Dynamic Pricing Strategies in a Finite Horizon Duopoly with Partial Information

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Abstract: In many applications the sale of perishable products is characterized by competitive settings and incomplete information. While prices of sellers are typically observable, the inventory levels of firms are mutually not observable. In this paper, we analyze stochastic dynamic pricing models in a finite horizon duopoly with partial information. We use a Hidden Markov Model approach to compute strategies that are applicable when the competitor's inventory level is not observable. Our approach utilizes feedback pricing strategies that are optimal if the competitor's inventory level is observable. We show that price reactions are balancing two effects: (i) to slightly undercut the competitor's price to sell more items, and (ii) to use high prices to promote a competitor's run-out and to act as a monopolist for the rest of the time horizon. Moreover, we compute heuristic strategies that can be applied when the number of competitors is large and their strategies are unknown. We find that expected profits are hardly affected by different information structures as long as the firms' information is symmetric.

1 INTRODUCTION

In many markets, firms offering their products have to deal with competition and limited information. Sellers are required to choose appropriate pricing decisions to maximize their expected profits. In e-commerce, it has become easy to observe and to change prices. Hence, dynamic pricing strategies that take into account the competitor's strategies will be more and more applied.

However, optimal price reactions are not easy to find. Applications can be found in a variety of contexts that involve perishable (e.g., airline tickets, accommodation services, seasonal products) as well as durable goods (e.g., technical devices, natural resources).

In this paper, we study duopoly pricing models in a stochastic dynamic framework. We focus on perishable goods. In our model, sales probabilities are allowed to be an arbitrary function of time and the competitor's prices. Our aim is to take into account scenarios in which (i) the competitor's inventory level *is* observable, (ii) the competitor's inventory level *is not* observable, and (iii) even the competitor's pricing strategy is unknown.

The best way to sell products is a classical application of revenue management theory. The problem is closely related to the field of dynamic pricing, which is summarized in the books by Talluri, van Ryzin (2004), Phillips (2005), and Yeoman, McMahon-Beattie (2011). The survey by Chen, Chen (2015) provides an excellent overview of recent pricing models under competition.

In the article by Gallego, Wang (2014) the authors consider a continuous time multi-product oligopoly for differentiated perishable goods. They use optimality conditions to reduce the multi-dimensional dynamic pure pricing problem to a one dimensional one. Gallego, Hu (2014) analyze structural properties of equilibrium strategies in more general oligopoly models for the sale of perishable products. Martinez-de-Albeniz, Talluri (2011) consider duopoly and oligopoly pricing models for identical products. They use a general stochastic counting process to model the demand of customers.

Further related models are studied by Yang, Xia (2013) and Wu, Wu (2015). Dynamic pricing models under competition that also include strategic customers are analyzed by Levin et al. (2009) and Liu, Zhang (2013). Competitive pricing models with limited demand information are studied by Tsai, Hung (2009), Adida, Perakis (2010), and Chung et al. (2012) using robust optimization and demand learning approaches. The effects of strategic interaction of

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data-driven strategies in competitive settings are studied by, e.g., Serth et al. (2017), using an interactive simulation platform.

In most existing models strong assumptions are made: (i) sales probabilities are assumed to be of a highly stylized form, (ii) the competitors' inventory levels are assumed to be observable, and (iii) competitors adjust their prices at the same point in time. While many papers concentrate on (the existence of) equilibrium strategies, we look for applicable solution algorithms that allow to compute effective response strategies in more realistic settings: Demand probabilities are allowed to generally depend on time as well as the prices of all market participants. Inventory levels do not have to be mutually observable. As in many practical applications, we assume sequential mutual price reactions with some delay. We consider a discrete time model which is based on the infinite horizon model by Schlosser, Boissier (2017). We extend their model by limited inventory levels as well as a finite horizon setting.

The main contribution of this paper is threefold. We (i) derive optimal pricing strategies when the competitor's inventory level is observable, (ii) derive near-optimal pricing strategies for the case that the competitor's inventory level is not observable, and (iii) we present a heuristic for the case that competitors' strategies are not known.

This paper is organized as follows. In Section 2, we describe the stochastic dynamic duopoly model for the sale of a finite number of perishable goods. We allow sales intensities to depend on the competitor's price as well as on time (seasonal effects). The state space of our model is characterized by time and the current competitors' prices. The stochastic dynamic control problem is expressed in discrete time.

In Section 3, we consider a duopoly competition, in which the inventory level of the competitor is observable. We assume that both competitors act rationally. We set up a firm's Hamilton-Jacobi-Bellman equation and use recursive methods (value iteration) to compute both firms' value functions. Finally, we are able to compute optimal feedback prices as well as expected profits of the two competing firms. By using numerical examples, we investigate typical properties of optimal pricing policies.

In Section 4, we analyze response strategies for cases where the inventory level of the competitor is not observable. Using a Hidden Markov Model, we show how to compute efficient pricing strategies and how to evaluate expected profits. Our proposed solution approach is based on the results of the full information model introduced in the previous section. The key idea is to let the competing firms mutually estimate their competitor's remaining inventory level. In Section 5, we show how to derive applicable dynamic pricing heuristics for cases in which the competitor's inventory level as well as its pricing strategy are completely unknown.

Finally, in Section 6, we compare the different strategies derived in this paper. Conclusions are offered in the final section.

2 MODEL DESCRIPTION

We consider the situation in which a firm wants to sell a finite number of goods (e.g., airline tickets, hotel tickets, etc.) on a digital market platform. We assume that a second seller competes for the same market. In our model, we allow customers to compare prices of the two different competitors.

The initial number of items of firm 1 and firm 2 are denoted by $N^{(1)}$ and $N^{(2)}$, respectively, $N^{(1)},N^{(2)} < \infty$. We assume that items cannot be reproduced or reordered. The time horizon T is finite, $T < \infty$. If firm k sells one item shipping costs $c^{(k)}$ have to be paid, $k = 1,2$. A sale of one of firm k's items at price a leads to a net revenue of $a - c^{(k)}$. Discounting is also included in the model. For the length of one period we use the discount factor δ, $0 < \delta \leq 1$..

Due to customer choice the sales probabilities of a firm should depend on its offer price a and the competitor's price p. We also allow the sales probabilities to depend on time.

The (joint) probability that between time t and $t+\Delta$ firm 1 can sell exactly i items at a price a, $a \geq 0$, while firm 2 can sell j items at price p, $p \geq 0$, is denoted by, $0 \leq t < T$, $i, j = 0,1,2,...$, $\Delta > 0$,

$$P_t^{(\Delta)}(i,j,a,p)$$

Without loss of generality, in the following, we assume Poisson distributed sales probabilities, i.e.,

$$P_t^{(\Delta)}(i,j,a,p) := \frac{\Lambda_{t,\Delta}^{(1)}(a,p)^i}{i!} \cdot e^{-\Lambda_{t,\Delta}^{(1)}(a,p)} \cdot \frac{\Lambda_{t,\Delta}^{(2)}(p,a)^j}{j!} \cdot e^{-\Lambda_{t,\Delta}^{(2)}(p,a)}, \quad (1)$$

where $\Lambda_{t,\Delta}^{(k)}(a,p) := \int_t^{t+\Delta} \lambda_s^{(k)}(a,p)ds$, $k = 1,2$, $a,p \geq 0$; the sales intensity of a firm k's product is denoted by $\lambda^{(k)}$. In our model, the sales intensity of firm k, $k = 1,2$, $t = [0,T]$, $a \geq 0$, $p \geq 0$,

$$\lambda_t^{(k)}(a,p) \quad (2)$$

is a general function of time t, offer price a, and the competitor's price p. The random inventory level of firm k at time t is denoted by $X_t^{(k)}$, $0 \leq t \leq T$. The end of sale for firm k is the random time $\tau^{(k)}$, when all of its items are sold, that is $\tau^{(k)} := \min_{0 \leq t \leq T} \left\{ t : X_t^{(k)} = 0 \right\} \wedge T$; for all remaining $t \geq \tau$ we let a firm's price $a_t := 0$ and $\lambda_t^{(k)}(0, \cdot) := 0$, $k = 1,2$. As long as a firm has items left to sell, for each period t, a price a has to be chosen.

We call strategies $(a_t)_t$ admissible if they belong to the class of Markovian feedback policies; i.e., pricing decisions $a_t \geq 0$ may depend on time t, the current own inventory level, the current prices of the competitor, and (if observable) the inventory level of the competitor. By A we denote the set of admissible prices. A list of variables and parameters is given in the Appendix, see Table 5.

In some applications, sellers are able to anticipate transitions of the market situation. In particular, the price responses of competitors as well as their reaction time can be taken into account. In this case, a change of the competitor's price p can take place within one period. A typical scenario is that a competitor adjusts its price in response to our price adjustment with a certain delay.

In the following two sections, we assume that the pricing strategy and the reaction time of competitors are known; i.e., we assume that choosing a price a at time t is followed by a state transition (e.g., a competitor's price reaction) and the current price p changes to a subsequent price reaction, which may depend on the current price decision a.

We assume that the state of the system is characterized by the inventory levels of both firms and the current competitor's price. In real-life applications, a firm is not able to adjust its prices immediately after the price reaction of the competing firm. Hence, we assume that in each period the price reaction of the competing firm takes place with a delay of h periods, $0 < h < 1$. I.e., after an interval of size h the competitor adjusts its price, see Figure 1.

Thus in period t the probability to sell exactly i items during the first interval of size h, i.e., $[t, t+h]$, is $P_t^{(h)}(i, j, a_t, p_{t-1+h})$, $t = 0, 1, ..., T-1$. Due to the competitor's price reaction for the rest of the period $[t+h, t+1]$ the sales probability changes to $P_{t+h}^{(1-h)}(i, j, a_t, p_{t+h})$, $t = 0, 1, ..., T-1$.

For single intervals $[0, h]$ and $[T, T+h]$, we assume that there is no demand and we let $P_0^{(h)}(i, j, a_0, p_0) = P_T^{(h)}(i, j, a_T, p_{T-1+h}) := 1_{\{i=j=0\}}$.

The evolution of the cumulated profits of firm k, $k = 1,2$, is connected to its inventory process $X_t^{(k)}$ and characterized by each period's realized net revenues. Depending on the chosen pricing strategy $(a_t)_t$ of firm 1 and the strategy $(p_t)_t$ of firm 2, the random accu-

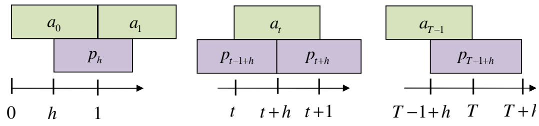

Figure 1: Sequence of price reactions in a duopoly with response time h, $0 < h < 1$.

mulated profit of firm k from time t on (discounted on time t) amounts to, $0 \leq t \leq T$, $k = 1,2$,

$$G_t^{(k)} := \sum_{s=t}^{T-1} \delta^{s-t} \cdot (a_s - c^{(k)}) \cdot \left(X_s^{(k)} - X_{s+1}^{(k)} \right). \quad (3)$$

Each firm k seeks to determine a non-anticipating (Markovian) pricing policy that maximizes its expected total profit, $k = 1,2$,

$$E\left(G_0^{(k)} \middle| X_0^{(1)} = N^{(1)}, X_0^{(2)} = N^{(2)} \right). \quad (4)$$

In the following sections, we will solve dynamic pricing problems that are related to (1) - (4). In the next section, we consider competitive duopoly markets with complete information. In Section 4, we compute pricing strategies for scenarios with incomplete information and partially observable states, i.e., we assume that the competitor's inventory level is not observable. In Section 5, we additionally assume that the competitor's strategy is unknown. In Section 6, we compare the results of the three different models.

3 OPTIMAL DYNAMIC PRICING STRATEGIES IN A DUOPOLY WITH OBSERVABLE STATES

3.1 Solution with Full Knowledge

In this section, we want to derive mutual optimal price response strategies. We assume that both firms can mutually observe their inventory levels. Following the Bellman approach, the best expected future profits of firm 1 and firm 2, i.e., $E(G_t^{(1)} | X_t^{(1)} = n, X_t^{(2)} = m, p_t = p)$ and $E(G_{t+h}^{(2)} | X_{t+h}^{(1)} = n, X_{t+h}^{(2)} = m, a_{t+h} = a)$, respectively, cf. (4), are described by the value functions $V_t^*(n, m, p)$ and $W_{t+h}^*(n, m, a)$, $t = 0, 1, ..., T$. The set of admissible prices A can be continuous or discrete. If either all items are sold or the time is up, no future profits can be made, i.e., the natural boundary condition for the value functions V and W are given by, $n = 0, 1, ..., N^{(1)}$, $m = 0, 1, ..., N^{(2)}$, $a, p \in A$, $t = 0, 1, ..., T-1$,

$$V_t^*(0, m, p) = 0, \quad \textit{and} \quad V_T^*(n, m, p) = 0, \quad (5)$$

$$W^*_{t+h}(n,0,a)=0, \quad and \quad W^*_{T+h}(n,m,a)=0. \quad (6)$$

We assume that in case of a run-out a firm sets its price equal to zero for the rest of the time horizon. The Hamilton-Jacobi-Bellman (HJB) equation of firm 1 can be written as, $t=0,1,...,T-1$, $n=1,...,N^{(1)}$, $m=0,...,N^{(2)}$, $0<h<1$, $a,p\in A$,

$$V^*_t(n,m,p)=\max_{a\in A}\left\{\sum_{i_1,j_1\geq 0}P^{(h)}_t(i_1,j_1,a,p)\right.$$
$$\cdot\sum_{i_2,j_2\geq 0}P^{(1-h)}_{t+h}\left(i_2,j_2,1_{\{n-i_1>0\}}\cdot a,\right.$$
$$\left.p^*_{t+h}\left((n-i_1)^+,(m-j_1)^+,1_{\{n-i_1>0\}}\cdot a\right)\right)$$
$$\cdot\Big((a-c^{(1)})\cdot\min(n,i_1+i_2)+$$
$$+\delta\cdot V^*_{t+1}\left((n-i_1-i_2)^+,(m-j_1-j_2)^+,1_{\{m-j_1-j_2>0\}}\right.$$
$$\left.\left.\cdot p^*_{t+h}\left((n-i_1)^+,(m-j_1)^+,1_{\{n-i_1>0\}}\cdot a\right)\right)\Big)\right\}. \quad (7)$$

Note, (7) mirrors all possible sales scenarios within one period of time and takes the corresponding inventory transitions as well as the anticipated optimal price reactions of the competitor into account.

The HJB of firm 2 is given by, $t=0,1,...,T-1$, $n=0,...,N^{(1)}$, $m=1,...,N^{(2)}$, $0<h<1$, $a,p\in A$,

$$W^*_{t+h}(n,m,a)=\max_{p\in A}\left\{\sum_{i_2,j_2\geq 0}P^{(1-h)}_{t+h}(i_2,j_2,a,p)\right.$$
$$\cdot\sum_{i_1,j_1\geq 0}P^{(h)}_{t+1}\left(i_1,j_1,\right.$$
$$\left.a^*_{t+1}\left((n-i_1)^+,(m-j_1)^+,1_{\{m-j_1>0\}}\cdot p\right),1_{\{m-j_1>0\}}\cdot p\right)$$
$$\cdot\Big((p-c^{(2)})\cdot\min(m,j_1+j_2)+$$
$$+\delta\cdot W^*_{t+1+h}\left((n-i_1-i_2)^+,(m-j_1-j_2)^+,1_{\{n-i_1-i_2>0\}}\right.$$
$$\left.\left.\cdot a^*_{t+1}\left((n-i_1)^+,(m-j_1)^+,1_{\{m-j_1>0\}}\cdot p\right)\right)\Big)\right\}. \quad (8)$$

The associated prices of both firms are given by the arg max of (7) and (8), respectively, i.e., $n,m>0$, $t=0,1,...,T-1$,

$$a^*_t(n,m,p)=\arg\max_{a\in A}\{...\}, \quad (9)$$

$$p^*_{t+h}(n,m,a)=\arg\max_{p\in A}\{...\}. \quad (10)$$

If a firm runs out of inventory, we set the price 0, i.e., for all m, p we let $a^*_t(0,m,p)=0$ and for all n, a, we let $p^*_{t+h}(n,0,a)=0$. The coupled value functions and the optimal feedback policies of the two competing firms can be computed in the following recursive order:

$$p^*_{T-1+h}(n,m,a),W^*_{T-1+h}(n,m,a)\rightarrow$$
$$a^*_{T-1}(n,m,p),V^*_{T-1}(n,m,p)\rightarrow\dots$$
$$\dots\rightarrow p^*_h(n,m,a),W^*_h(n,m,a)$$
$$\rightarrow a^*_0(n,m,p),V^*_0(n,m,p). \quad (11)$$

3.2 Numerical Examples

To illustrate the approach, cf. (7) - (11), in the following, we consider a numerical example.

Example 3.1. We assume a duopoly. Let $T=50$, $c^{(1)}=c^{(2)}=10$, $N^{(1)}=N^{(2)}=10$, $\delta=1$, $h=0.5$, and $a\in A:=(10,20,...,400)$. We assume Poisson distributed sales probabilities $P^{(h)}_t(i,j,a,p)$, which are determined by $t=0,h,1,...,T$, $k=1,2$, cf. (1), $\Lambda^{(k)}_{t,h}(a,p):=h\cdot\left(1-e^{-10^5\cdot a^{-2.5+t/T}}\right)\cdot\beta(a,p)$, and the factor $\beta(a,p):=\frac{1_{\{a>0\}}\cdot(p-L\cdot\min(a,p))}{a+p-2\cdot L\cdot\min(a,p)}$, $L:=0.8<1$.

Table 1 illustrates the expected profits of firm 1 for different inventory levels n and different points in time t (for the case that firm 2's price is $p=100$ and its inventory level is $N^{(2)}=10$). We observe that the expected future profits are decreasing in time and increasing-decreasing in the number of items left to sell. The optimal expected profits of the second firm have the same characteristics. Compared to firm 1 the total expected profits of firm 2 are slightly larger ($W^*_h(10,10,a^*_0(10,10,0))=1769$).

Table 1: Expected profits $V^*_t(n,10,100)$, Example 3.1.

$n\backslash t$	0	10	20	30	40	45
1	363	362	359	348	306	252
2	654	652	640	601	494	368
3	877	872	852	788	628	423
5	1213	1202	1166	1056	782	381
7	1464	1449	1396	1233	737	381
10	1754	1726	1638	1348	723	381

Table 2 illustrates the feedback prices of firm 1 for different competitor's inventory levels m and different prices p (for the case that $t=20$ and firm 1's inventory level is $N^{(1)}=10$). We observe that optimal response prices are decreasing-increasing in the competitor's price and decreasing in the competitor's inventory level. I.e., in general, there is an incentive to (slightly) *undercut* the competitor's price.

However, if the competitor has a small price and a small inventory level then it is more advantageous to set *high* prices such that the competitor is likely to sell all of its items, and in turn, our firm becomes a *monopolist* for the rest of the time horizon. If the competitor's inventory level is small, the optimal price can even dominate the monopoly price, cf. $a_{20}^{*}(10,0,0) = 260$ in Table 2!

Table 2: Expected profits $a_{20}^{*}(10,m,p)$, Example 3.1.

$p\backslash m$	0	1	2	3	5	7	10
0	260	.	.	.	.	.	.
50	.	400	390	300	220	200	160
100	.	400	390	300	220	200	160
150	.	400	310	300	220	190	140
200	.	400	280	250	190	180	150
250	.	340	260	200	190	180	150
300	.	240	210	200	190	180	150
400	.	220	200	200	190	180	150

Remark 3.1.

(i) The expected profits are increasing-decreasing in their own inventory level.

(ii) The expected profits are decreasing in the competitor's inventory level.

(iii) If there is no discounting then the expected profits are increasing in the time-to-go.

(iv) The expected profits are increasing-decreasing in the current competitor's price.

Remark 3.2.

(i) The optimal prices are not necessarily decreasing in their own inventory level.

(ii) The optimal prices are decreasing in the competitor's inventory level.

(iii) If demand is not increasing in time then the optimal prices are decreasing in the time.

(iv) The optimal prices are decreasing-increasing in the current competitor's price.

Figure 2 illustrates simulated sales processes in the context of Example 3.1. Figure 2a illustrates price trajectories of the two competing firms. Figure 2b shows the associated evolutions of the inventory levels. As demand is increasing in time, on average, prices as well as the number of sales increase at the end of the time horizon.

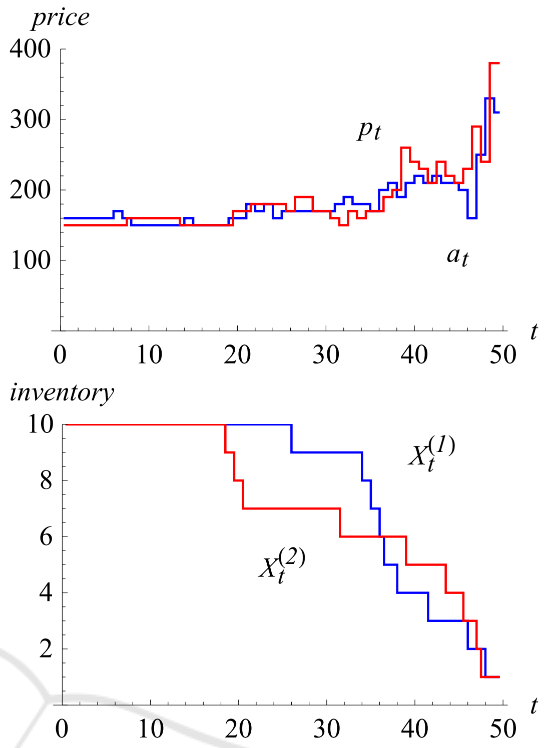

Figure 2: Simulated price paths (upper window 2a) and associated inventory levels over time (lower window 2b); Example 3.1.

4 A HIDDEN MARKOV MODEL WITH PARTIALLY OBSERVABLE STATES

4.1 Theoretical Solution

In this section, we will assume that the competitor's inventory level cannot be observed. To derive feedback pricing strategies we use a Hidden Markov Model. We will use probability distributions for the competitor's inventory level, which are based on the observable price paths of both firms.

Let $\pi_t(m)$ denote the (estimated) probability that firm 2 has exactly m items left at time t; let $\varpi_t(n)$ denote the probability that firm 1 has exactly n items left at time t. We assume that the initial inventory levels of both competitors are common knowledge; i.e., the starting distributions are $\pi_0(m) = \pi_h(m) = 1_{\{m=N^{(2)}\}}$ and $\omega_0(n) = \omega_h(n) = 1_{\{n=N^{(1)}\}}$. Furthermore, a run-out is observable, since we assume that in case of a run-out a firm has to set its price equal to zero. The evolutions of the probabilities $\pi_t(m)$ and $\varpi_t(n)$ are given by, $n = 0,...,N^{(1)}$, $m = 0,...,N^{(2)}$, $a_t, p_t, a_{t-1+h}, p_{t-1+h} \in A$, $t = 0,1,...,T$,

$$\pi_{t+h}(m; a_t, p_t) = \sum_{\substack{i_1, j_1 \geq 0, 0 \leq m^- \leq N^{(2)}: \\ m = (m^- - j_1)^+}} P_t^{(h)}(i_1, j_1, a_t, p_t) \cdot \pi_t(m^-)$$

$$\pi_t(m; a_{t-1+h}, p_{t-1+h}) = \sum_{\substack{i_2, j_2 \geq 0, \\ 0 \leq m^- \leq N^{(2)}: \\ m = (m^- - j_2)^+}} P_{t-1+h}^{(1-h)}(i_2, j_2, a_{t-1+h}, p_{t-1+h}) \cdot \pi_{t-1+h}(m^-) \tag{12}$$

$$\varpi_{t+h}(n; a_t, p_t) = \sum_{\substack{i_1, j_1 \geq 0, 0 \leq n^- \leq N^{(1)}: \\ n = (n^- - i_1)^+}} P_t^{(h)}(i_1, j_1, a_t, p_t) \cdot \varpi_t(n^-)$$

$$\varpi_t(n; a_{t-1+h}, p_{t-1+h}) = \sum_{\substack{i_2, j_2 \geq 0, \\ 0 \leq n^- \leq N^{(1)}: \\ n = (n^- - i_2)^+}} P_{t-1+h}^{(1-h)}(i_2, j_2, a_{t-1+h}, p_{t-1+h}) \cdot \varpi_{t-1+h}(n^-). \tag{13}$$

Note, (12) and (13) are relevant for both firms as they might try to estimate (i) the competitor's inventory level as well as (ii) the competitor's beliefs concerning the own inventory. This way the competitor's price reactions can be anticipated via a probability distribution.

Both firms are assumed to act rationally. Pricing decisions are such that no firm has an advantage to deviate from its strategy. Due to the defined sequence of events, theoretically, optimal decisions can be recursively inferred. The corresponding value functions of both firms, denoted by

$$V_t^{(*)}(n, p, \vec{\pi}_t, \vec{\omega}_t) \tag{14}$$

$$W_{t+h}^{(*)}(m, a, \vec{\pi}_{t+h}, \vec{\omega}_{t+h}), \tag{15}$$

are determined by the usual boundary conditions $V_t^{(*)}(0, \cdot, \cdot, \cdot) = 0$, $V_T^{(*)}(\cdot, \cdot, \cdot, \cdot) = 0$ (for firm 1) and $W_{t+h}^{(*)}(0, \cdot, \cdot, \cdot) = 0$, $W_{T+h}^{(*)}(\cdot, \cdot, \cdot, \cdot) = 0$ (for firm 2) as well as an associated system of Bellman equations similar to (7)-(8) extended by transitions for the beliefs, cf. (12)-(13). The corresponding optimal feedback policies $a_t^{(*)}(n, p, \vec{\pi}_t, \vec{\omega}_t)$ and $p_{t+h}^{(*)}(m, a, \vec{\pi}_{t+h}, \vec{\omega}_{t+h})$ of the two competing firms can be computed in recursive order (similar to (9)-(11)).

However, optimal policies *cannot* be computed in practical applications. Note, the size of the state space is exploding as the probability distributions $\vec{\pi}$ and $\vec{\omega}$ are involved (curse of dimensionality). Hence, heuristic solutions are needed.

In the following subsection, we present an approach to compute viable heuristic feedback pricing strategies for the model with partially observable states. The key idea is to approximate the functions $V_t^{(*)}(n, p, \vec{\pi}_t, \vec{\omega}_t)$ and $W_{t+h}^{(*)}(m, a, \vec{\pi}_{t+h}, \vec{\omega}_{t+h})$ by using weighted expressions of the value functions $V_t^*(n, m, p)$ and $W_t^*(n, m, a)$ (of the model with full knowledge) and their associated policies $a_t^*(n, m, p)$ and $p_t^*(n, m, a)$ derived in Section 3.

4.2 Solution with Partial Knowledge

Motivated by the Hidden Markov Model (HMM), cf. Section 4.1, in which the competitor's inventory level cannot be observed, next, we want to define viable heuristic pricing strategies for the two competing firms. Based on current beliefs, we approximate the correct value functions (14) - (15) (and related controls) using price reactions (9) - (10) and future profits (7) - (8) of the fully observable model. As the value functions of the fully observable model might systematically overestimate the correct values (14) - (15), we include an additional positive penalty factor z. If z is smaller than 1, future profits (7) - (8) are reduced.

For firm 1 we define the feedback prices, $t = 0, 1, ..., T-1$, $n = 1, ..., N^{(1)}$, $p \in A$,

$$\tilde{a}_t(n, p; \vec{\pi}_t, \vec{\omega}_t) = \underset{a \in A}{\arg\max} \left\{ \sum_{i_1, j_1 \geq 0} P_t^{(h)}(i_1, j_1, a, p) \right.$$

$$\cdot \sum_{0 \leq \tilde{m} \leq N^{(2)}} \pi_t(\tilde{m}) \cdot \sum_{0 \leq \tilde{n} \leq N^{(1)}} \varpi_t(\tilde{n}) \cdot \sum_{i_2, j_2 \geq 0} P_{t+h}^{(1-h)}(i_2, j_2,$$

$$1_{\{\tilde{n}-i_1>0\}} \cdot a, p_{t+h}^*\left((\tilde{n}-i_1)^+, (\tilde{m}-j_1)^+, 1_{\{\tilde{n}-i_1>0\}} \cdot a\right)\Big)$$

$$\cdot \Big((a - c^{(1)}) \cdot \min(n, i_1 + i_2) + \delta \cdot z$$

$$\cdot V_{t+1}^*\left((n-i_1-i_2)^+, (\tilde{m}-j_1-j_2)^+, 1_{\{\tilde{m}-j_1-j_2>0\}}\right.$$

$$\left.\left.\cdot p_{t+h}^*\left((\tilde{n}-i_1)^+, (\tilde{m}-j_1)^+, 1_{\{\tilde{n}-i_1>0\}} \cdot a\right)\right)\Big)\right\}. \tag{16}$$

Note, (16) mirrors the beliefs for both inventory levels and the corresponding transitions. For anticipated price reactions we use p^*, cf. (10). To estimate future profits we use $z \cdot V^*$, cf. (7).

Similarly, the prices of firm 2 are given by, $t = 0, 1, ..., T-1$, $m = 1, ..., N^{(2)}$, $a \in A$,

$$\tilde{p}_{t+h}(m,a;\vec{\pi}_t,\vec{\omega}_t) = \arg\max_{p\in A} \left\{ \sum_{i_1,j_1\geq 0} P_{t+h}^{(1-h)}(i_1,j_1,a,p) \right.$$

$$\cdot \sum_{0\leq\tilde{m}\leq N^{(2)}} \pi_{t+h}(\tilde{m}) \cdot \sum_{0\leq\tilde{n}\leq N^{(1)}} \varpi_{t+h}(\tilde{n}) \cdot \sum_{i_2,j_2\geq 0} P_{t+1}^{(h)}(i_2, j_2,$$

$$a_{t+1}^{*}\left((\tilde{n}-i_1)^+,(\tilde{m}-j_1)^+,1_{\{\tilde{m}-j_1>0\}}\cdot p\right),1_{\{\tilde{m}-j_1>0\}}\cdot p)$$

$$\cdot\Big((p-c^{(2)})\cdot\min(m,j_1+j_2)+\delta\cdot z$$

$$\cdot W_{t+1+h}^{*}\left((\tilde{n}-i_1-i_2)^+,(m-j_1-j_2)^+,1_{\{\tilde{n}-i_1-i_2>0\}}\right.$$

$$\left.\left.\cdot a_{t+1}^{*}\left((\tilde{n}-i_1)^+,(\tilde{m}-j_1)^+,1_{\{\tilde{m}-j_1>0\}}\cdot p\right)\right)\Big)\right\}. \quad (17)$$

In each period, realized sales are used to update the beliefs π and ω such that the prices (16) and (17) can be computed during the sales process, i.e.:

$$\tilde{a}_0(N^{(1)},0;\vec{\pi}_0,\vec{\omega}_0)\rightarrow\vec{\pi}_h,\vec{\omega}_h\rightarrow\tilde{p}_h(N^{(2)},a_h;\vec{\pi}_h,\vec{\omega}_h)$$

$$\rightarrow\vec{\pi}_1,\vec{\omega}_1\rightarrow\tilde{a}_1(X_1^{(1)},p_1;\vec{\pi}_1,\vec{\omega}_1)\rightarrow\ldots$$

$$\ldots\tilde{a}_{T-1}(X_{T-1}^{(1)},p_{T-1};\vec{\pi}_{T-1},\vec{\omega}_{T-1})\rightarrow\vec{\pi}_{T-1+h},\vec{\omega}_{T-1+h}$$

$$\rightarrow\tilde{p}_{T-1+h}(X_{T-1+h}^{(2)},a_{T-1+h};\vec{\pi}_{T-1+h},\vec{\omega}_{T-1+h}). \quad (18)$$

Using simulations both firms' expected profits as well as their distributions can be easily approximated. Evaluating different z values makes it possible to identify the (mutual) best z value.

4.3 Numerical Example

To illustrate our approach, in this subsection, we consider a numerical example.

Example 4.1. We assume the setting of Example 3.1. Both firms use the heuristic Hidden Markov strategies, cf. (16) - (18), for different parameter values z, $0.2 \leq z \leq 1.5$.

We observe that z has an impact on the expected profits of both competing firms. In our example, the simulated average profits of both firms are maximized for $z = 0.8$. Note, the lower z is the more risk averse (or aggressive) are the pricing policies (see standard deviations σ), cf. Table 3.

Table 3: Simulated expected profits and its standard deviations of both firms for different z values, Example 4.1.

z	$EG_0^{(1)}$	$EG_0^{(2)}$	$EX_T^{(1)}$	$EX_T^{(2)}$	$\sigma(G_0^{(1)})$	$\sigma(G_0^{(2)})$
0.2	1141	1104	0.00	0.00	209	188
0.5	1679	1701	0.44	0.42	249	258
0.6	1743	1741	0.70	0.57	320	283
0.7	1742	1756	0.89	0.79	351	338
0.8	1739	1770	1.15	0.90	397	359
0.9	1732	1753	1.19	1.29	393	420
1.0	1716	1748	1.43	1.40	419	426
1.1	1686	1740	1.72	1.39	452	417
1.2	1668	1715	1.90	1.59	456	427
1.5	1647	1639	2.07	2.31	454	470

Remark 4.1. (Parallelization.)
The computation of feedback policies and particularly extensive simulation studies can become CPU-intensive. Parallelization can be used to compute results more efficiently:

(i) Feedback prices for the same point in time can run in parallel.

(ii) Simulations can be computed independent from each other.

Figure 3 illustrates simulated sales processes in the context of Example 4.1. Figure 3a illustrates price trajectories of the two competing firms. Figure 3b

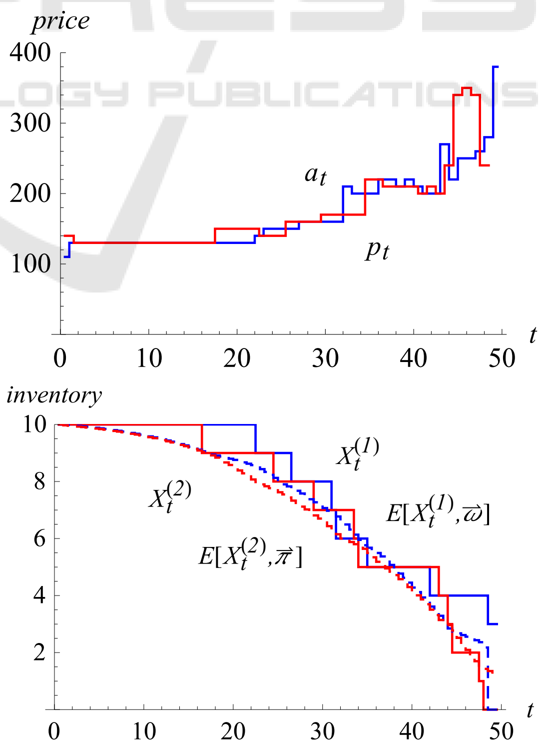

Figure 3: Simulated price paths (upper window 3a) and associated (estimated) inventory levels over time (lower window 3b), $z = 0.8$; Example 4.1.

shows the associated evolutions of the inventory levels and the (mutually) estimated inventory levels of the competitor (dashed plots).

5 UNKNOWN STRATEGIES

In this section, we want to present another heuristic approach to derive effective pricing strategies in competitive markets with limited information. We assume that the strategy of the competitor is completely unknown.

Our key idea to deal with unknown price reactions is to assume sticky prices. For firm 1, we define the following value function, $p \in A$, $n \geq 1$, $t = 0, 1, ..., T-1$, $\bar{V}_t(0, p) = 0$, $\bar{V}_T(n, p) = 0$,

$$\bar{V}_t(n,p) = \max_{a \in A} \left\{ \sum_{i_1, j_1} P_t^{(h)}(i_1, j_1, a, p) \right.$$

$$\cdot \sum_{i_2, j_2} P_{t+h}^{(1-h)}(i_2, j_2, a, p) \cdot \Big((a - c^{(1)}) \cdot \min(n, i_1 + i_2)$$

$$\left. + \delta \cdot \bar{V}_{t+1}\left((n - i_1 - i_2)^+, p \right) \Big) \right\}. \qquad (19)$$

The heuristic strategy $\bar{a}_t(n, p)$ – determined by the arg max of (19) – only depends on t, n, and p. Similarly, the corresponding pricing strategy $\bar{p}_t(m, a)$ of firm 2 is determined by the arg max of, $a \in A$, $m \geq 1$, $t = 0, 1, ..., T-1$, $\bar{W}_{t+h}(0, a) = 0$, $\bar{W}_{T+h}(m, a) = 0$,

$$\bar{W}_{t+h}(m,a) = \max_{p \in A} \left\{ \sum_{i_2, j_2} P_{t+h}^{(1-h)}(i_2, j_2, a, p) \right.$$

$$\cdot \sum_{i_1, j_1} P_{t+1}^{(h)}(i_1, j_1, a, p) \cdot \Big((p - c^{(2)}) \cdot \min(m, j_1 + j_2)$$

$$\left. + \delta \cdot \bar{W}_{t+1+h}\left((m - j_1 - j_2)^+, a \right) \Big) \right\}. \qquad (20)$$

The advantage of this approach is that the value function does not need to be computed for all competitors' prices p in advance. The value function and the associated pricing policy can be computed separately for single prices p (e.g., just when they occur). If the competitor's strategy is not known (which is often the case) it is not possible to anticipate potential price adjustments. This feedback strategy is able to react immediately if a change of the competitor's price takes place. In such an event, the value function (19) - (20) and the associated prices have to be computed for the new state.

Remark 5.1. (Oligopoly Competition.)
Note, due to the curse of dimensionality, the strategies derived in Section 3 and 4 are just applicable when the number of competitors is small. The heuristic strategy described above, however, can still be applied when the number of competitors is large! In case of K competitors, the state p in (19) just have to be replaced by $\vec{p} = (p^{(1)}, ..., p^{(K)})$, $p^{(k)} \in A$, $k = 1, ..., K$.

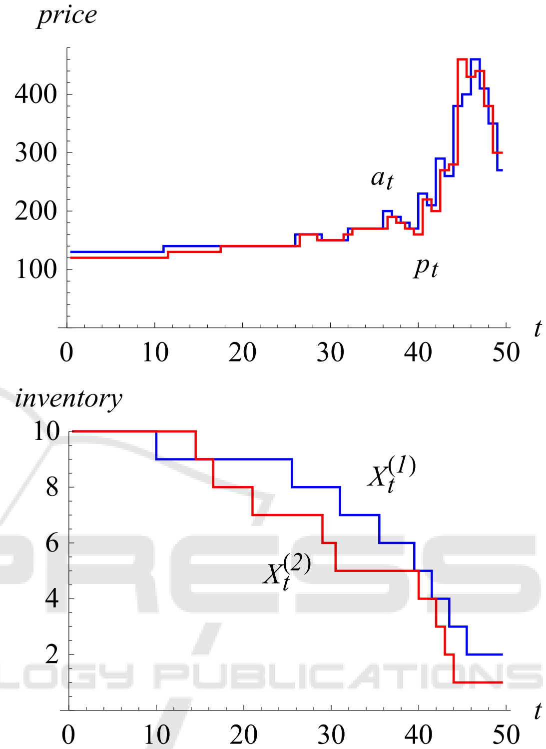

Figure 4: Simulated price paths (upper window 4a) and associated inventory levels over time (lower window 4b); setting of Example 3.1.

For the case that the competitor's strategy is unknown, Figure 4 illustrates simulated sales processes based on the heuristic, cf. (19) - (20), in the context of Example 3.1. Figure 4a illustrates price trajectories of the two competing firms. We observe that firms either raise the price or undercut the competitor's price. Figure 4b shows the corresponding inventory levels.

6 STRATEGY COMPARISON

In this section, we want to compare the outcome of our different solution strategies which take advantage of different kind of information.

If pricing strategies are allowed to use full information, i.e., the own inventory level, the competitor's inventory, and the competitor's price, then the optimal expected profits can be computed analytically, cf.

Section 3. In case the competitor's inventory level is not known, we presented an approach to compute viable strategies via a Hidden Markov Model, cf. Section 4. If the competitor's inventory is not known and his/her pricing strategy as well as his/her reaction time is not known, we proposed an efficient heuristic.

By S_{FK}, we denote the strategy derived in Section 3 (full knowledge). By S_{PK}, we denote the response strategy derived in Section 4 (partial knowledge) with $z = 0.8$. By S_{UK}, we denote the heuristic strategy, cf. Section 5, in case that the competitor's strategy is unknown. Considering the setting of Example 3.1 and Example 4.1, the expected profits of the different symmetric strategy combinations are summarized in Table 4.

Table 4: Expected profits $EG_0^{(1)}$ (of firm 1) and $EG_0^{(2)}$ (of firm 2), when firm 1 and firm 2 play different pairs of strategies: S_{FK} (both use full knowledge), S_{PK} (both use partial knowledge), S_{UK} (mutually unknown strategies), cf. Example 3.1 - 4.1.

Case	$EG_0^{(1)}$	$EG_0^{(2)}$	$EX_T^{(1)}$	$EX_T^{(2)}$	$\sigma(G_0^{(1)})$	$\sigma(G_0^{(2)})$
FK	1754	1769	1.51	1.51	467	469
PK	1739	1770	1.15	0.90	397	359
UK	1771	1768	0.78	0.47	329	312

In the three cases expected total profits, expected remaining inventory, and standard deviations of total profits have been approximated using simulations. Surprisingly, we observe that in all three scenarios both firms can expect similar profits. It turns out that as long as information structure is symmetric, a lack of information does not necessarily result in smaller expected profits.

The number of unsold items (cf. load factor), as well as the variance of profits, however, have significant differences. In case of fully observable states (S_{FK} vs. S_{FK}) the remaining inventory and the variance of profits is comparably high. Both firms can expect almost equal results. In the second case with partially observable states (S_{PK} vs. S_{PK}) we observe that the load factor of both firms is higher and the variation of profits is much smaller. Since less information is available the competition between both firms is less intense.

In case of mutual unknown strategies (S_{UK} vs. S_{UK}) we obtain a similar result. Furthermore, we can assume that the heuristic S_{UK} strategy will yield robust results when played against various other strategies. The other two strategies are optimized to play against a specific strategy. Hence, they might perform less well, when the competitor is playing a different strategy. Moreover, the efficient computation of our heuristic S_{UK} allows for fast computation times, and in turn a high price reaction frequency, which is also a competitive advantage.

7 CONCLUSION

In e-commerce, it has become easier to observe and adjust prices automatically. Consequently, there exists an increased demand for dynamic pricing. The computation of suitable pricing strategies is highly challenging as soon as strategic competitors are involved and remaining inventory levels play a major role. In this paper, we analyzed stochastic dynamic finite horizon duopoly models characterized by price responses in discrete time. We allow sales probabilities to generally depend on time as well as the competitors' prices. Further, we are able to model different reaction times.

We have considered three different types of information structure. In the first setting, we assume that the inventory levels of the competing firms are mutually observable. We show that optimal price reaction strategies – which are based on mutual price anticipations – can be derived using standard methods (e.g., backward induction). Examples are used to identify structural properties of expected profits and feedback pricing strategies. Optimal prices are balancing two effects: (i) slightly undercut the competitor's price in order to sell more items, and (ii) the use of high prices in order to promote a competitor's run-out and to act as a monopolist for the rest of the time horizon.

In the second setting, we assume that the inventory of the competitor is not observable. Based on observable prices, we compute probability distributions (beliefs) for the number of items the competitor might have left to sell. We propose a simplified Hidden Markov Model to be able to compute applicable feedback pricing strategies. Our examples show that the resulting expected profits of both firms are similar to those obtained in the model with full knowledge. The variance of profits and the average number of remaining items, however, is significantly lower.

In the third setting, we assume that the competitor's strategy is completely unknown, i.e., competitors cannot anticipate price responses. We propose an efficient decomposition approach to circumvent the curse of dimensionality and demonstrate how to compute powerful pricing strategies. We verify that – when applied by both competitors – the heuristic yields the same expected profits as in the two other settings, in which more information is available.

To this end, we have shown how to compute applicable reaction strategies for real-life scenarios with different information structures. We find that expected profits are hardly affected by less information as long as the information structure is symmetric.

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APPENDIX

Table 5: List of variables and parameters.

t	time / period
$c^{(k)}$	shipping costs of firm k, $k=1,2$
$G_t^{(k)}$	random future profits of firm k
$N_t^{(k)}$	initial number of sold items of firm k
$X_t^{(k)}$	random inventory level of firm k
δ	discount factor
h	reaction time
$P_t^{(h)}$	sales probability for $(t, t+h)$
A	set of admissible prices
V	value function of firm 1
W	value function of firm 2
a	offer price of firm 1
p	offer price of firm 2
n	inventory state of firm 1
m	inventory state of firm 2
$\pi(m)$	beliefs of firm 1
$\omega(n)$	beliefs of firm 2
a^*, p^*	strategies (full knowledge)
$\tilde{a}, \tilde{p}$	strategies (partial knowledge)
$\bar{a}, \bar{p}$	strategies (no knowledge)