Keywords: Navier-Stokes Equation

Abstract: This research discusses about stability on three dimensional incompressible Navier-Stokes equations in steady state \((0, p_0)\) and with Navier boundary condition. The analysis is performed in a region geometrically of the form box hollow. The result shows the shape of stability (or instability) depends on energy, and strengthen the slip length and viscosity. With the presence of critical viscosity, it can also be shown the stability in three-dimensional domain hold by using of normed spaces.

1 INTRODUCTION

In mathematically, the Navier-Stokes equations in three dimensions are formed by viscosity. So, the equations from \(F\) is described by the following system

\[
\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = \mu \nabla^2 v - \nabla p + \mathbf{F}(x,t)
\]

where \(t\) is the time, \(x\) is the point of \(\Omega\); \(\rho\) is the density, \(v\) is the velocity, \(\pi\) is the corresponding pressure and the positive constant \(\mu\) is the velocity coefficient. So,

\[
v \cdot \nabla v = \sum_{i=1}^{3} v_i \frac{\partial v}{\partial x_i}
\]

The basic of stability analysis depends on which is function \(f(x,t)\). In the case, the autonomous system means that the systems are not depend on the time \(t\). Therefore,

\[
\dot{x} = f(x), \quad t \geq 0, \quad x(0) = x_0
\]

The stability is defined by two Lyapunov Stability and Asymptotic Stability (Jiang F and Jiang S, 2014). The concept of Lyapunov stability is \(\dot{x}(t)\) is said to be stable if given \(\varepsilon > 0\), there exist a \(\delta = \delta(\varepsilon) > 0\) such that, for any other solution, \(y(t)\) satisfying \(|\dot{x}(t_o) - y(t_o)| < \delta\), then \(|\dot{x} - y(t)| < \varepsilon\) for \(t > t_0\), \(t_0 \in \mathbb{R}\). The Asymptotic Stability is defined by if there exist a constant \(b > 0\) such that, if \(|\dot{x}(t_o) - y(t_o)| < b\), the \(\lim_{t \to \infty} |\dot{x}(t_o) - y(t_o)| = 0\).

There are so many researches about stability analysis in Navier-Stokes equations, the nonlinear instability in inhomogeneous incompressible flow and stability and instability of gravity (Tulus, 2012).

Figure 1: \(\mathbb{R} \times (0,1) \times (0,1)\).
2 METHOD

The method used in this research is
1. Define 3 dimensional incompressible Navier-Stokes equations (Tulus et al., 2017).
2. Define the theorems of stability and instability (linear or nonlinear).
3. Analysis of the Navier-Stokes equation on incompressible flow with linear stability analysis (Tulus et al., 2018).

3 DISCUSSION

3.1 3-Dimensional Incompressible Navier-Stokes

Given the Navier-Stokes equations for incompressible flow
\[
\begin{aligned}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &+ \nabla p = 0, \\
\nabla \cdot \mathbf{v} & = 0,
\end{aligned}
\]
(1)

Where \( p \) is identified as the pressure and \( \mathbf{v} \) is the vector field of velocity (Marpaung et al., 2018).

In this research, the Navier boundary condition is defined on domain \( \Omega = \mathbb{R} \times (0,1) \times (0,1) \), then
\[
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = 0; \quad \text{on } \mathbb{R} \times (0,1) \times t \geq 0,
\]
(2)

3.2 Linearization the Equation

On this research, for analysis the stability we use the strong solution on \( (v_0 = p_0) \) (Tulus et al., 2006). Let \( \mathbf{u} \) is the strong solution for linearization the equation with steady state \( (0, p_0) \), \( \alpha \) is the slip length, the kinetic energy system is \( E = \frac{1}{2} \| \mathbf{u}(t) \|^2_{L^2(\Omega)} \). So, based on the law of energy is
\[
E(t) = E(0) - \mu \int_0^t \int_{\Omega} |\nabla \mathbf{u}|^2 \, dx \, ds + \alpha \int_0^t \int_{\Omega} |\mathbf{u}|^2 \, ds.
\]
With \( \alpha \leq 0 \). There are the following steps for linearization the equation (2).
\[
\mathbf{u} = \mathbf{v} - 0, \quad q = p - p_0.
\]
Then, \( (\mathbf{u}, q) \) satisfies the following perturbed equations:
\[
\begin{aligned}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla q - \mu \Delta \mathbf{u} = 0, \\
\nabla \mathbf{u} & = 0.
\end{aligned}
\]
Therefore, the Navier boundary condition:
\[
\begin{aligned}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla q - \mu \Delta \mathbf{u} & = 0, \\
\nabla \mathbf{u} & = 0.
\end{aligned}
\]
(3)

3.3 Stability Analysis

For facilitate the analysis process, we will introduce some notations,
\[
\Omega \equiv \mathbb{R} \times (0,1) \times (0,1), \quad L^p := L^p(\Omega), \quad H^k := W^{2,k}(\Omega),
\]
(9)

Where \( H_0^k(0,1) \times (0,1) \) and \( H^2(0,1) \times (0,1) \) can be written by \( H_0^k \) and \( H^2 \).

We will define the steady state on the equations. From the equations (4)-(8), we obtained
\[
\psi(x, y, z; t) = w(x, y, z) e^{\lambda t}, q(x, y, z; t) = p(x, y, z) e^{\lambda t}
\]
(10)

For \( \lambda > 0 \), substituted \( \lambda \) and \( w \) on the equation,
\[
\begin{aligned}
\lambda \mathbf{w} + \nabla \tilde{p} - \mu \nabla w & = 0, \quad \text{on } \Omega, \\
\nabla \mathbf{w} & = 0, \quad \text{on } \Omega.
\end{aligned}
\]
(11)

We can write the \( \mathbf{w} \) and \( \tilde{p} \) in the terminology, \( \psi, \phi, \pi : (0,1) \times (0,1) \to \mathbb{R} \) for all \( \xi \), we obtained:
\[
\psi(x, y, z) = -i \phi(z) e^{i x \xi}, w(x, y, z) = n(z) e^{i x \xi},
\]
(12)

Elimination \( \pi \) for the second equations (4) then substitution the fourth order from the ODE system for \( \psi \), then:
\[-\lambda (\xi^2 \psi - \psi^n) = \mu (\psi^4) - 2\xi^2 \psi^n + \xi^4 \psi, \quad z \in (0,1) \times (0,1), \quad z \in \mathbb{R}^2 \]

\[-\lambda = \inf_{H^1_0 \cap H^2} \frac{E(\psi)}{f(\psi)}. \quad (14)\]

In this case

\[E(\psi) = \frac{\mu}{2} \int_0^1 \int_0^1 [(\psi'')^2 + 2\xi^2 (\psi''^2 + \xi^4 \psi^2)] - \frac{k_1}{2} (\psi'(1,z))^2 - \frac{k_0}{2} (\psi'(0,z))^2 - \frac{k_2}{2} (\psi'(y,1))^2 - \frac{k_2}{2} (\psi'(y,0))^2\]

and

\[f(\psi) = \frac{1}{2} \int_0^1 [(\xi^2 \psi^2 + (\psi')^2)] \]

are well defined on the space \(H^1_0 \cap H^2\).

For the positive \(\lambda, \lambda(=\lambda(\xi^2)), \)

\[\mu \int \int_0^1 [(\psi')^2 - \frac{k_1}{2} (\psi'(1,z))^2 - \frac{k_0}{2} (\psi'(0,z))^2 - \frac{k_2}{2} (\psi'(y,0))^2 \]

is negative for the viscosity \(\mu\). Therefore \(E(\psi)\) is negative on \(\xi\). So, the critical viscosity is defined by

\[\mu_c = \sup_{\psi \in \{\psi \in \mathcal{F} \cup \mathcal{G} \} ; \psi' \neq 0} \frac{k_1 (\psi'(1,z))^2 + k_0 (\psi'(0,z))^2}{\int \int_0^1 (\psi')^2 + k_2 (\psi'(y,1))^2 + k_2 (\psi'(y,0))^2} \quad (16)\]

\[y = \{ \psi \in H^1_0 \cap H^2 ; \int_0^1 (\psi'(0))^2 = 1 \} \]

Then, for every \(f \in H^1_0 \cap H^2\),

\[\|f''\|^2_{L^2} \geq \|f''\|^2_{L^2}, \quad (19)\]

with applied the Poincare inequality and \(f \in H^1_0\), we obtained the proof.

So that, for every \(\psi \in \mathcal{Y}\),

\[|Z(\psi)| = \frac{1}{2} \int \int_0^1 \left[ \left( (k_1 + k_0) y - k_0 \right) (\psi')^2 \right] \right| \quad (17)\]

\[\int \int_0^1 \left[ \frac{1}{2} \left[ \left( (k_1 + k_0) y - k_0 \right) (\psi')^2 \right] \right] = \left[ \int \int_0^1 \left( \psi''^2 \right) \right] dydz \]

\[\leq \frac{1}{2} \int \int_0^1 \left( 1 + \psi''^2 \right) \right] dydz = C_3\]

for every positive constant \(C_1, C_2, C_3\), depending on \(k_1, k_2, k_3\). This statement proof \(\sup_{\psi \in \mathcal{Y}} Z(\psi)\) is exist and bounded. For \(\mu_c := \sup_{\psi \in \mathcal{Y}} Z(\psi)\), we can distinguish become two propositions.

**Propositions.** For \(\mu_c \in (\mu \rightarrow +\infty)\). If \(\max_{v_0, v_1, v_2, v_3} > 0 \) then \(\mu_c > 0\)

**Proof.** Let \(v_0, v_1, v_2, v_3\) are non-positive. Then we have

\[\mu_c = \sup_{\psi \in \mathcal{Y}} Z(\psi) \quad (18)\]

\[\Psi(x) = \left\{ \begin{array}{ll}
0, & x \in \left[ 0, \frac{1}{4} \right); \\
\alpha \exp \left( \frac{1}{(x - \frac{1}{2})^2} \right), & x \in \left[ \frac{1}{4}, \frac{3}{4} \right); \\
0, & x \in \left[ \frac{3}{4}, 1 \right);
\end{array} \right. \quad (21)\]

on \(y\), moreover \(\Psi \in C_0^{\infty}([0,1]) \) and \(\Psi'_1(1,1) = \Psi'_0(0,0) = 0 \) \(0 \leq z \leq 1\), which implies \(Z(\Psi_1) = 0\) implies that \(\mu_c = 0\).

**4 CONCLUSIONS**

On this research the linear stability analysis is obtained from the instability depends on the strength viscosity and slip length. There are three theorems,
which are linear instability theorem, nonlinear instability theorem, and linear stability asymptotic. Moreover, there are the critical viscosity which distinguish the linear stability and nonlinear instability.

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REFERENCES


