; Addis et al., 2008) or polar and cartesian coordinates combined (Mladenović et al., 2005; López and Beasley, 2013). In this paper, we use direct trigonometry in cartesian coordinates to ensure non-overlapping between items.

Both for circle packing problems and polygon packing problems, different techniques of solution have been developed: there are heuristics (Liu et al., 2016; Oliveira and Ferreira, 1993), linear programming models (Galiev and Lisafina, 2013), linear mixed integer programming models (Toledo et al., 2013), nonlinear mixed integer programming models (George et al., 1995), and nonlinear programming models (Birgin et al., 2013; Stoyan et al., 2016). For the problem of packing circles and polygons simultaneously, there are fewer works, such as (Kallrath, 2009; Stoyan et al., 2012; Mundim et al., 2017). In (Kallrath, 2009) and (Stoyan et al., 2012), nonlinear programming models where the items can be both translated and rotated were proposed. In (Mundim et al., 2017), two heuristics based on bottom-left moves and the no-fit raster concept were proposed; in these, the items can not be freely rotated.

In (Kallrath, 2009) a model for two cases of packing of circles and convex polygons was developed. In the first case, the objective is to pack all items provided minimizing the area of the rectangular enve-

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lope, as it will be done here. In the second case, the objective is to pack the items on stocked rectangles of known geometric dimensions. Separation lines are used to ensure non-overlapping. In (Kallrath, 2009), the problem is addressed using Branch&Reduce Optimization Navigator (BARON) (Tawarmalani and Sahinidis, 2005; Sahinidis, 2014) and LindoGlobal from Lindo Systems, Inc., which is part of the GAMS 22.5 distributions.

In (Stoyan et al., 2012) a model for packing problem of circles and non-convex polygons into a strip with prohibited regions was proposed. The objective of the problem is to maximize space utilization. Furthermore, the authors ensures non-overlapping by using the phi-function.

In (Mundim et al., 2017) two heuristics for packing circles and non-convex polygons into two-dimensional bin in order to minimize one or both dimensions of the bin are proposed. These heuristics are based on bottom-left moves and the no-fit raster concept. For the development of the heuristics, the sequence in which items are packed must be determined and this is done by a Biased Random Key Genetic Algorithm.

The model proposed here is an extension of the one presented in (Peralta et al., 2017), which is a nonlinear mathematical model for an irregular strip packing problem of polygons, which may be convex or non-convex, that can rotate freely. This model yielded good solutions for large instances in rather reasonable execution time, and it should be noted that this model has a significantly smaller number of variables, when compared to the model proposed in (Kallrath, 2009). Here, we propose a nonlinear mathematical model for a packing problem in which the items are not only polygons, but also circles. The container is no longer a strip, but a rectangular envelope; the objective is to minimize the area of the rectangular envelope. In this model, we use direct trigonometry, in particular separation lines, to ensure the non-overlapping between items. A separation line is a straight line such that, given two items, one of them is on one side of the line and the other on the opposite side. A polygon is on one side of the line if all its vertices are on that side of the line or on the line. A circle is on one side of the line if its center is on that side of the line and if the distance from center to the line is greater than or equal to its radius. The use of separation lines allows us to find good solutions, even when dealing with non-convex polygons, since in this case the non-convex polygons are partitioned into convex polygons and we must have lines separating each convex sub-polygon belonging to the partition of the remaining items.

Both the polygons and the separation lines can rotate freely. We use a code for nonlinear programming to solve the problem, IPOPT (Wächter and Biegler, 2006).

This paper is organized as follows. In the next section we develop a model for packing circles and polygons from one rectangular envelope, which considers free rotations of the polygons. In Section 3, the numerical results obtained are presented and discussed. We end up presenting some conclusions in Section 4.

2 MODEL FOR A CIRCLES AND POLYGONS PACKING PROBLEM

We assume that n items, which can rotate freely, should be positioned into one rectangular envelope, obeying containment constraints and non-overlapping constraints, in order to minimize the area of the rectangular envelope. Items can be circles or irregular (convex or non-convex) polygons. The ideas used to model this problem are based in the ones presented in (Peralta et al., 2017).

If a polygon is non-convex, it is partitioned into convex polygons. The coordinates of a vertex of a non-convex polygon P_i are given by:

$$(v^l_{x_{i_k}}, v^l_{y_{i_k}}),$$

with $k = 1, \dots, p_i$ and $l = 1, \dots, v_{i_k}$, being p_i the number of convex polygons belonging to the partition of the non-convex polygon P_i and v_{i_k} the number of vertices of the convex polygon P_{i_k}.

The problem is modeled using the following variables: for each circle i, the coordinates of its center (x^{c_i}, y^{c_i}); for each polygon i, the coordinates of its reference point (x^{P_i}, y^{P_i}) and its angle of rotation θ_i; for each separation line, which separates item i from item j, the coordinates of its reference point $(x^{\ell_{i,j}}, y^{\ell_{i,j}})$ and its rotation angle $\alpha_{i,j}$; and, the width W and the length L of the rectangular envelope.

The reference point of a polygon is used to represent all the vertices of the polygon, even after translations and/or rotations are undergone; a separation line is given by two points, one is its reference point and the other is rewritten in terms of its reference point (see more details in (Peralta et al., 2017)).

The nonlinear mathematical model which is proposed here, as said before, consists of minimizing the area $(W \times L)$ of the rectangular envelope, subject to constraints ensuring that all items do not exceed the bounds of the rectangular envelope (that are presented in Section 2.1), and others ensuring that the items do not overlap (that are presented in Section 2.2).

2.1 Containment Constraints

We have to ensure that all items are fully inside the rectangular envelope.

To garantee that the circles are inside the rectangular envelope, we require that:

$$r_i - x^{c_i} \leq 0, \quad (1)$$

$$r_i + x^{c_i} \leq W, \quad (2)$$

$$r_i - y^{c_i} \leq 0, \quad (3)$$

and,

$$r_i + y^{c_i} \leq L, \quad (4)$$

where (x^{c_i}, y^{c_i}) are the coordinates of the center of circle i and r_i is its radius, for $i = 1, ..., n_c$, with n_c the number of circles to be packed.

To garantee that the polygons are inside the rectangular envelope, we require that:

$$0 \leq v^l_{x_{i_k}} \cos\theta_i - v^l_{y_{i_k}} \sin\theta_i + x^{P_i} \leq W, \quad (5)$$

and,

$$0 \leq v^l_{x_{i_k}} \sin\theta_i + v^l_{y_{i_k}} \cos\theta_i + y^{P_i} \leq L, \quad (6)$$

for $i = 1, \dots, n_p$, where n_p is the number of polygons to be packed; $k = 1, \dots, p_i$ and $l = 1, \dots, v_{i_k}$.

2.2 Non-overlapping Constraints

In the model, separation lines are used to ensure non-overlapping. The equation that ensures that a line separates item i from item j is given by:

$$y = c_{i,j}x + d_{i,j}, \quad (7)$$

with

$$c_{i,j} = \frac{(x_{k+1} - x_k)\sin\alpha_{i,j} + (y_{k+1} - y_k)\cos\alpha_{i,j}}{(x_{k+1} - x_k)\cos\alpha_{i,j} + (y_k - y_{k+1})\sin\alpha_{i,j}}$$

and

$$d_{i,j} = y^{\ell_{i,j}} - c_{i,j}x^{\ell_{i,j}},$$

being (x_k, y_k) and (x_{k+1}, y_{k+1}), the two points that determine the initial separation line; $(x^{\ell_{i,j}}, y^{\ell_{i,j}})$ and $\alpha_{i,j}$, are the reference point and the rotation angle of the separation line, respectively. When we are dealing with non-convex polygons, we must have lines separating each pair of convex polygons, belonging to the partition of two different non-convex polygons. To see more details of how separation lines are used when it comes to non-convex polygons, see (Peralta et al., 2017). The non-overlapping constraints are given below.

2.2.1 Avoiding Overlap Between Circles

In the constraints of non-overlap between circles, the use of separation lines is not necessary. We ensure that two circles do not overlap by enforcing that the distance between their centers are greater than or equal to the sum of their radii. Therefore, for two circles C_i and C_j, $i = 1, \dots, n_c - 1$ and $j = i+1, \dots, n_c$, the constraint is given by

$$(x^{c_i} - x^{c_j})^2 + (y^{c_i} - y^{c_j})^2 \geq (r_i + r_j)^2. \quad (8)$$

2.2.2 Avoiding Overlap Between Polygons

Non-overlap between two polygons P_i and P_j is enforced by the condition that all vertices of P_i and P_j are on different sides of, or on a separation line.

Let

$$X^l_i = v^l_{x_i}\cos\theta_i - v^l_{y_i}\sin\theta_i + x^{P_i},$$

and

$$Y^l_i = v^l_{x_i}\sin\theta_i + v^l_{y_i}\cos\theta_i + y^{P_i},$$

be the coordinates of vertice l of a polygon P_i. Then, the constraints that ensure non-overlap between two polygons are given by

$$Y^l_i - c_{i,j}X^l_i - d_{i,j} \leq 0, \text{ for } l = 1, \dots, v_i \quad (9)$$

and

$$Y^l_j - c_{i,j}X^l_j - d_{i,j} \geq 0, \text{ for } l = 1, \dots, v_j. \quad (10)$$

2.2.3 Avoiding Overlap Between Circles and Polygons

We ensure that a circle and a polygon are not overlapping by using a separation line. A separation line separates a circle C_i and a polygon P_j if (i) all vertices of P_j are on one side of (or on) the separation line, and (ii) the center of C_i is on the other side of the separation line and distant from at least the radius of C_i. The constraints that ensure non-overlapping between a circle and a polygon are given by

$$Y^l_j - c_{i,j}X^l_j - d_{i,j} \geq 0, \text{ for } l = 1, \dots, v_j, \quad (11)$$

$$y^c_i - c_{i,j}x^c_i - d_{i,j} \leq 0, \quad (12)$$

and

$$\Delta_{i,j} \geq r_i, \quad (13)$$

with $\Delta_{i,j} = \dfrac{|c_{i,j}x^{c_i} - y^{c_i} + d_{i,j}|}{\sqrt{c^2_{i,j} + 1}}$ the distance from the center of C_i to the line that separates C_i from P_j.

3 NUMERICAL RESULTS

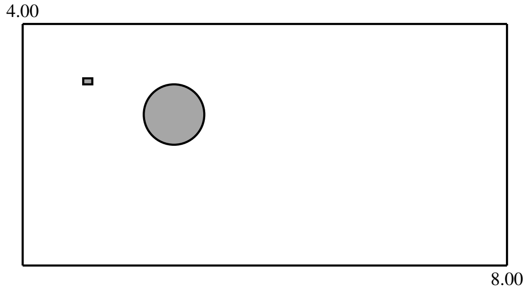
In this section we present and compare the results obtained when applying our model to solve the problems presented in (Kallrath, 2009) and (Stoyan et al., 2012), which pack circles and polygons simultaneously. All the numerical experiments have been performed using a code for nonlinear programming to solve the problem, IPOPT (Wächter and Biegler, 2006) (an algorithm of interior points type designed to find local solutions of mathematical optimization problems), which is part of the COIN-OR (Wächter and Biegler, 2015), on an Ubuntu 12.04 laptop with an Intel Core I7-4510U CPU @ 2.1GHz processor and 8 GB of memory.

In (Kallrath, 2009), as here, the items should be positioned into a rectangular envelope. The rectangular envelope is subject to lower and upper bounds of its width and length. We tested all those instances in (Kallrath, 2009) with circles and polygons which has data available and used the same bounds presented there, which are $\bar{W} = 4$ for the upper bound of the width and $\bar{L} = 8$ for the upper bound of the length for all instances, except for instance *c1p1-3*, for which we have $\bar{W} = 2.5$ and $\bar{L} = 3$, and for instance *c6p3*, for which we have $\bar{W} = 6$ and $\bar{L} = 10$.

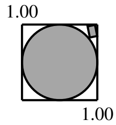
The results obtained by solving these instances using the model presented in Section 2 and those presented in (Kallrath, 2009) are summarized in Table 1. The names of the instances are presented in the first column. The number of circles and polygons are presented in second and third columns, respectively. From the fourth to the seventh column, the data and results in (Kallrath, 2009) are presented. From the eighth to the eleventh column, our data and results are presented. In these, n is the number of variables of the model; m is the number of constraints of the model; CPU is the execution time in seconds; and a is the area of the rectangular envelope. In all instances, IPOPT stopped in optimal local solutions. The solution obtained by the proposed model and the starting point used to find it, for each instance, are displayed from Figures 1 to 8.

We can observe that the number of variables and constraints in our model is significantly smaller than the number of variables and constraints of the model presented in (Kallrath, 2009). The execution time is considerably reduced, which can be attributed, not only to the reduction of variables and constraints in the model, but also to the differences in operating system, hardware architecture, among others. We can also observe in our results that the area is smaller than or equal to the ones obtained in (Kallrath, 2009) in most instances. This reduction in the number of variables, constraints and execution time allows us to solve larger instances, as the ones presented in (Stoyan et al., 2012).

In (Stoyan et al., 2012), the items should be positioned into a strip which can have prohibited regions; only one instance without prohibited regions was presented in this paper. We use this instance to test our model, fixing the width W of the strip in the value

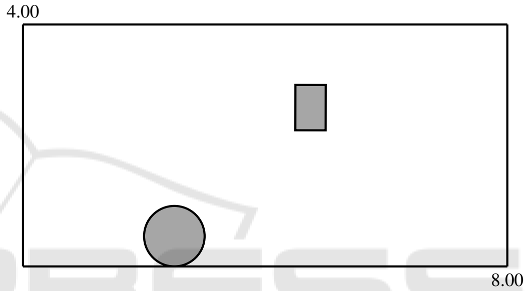


(a) Starting point

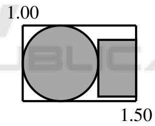


(b) Solution

Figure 1: Instance: c1p1-1.

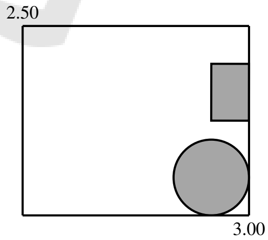


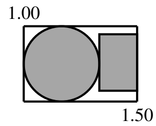
(a) Starting point



(b) Solution

Figure 2: Instance: c1p1-2.



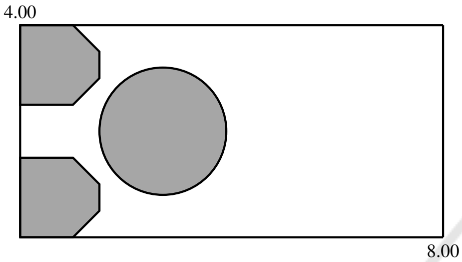


(a) Starting point (b) Solution

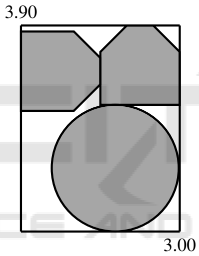
Figure 3: Instance: c1p1-3.

Table 1: Comparison of our results to those in (Kallrath, 2009).

			(Kallrath, 2009)				Our approach			
Instance	C	P	n	m	CPU	a	n	m	CPU	a
c1p1-1	1	1	47	54	0.10	1.00	10	26	0.01	1.00
c1p1-2	1	1	47	54	42.00	1.50	10	26	0.02	1.50
c1p1-3	1	1	47	54	368.00	2.50	10	26	0.01	1.50
c1p5a	1	2	767	799	2000.00	15.90	19	80	0.89	11.70
c1p5b	1	5	791	838	1800.00	18.55	64	284	0.06	19.73
c1p6a	1	6	1110	1161	1800.00	20.37	85	376	0.05	20.06
c3p3	3	3	489	516	2000.00	16.75	53	177	0.03	16.86
c6p3	6	3	628	661	33000.00	35.10	86	219	0.04	33.19

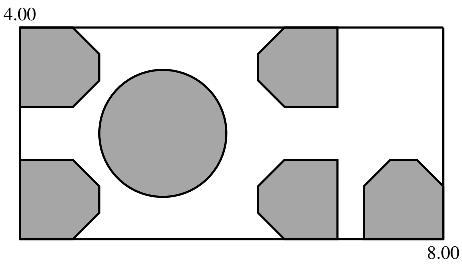


(a) Starting point

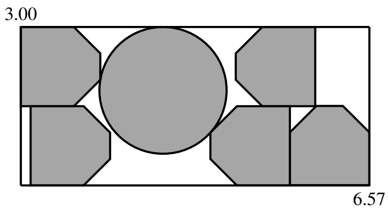


(b) Solution

Figure 4: Instance: c1p5a.

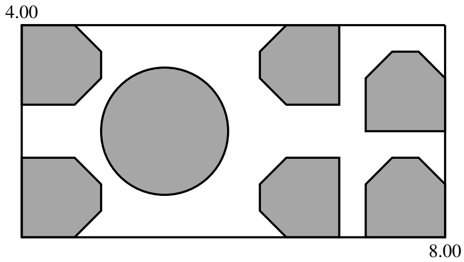


(a) Starting Point

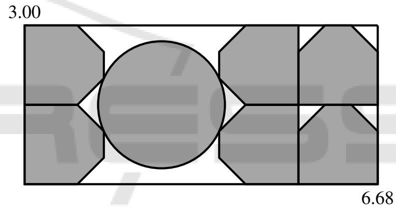


(b) Solution

Figure 5: Instance: c1p5b.

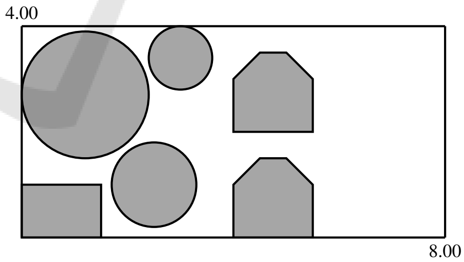


(a) Starting point

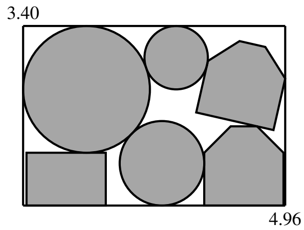


(b) Solution

Figure 6: Instance: c1p6a.

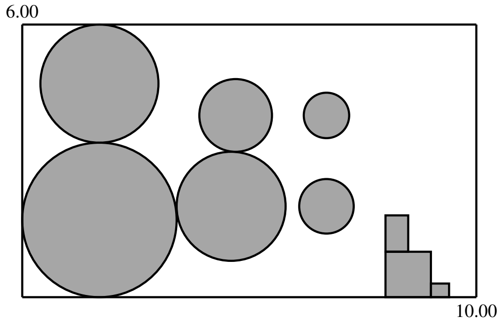


(a) Starting point

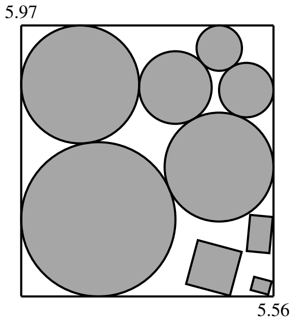


(b) Solution

Figure 7: Instance: c3p3.



(a) Starting point



(b) Solution

Figure 8: Instance: c6p3.

presented in the paper, $W = 25$; for this, we setup the option of IPOPT “fixed variable treatment” which includes equality constraints to fix the variable. This instance consists of 44 polygons, being some non-convex, and 5 circles of different size. To model this instance, 19335 variables and 54860 restrictions were used. In (Stoyan et al., 2012), a length of 33.9 in two hours of runtime was achieved. We achieved a length of 36.00 in 521.64 seconds. The layout obtained is displayed in Figure 10.

4 CONCLUSIONS

This work deals with a mathematical model for a packing problem of circles and non-convex polygons with continuous rotations in a rectangular envelope; the objective of the problem is to minimize the area of the rectangular envelope. In the model, we use separation lines to avoid overlap of the items. The modeling of separation lines uses the general equation of the line. We compare our model with the one presented in (Kallrath, 2009) for packing problem of circles and convex polygons, in which separation lines are also used, but these are modeled using the vector equation of the line. The vector equation of the line implies the employment of many variables and constraints. With the use of the general equation of the line the number of variables and constraints in the model is significantly reduced, and therefore, the execution time is also reduced; with these reductions, we were able to extend the model to non-convex polygons and to solve larger problems.

We tested seven instances of those presented in (Kallrath, 2009), and to verify the effectiveness of our model for larger problems with non-convex polygons, we tested the only instance for packing problem of circles and polygons without prohibited regions presented in (Stoyan et al., 2012). We achieved a very good solution in a short runtime for this instance. To solve this problem, we use a code for nonlinear programming, IPOPT (Wächter and Biegler, 2006), which depends substantially on a starting point. In all the instances we have tested, we use a random starting point. We will work on a method that constructs starting points, with which we believe the results could be improved.

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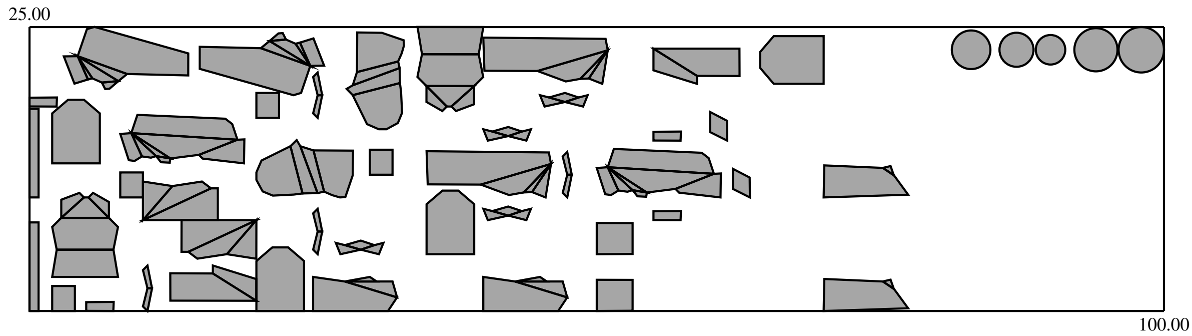

Figure 9: Starting point used to solve the instance from (Stoyan et al., 2012).

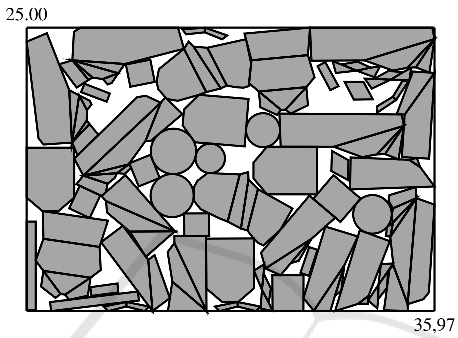

Figure 10: Solution found for the instance from (Stoyan et al., 2012).

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