A Single-source Weber Problem with Continuous Piecewise Fixed Cost

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Abstract: This paper analyzes the location of a distribution center in an urban area using a single-source Weber problem with continuous piecewise fixed cost to find a global optimal location. The fixed cost is characterized by a Kriging interpolation method. To make the fixed cost tractable, we approximate this interpolation with a continuous piecewise function that is convex in each piece, using Delaunay triangulation. We present a decomposition formulation, a decomposition conic formulation and a conic logarithmic disaggregated convex combination model to optimally solve the single-source Weber problem with continuous piecewise fixed cost. Although our continuous approach does not guarantee the global optimal feasible location, it allows us to delimit a zone where we can intensify the search of feasible points. For instances we tested, computational results show that our continuous approach found better locations than the discrete approach in 23.25% of the instances and that the decomposition formulation is the best one, in terms of CPU time.

1 INTRODUCTION

The location of a distribution center (DC) in an urban area, considering the transportation and installation costs, can be treated as an uncapacitated facility location problem (UFLP) or as a Weber problem with fixed cost. It is known that the solution of the UFLP is feasible but not necessarily optimal, due to the use of an incomplete set of possible locations. On the other hand, the Weber problem with fixed cost gives an optimal location probably not feasible.

This paper analyzes the installation of a single DC in an urban area, using the single-source Weber problem with fixed cost to find an optimal location that allows us to delimit a zone around the optimal location previously found, but smaller than the original one. This way, we can focus the search of feasible points, obtaining a more reliable and complete set of possible locations such that, when an UFLP is applied, we find the *optimal feasible location*.

To the best of our knowledge, few papers deal with the inclusion of the fixed costs into the Weber problem. Fixed costs have been considered as a constant cost for all the plane (Brimberg et al., 2004), as zone dependent with a constant cost in a specific convex polygon (Brimberg and Salhi, 2005),(Hosseininezhad et al., 2015), or as a proportion between the fixed cost of two zones and their relative distance, (Luis et al., 2015). To consider that a plane can be partitioned in a finite set of convex polygons, each one with constant fixed costs, is considered a good first approximation to characterize the variating nature of this cost. In this paper we propose that the fixed cost on each convex polygon is a function of its vertices, allowing us to better model the fixed costs in an urban area.

The objective of this paper is to find the best formulation to locate a single DC in an urban area, where the fixed costs depend on the location in a continuous way. The fixed cost function is characterized by a Kriging interpolation method using a set of nodes where the cost is known. To make the formulation tractable, we approximate the interpolation with a continuous piecewise function that is convex in each piece. This is constructed through a convex combination of the vertices of a mesh created with a Delaunay triangulation. The sinlge-source Weber problem with continuous piecewise fixed cost is formulated as an MINLP problem. We take advantage of the problem's structure to propose three solution approaches. The first approach considers a conic reformulation of the single-source Weber problem with continuous piecewise fixed cost using a *logarithmic disaggregated convex combination model*. The second consists of a decomposition method, where we solve a non-linear convex problem for each Delaunay triangle, and using complete enumeration we determine the optimal solution. The last one, consider a conic reformulation for each sub-problem of the sub-

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sequent decomposition formulation. Each approach was implemented in a series of experiments to compare their performance in CPU time.

The main contributions of this paper are: (*i*) a new way to represent the fixed costs in an urban area and (*ii*) to identify the best solution approach for the single-source Weber problem with continuous piecewise fixed cost.

The paper is organized as follows: Section 2 presents our related work. Section 3 presents a single-source Weber problem with continuous piecewise fixed cost. Section 4 presents three different approaches to solve the problem defined in the previous section. Section 5 presents some experimental results for all the different approaches. Our conclusions and highlights are presented in section 6.

2 RELATED WORK

The continuous location problem for a single-source or single-source Weber problem, described in Weber and Friedrich (1929), has been extensively studied. To find the solution there are different approaches: a one-point iterative method better known as the Weiszfeld algorithm (Weiszfeld and Plastria, 2009), a unified cutting plane method (Plastria, 1987), a dual method (Planchart and Hurter, 1975), a primal-dual algorithm involving mixed norms (Michelot and Lefebvre, 1987), or a primal-dual potential reduction algorithms with the problem formulated in conic form (Xue and Ye, 1997). A comprehensive review of the Weber problem can be found in Drezner et al. (2002).

The multi-source Weber problem, or location-allocation problem, is an NP-hard problem (Megiddo and Supowit, 1984). There are few heuristics that solve it to optimality but they work only in small problems (Cooper, 1972), (Sherali et al., 2002), (Chen et al., 1998). For the heuristic approach to solve the problem to near optimum, there are more publications: Cooper (1964) explored different algorithms with computational experiments. The alternating location-allocation heuristic is used by Cooper (1972). The method used by Bongartz et al. (1994) relaxes the binary constraints on the allocations, and solves both location and allocation simultaneously. An approach based on a nonlinear second-order cone program reformulation is found in Chen et al. (2011). The approach to use the discrete models in solving the continuous location-allocation problems is widely used by Hansen et al. (1998), Brimberg et al. (2014), and others. For this, a survey in the p-median problem with the aim in procedures based on metaheuristics rules (Mladenovic et al., 2007) is useful. For a survey on the multi-source Weber problem there is Brimberg et al. (2000) and Brimberg et al. (2008).

The inclusion of the fixed cost to the Weber problem has little reviews, there are four papers to the best of our knowledge. First it is included as a constant cost for all plane in Brimberg et al. (2004). Later, in Brimberg and Salhi (2005), it was extended to a zone-dependent fixed cost, where zones are non-overlapping convex polygons with a constant fixed cost for each zone. In Hosseininezhad et al. (2015) is developed a metaheuristic Cross Entropy for a continuous location problem, with an fixed cost depending on the zone and on the facility to install. And Luis et al. (2015), proposed a multi-source Weber problem with capacity and zone-dependent fixed cost using the second-order Voronoi regions.

In general, data gathering is expensive in terms of monetary and time-consuming costs (Helbich et al., 2013). Therefore, there is a necessity to estimate the land values in unvisited locations, as geostatistical methods Luo (2004), Cellmer et al. (2014). Here, we use a Kriging method of interpolation (Oliver and Webster, 1990). This method was recommended over other interpolation approaches in Anselin and Le Gallo (2006) and Fernández-Avilés et al. (2012), in an air quality and pollution studies, respectively. The possibilities and limitations of geostatistical methods to approximate the land values are discussed in Cellmer (2014). A comparison between Kriging methods for the real estate market is discussed in Kuntz and Helbich (2014). The Kriging interpolation is used to find the value of land for different cities by Liang and Yi (2012), Hu et al. (2015), Larraz and Poblacin (2013).

In summary, there are few previous works on single-source and multi-source Weber problem that include a second order cone formulation and, to the best of our knowledge, only one paper presents a solution approach. The few papers that include fixed costs make a simplistic representation of them that do not reflect their variations in an urban area. Unlike them, we make a more realistic representation of the fixed costs, considering different possible approaches for the Weber problem with fixed costs.

3 MODEL FORMULATION

3.1 Single-source Weber Problem with Continuous Piecewise Fixed Costs

The generalized single-source Weber problem with fixed costs considers the localization of a single source with coordinates $(\bar{x},\bar{y}) \in \mathbb{R}^2$. This source must supply a set J of customers with known coordinates, (xc_j, yc_j) for every $j \in J$. Let $f(\bar{x},\bar{y})$ be the fixed cost incurred when the source is installed in $(\bar{x},\bar{y})$. Let w_j be the the expected demand weighted by the transportation ratios, for all $j \in J$. The problem is to determine the optimal location for the single source such that the transportation and the fixed costs are minimized. The generalized single-source Weber problem with fixed costs can be expressed as follows:

$$\min_{(\bar{x},\bar{y})} \quad \sum_{j\in J} w_j \sqrt{(\bar{x}-xc_j)^2 + (\bar{y}-yc_j)^2} + f(\bar{x},\bar{y}) \quad (1)$$

$$\text{s.t.} \quad (\bar{x},\bar{y}) \in \mathbb{R}^2 \quad (2)$$

We consider a convenient set I of nodes with known information of their fixed costs, C_i, and their coordinates, (xi_i, yi_i), for every $i \in I$. In our paper, the way to address the fixed costs is by applying a Kriging interpolation method and defining a continuous function for the cost in every point of the convex hull of I. This cost function is not simple and could not be convex. To make the continuous fixed cost function tractable, we are going to approximate the Kriging interpolation with a piecewise function that is convex in each piece. For this, we partition the convex hull of I through a polyhedral mesh and defined the continuous piecewise fixed cost function as the convex combination of the vertices of the mesh. To the best of our knowledge, it is better to use the smallest subset of information nodes possible with empty interior to create the polyhedra, i.e, using Delaunay triangulation.

We applied a Delaunay triangulation over the set I obtaining a set K of triangles; each triangle k-th will be denoted as P_k, with $P_k = \{(x,y) \in \mathbb{R}^2 | (x,y) = \sum_{l=1}^{3} \lambda^{k_l}(x_{k_l}, y_{k_l}), \sum_{l=1}^{3} \lambda^{k_l} = 1, \forall \lambda^{k_l} : \lambda^{k_l} \geq 0\}$, where (x_{k_1}, y_{k_1}), (x_{k_2}, y_{k_2}) and (x_{k_3}, y_{k_3}) are the vertices of the k-th triangle and $C_{k_1}, C_{k_2}, C_{k_3}$ their fixed cost. We have λ^{k_l} as the convex combination vector for the vertices of the triangle $k \in K$ and $l = 1,2,3$ the vertices of the triangle. The set of all possible locations, $\bigcup_{k\in K} P_k$, can be non-convex if we clean the areas where we can not install, as a lake or a strictly residential area.

Given the above, the facility's location can be expressed as $(\bar{x},\bar{y}) = \sum_{k\in K} \sum_{l=1}^{3} \lambda^{k_l}(x_{k_l}, y_{k_l})$ and its fixed cost as a convex combination of the vertices of the triangles's costs, $\sum_{k\in K} \sum_{l=1}^{3} \mathbf{C}_{k_l}^T \lambda^{k_l}$. Let Z_k be a binary variable that forces the installation to be in only one triangle, being 1 if it is installed in the k-th triangle and 0 if it is not.

We can formulate the single-source Weber problem with continuous piecewise fixed cost as follows:

Problem $(P0)$*:*

$$\min_{\mathbf{Z},\lambda} \sum_{k\in K} \left(\sum_{l=1}^{3} \mathbf{C}_{k_l} \lambda^{k_l} + Z_k \sum_{j\in J} w_j \sqrt{(\sum_{k\in K} \sum_{l=1}^{3} \lambda^{k_l} x_{k_l} - xc_j)^2 + (\sum_{k\in K} \sum_{l=1}^{3} \lambda^{k_l} y_{k_l} - yc_j)^2} \right) \quad (3)$$

$$\text{s.t.} \quad \sum_{l=1}^{3} \lambda^{k_l} = Z_k, \qquad \forall k \in K \quad (4)$$

$$\sum_{k\in K} Z_k = 1 \quad (5)$$

$$\lambda^{k_l} \geq 0, \qquad \forall l \in \{1,2,3\}, k \in K \quad (6)$$

$$Z_k \in \{0,1\}, \quad \forall k \in K \quad (7)$$

In what follows, we present different ways to solve the problem $(P0)$.

4 SOLUTION APPROACH

We consider three distinct solution approaches for $(P0)$. For the first approach, we use a monolithic reformulation of $(P0)$. The second approach considers a decomposition of $(P0)$ by fixing the variable $\mathbf{Z}$ and solving the sub-problem generated; we evaluated all the possible values of $\mathbf{Z}$. The last approach considers a conic reformulation of the previous sub-problems.

4.1 Conic Logarithmic Disaggregated Convex Combination Model

Now, we reformulate $(P0)$ in two steps. First, we formulate the problem as a Conic Quadratic Non Linear problem (CQNLP). Afterwards, using the *logarithmic disaggregated convex combination model* (Vielma et al., 2010), we efficiently solve the continuous piecewise fixed cost function.

Next, we formulate the problem $(P0)$ as a CQNLP in order to eliminate the square root terms. First we introduce one set of nonnegative continuous variables, d_j, to represent the square root term in:

$$d_j = \sqrt{(\sum_{l=1}^{3} x_{k_l} \lambda^{k_l} - xc_j)^2 + (\sum_{l=1}^{3} y_{k_l} \lambda^{k_l} - yc_j)^2}, \forall j \in J \quad (8)$$

$$d_j \geq 0, \forall j \in J \quad (9)$$

For simplicity, we can add two more sets of auxiliary variables, v_j and r_j, leaving (8) as:

$$d_j^2 = z_j^2 + w_j^2, \qquad \forall j \in J \tag{10}$$

$$v_j = \sum_{l=1}^{3} x_{k_l}\lambda^{k_l} - xc_j, \qquad \forall j \in J \tag{11}$$

$$r_j = \sum_{l=1}^{3} y_{k_l}\lambda^{k_l} - yc_j, \qquad \forall j \in J \tag{12}$$

Because the nonnegative variables d_j are introduced in the objective function of $(P0)$ with positive coefficients, and this problem is a minimization problem, the equation can be further relaxed as the following inequalities:

$$d_j^2 \geq v_i^2 + r_i^2, \qquad \forall j \in J \tag{13}$$

Note that the constraints (9) and (13) define second-order cone constraints. The problem $(P0)$ can be expressed as the following conic problem:
Problem $(CP0)$*:*

$$\min_{\mathbf{Z},\lambda,\mathbf{d},\mathbf{v},\mathbf{r}} \sum_{k\in K}(Z_k \sum_{j\in J} w_j d_j + \sum_{l=1}^{3}\mathbf{C}_{k_l}\lambda^{k_l}) \tag{14}$$

$$\text{s.t. } (4),(5),(6),(7),(9),(11),(12),(13)$$

The *logarithmic disaggregated convex combination model* consists in replacing the piecewise function $f(\bar{x},\bar{y})$ for its epigraph $epi(f)$ and setting the coordinate $(\bar{x},\bar{y})$ to be contained by one and only one of the domains of f. For a minimization, solving the function f is equivalent to solving $epi(f)$. To construct a model with the least number of binary variables and constraints, we identify each triangle with a binary vector in $\{0,1\}^{\lceil log_2|K|\rceil}$ through an injective function $B: K \to \{0,1\}^{\lceil log_2|K|\rceil}$ and use $\lceil log_2|K|\rceil$ binary variables, $\mathbf{m} \in \{0,1\}^{\lceil log_2|K|\rceil}$, to ensure that the coordinates are in only one triangle. Let Q be $epi(f)$.

Using the *logarithmic disaggregated convex combination model* and a second order cone formulation to reformulate $(P0)$, leaves the following:
Problem $(DlogCP0)$*:*

$$\min_{\lambda,\mathbf{m},\mathbf{d},Q,\mathbf{v},\mathbf{r}} \sum_{j\in J} w_j d_j + Q \tag{15}$$

$$\text{s.t. } \sum_{k\in K}\sum_{l=1}^{3}\mathbf{C}_{k_l}\lambda^{k_l} \leq Q \tag{16}$$

$$\sum_{k\in K}\sum_{l=1}^{3}\lambda^{k_l} = 1 \tag{17}$$

$$\sum_{k\in K^+(B,t)}\sum_{l=1}^{3}\lambda^{k_l} \leq m_t, \qquad \forall t \in T(K) \tag{18}$$

$$\sum_{k\in K^0(B,t)}\sum_{l=1}^{3}\lambda^{k_l} \leq (1-m_t), \forall t \in T(K) \tag{19}$$

$$\lambda^{k_l} \geq 0 \qquad \forall l \in 1,2,3, k \in K \tag{20}$$

$$m_t \in \{0,1\} \qquad \forall t \in T(K) \tag{21}$$

$$(9),(11),(12),(13)$$

where $B: K \to \{0,1\}^{\lceil log_2|K|\rceil}$ is any injective function, $K^+(B,t) = \{k \in K : B(k)_t = 1\}$, $K^0(B,t) = \{k \in K : B(k)_t = 0\}$ and $T(K) = \{1,\ldots,\lceil log_2|K|\rceil\}$. This problem is a mixed integer conic quadratic nonlinear problem with a linear objective function and can be solved by solvers like GUROBI, CPLEX or MOSEK.

4.2 Decomposition Formulation

From the problem $(P0)$, we can observe that the variables λ and $\mathbf{Z}$ are related in only one constraint. And, fixing the variable $\mathbf{Z}$, the problem is separable in $|K|$ sub-problems where we force the localization of the DC to be in the k-th Delaunay triangle, i.e., forcing $Z_k = 1$ and $Z_{k'} = 0$ for all $k' \in K \setminus k$. Then the k-th sub-problem can be written as:
Sub-Problem $(SP0(k))$*:*

$$\min_{\lambda^k} \quad \sum_{j\in J} w_j\sqrt{(\bar{x}-xc_j)^2+(\bar{y}-yc_j)^2} + \sum_{l=1}^{3}\mathbf{C}_{k_l}\lambda^{k_l} \tag{22}$$

$$\text{s.t.} \quad \sum_{l=1}^{3}\lambda^{k_l} = 1 \tag{23}$$

$$\lambda^{k_l} \geq 0 \quad , \forall l \in \{1,2,3\} \tag{24}$$

This sub-problem $(SP0(k))$ is a convex nonlinear problem with linear constraint and can be efficiently solved by MINOS or IPOPT solvers.

Let λ^{k^*} be the optimal solution of the problem $(SP0(k))$; $FO_{(SP0(k))}(\lambda^{k^*})$ be the optimal cost of the objective function in the problem $(SP0(k))$, and let $(\bar{\lambda},\bar{\mathbf{Z}})$ be the optimal solution of the problem $(P0)$. The optimal solution for the $(P0)$ problem is the best solution for all of the sub-problems $(SP0(k))$, i.e. $\bar{\lambda} = \lambda^{k_\dagger}$, where $k_\dagger = \text{argmin}_{k\in K}\{FO_{(SP0(k))}(\lambda^{k^*})\}$. For $\bar{\mathbf{Z}}$, the value of $\bar{Z}_k = 1$ for $k = k_\dagger$ and $\bar{Z}_k = 0$ for every other k.

4.3 Decomposition Conic Formulation

The squared root term in the objective function of problem $(SP0(k))$ can give rise to difficulties in the optimization procedure. Following the logic exposed for the first approach, we reformulate $(SP0(k))$ as a CQNLP, leaving the following conic problem:

Sub-Problem $(SCP0(k))$*:*

$$\min_{\lambda^k, \mathbf{d}, \mathbf{v}, \mathbf{r}} \quad \sum_{j \in J} w_j d_j + \sum_{l=1}^{3} \mathbf{C}_{k_l} \lambda^{k_l} \tag{25}$$
$$\text{s.t. } (23), (24), (9), (11), (12), (13)$$

The problem $(SCP0(k))$ can be trivially shown to be equivalent to $(SP0(k))$, but it has now conic and nonlinear constraints with a more simple linear objective function. The optimal solution for $(P0)$ is the best solution for all the sub-problems $(SCP0(k))$, equivalently to the decomposition formulation.

The advantage of the CQNLP formulation is that it can be solved directly using standard optimization software packages such as CPLEX, GUROBI or MOSEK.

5 COMPUTATIONAL STUDY

In this section, we present our numerical study and its results. The main objectives of this computational study is to show which solution approach has the best performance in terms of CPU time, and to compare them to an UFLP. To characterize the different approaches, we carried out 400 instances that we denote *test set*. We also corroborate the installation of a single DC in every instance with the UFLP.

All the problems were programmed using AMPL. To solve the decomposition formulation we use the solver MINOS. For $(DlogCP0)$ and the decomposition conic formulation we solve it through CPLEX solver. The Kriging interpolation method and the Delaunay triangulation were made in MATLAB. The *test set* were run on a PC with AMD FX 4,00 GHz processor and 12 GB RAM, and the UFLP were run on a PC with Intel i3 2,10 Ghz and 4 GB RAM.

5.1 Test Set

In order to determine which one has the best performance in CPU time, we generated 100 experiments. In each experiment, we fixed the number of customer nodes and used 4 refinements of the triangulation. Therefore, we have 400 instances. For simplicity, we considered $w_j = 1$, for any $j \in J$.

Each experiment has the same initial set of 100 information nodes, generated randomly. For a better piecewise convex approximation of the continuous fixed cost function, we proposed the following refinement of the mesh. We consider the Delaunay triangulation of the initial set of information nodes as the first refinement, shown in figure 1. The second refinement is generated by creating additional information nodes where their location is at the center of the edge of every triangle and their fixed cost is determined by the Kriging interpolation. Then the Delaunay triangulation is used over the original set I plus the additional information nodes. The third and fourth refinements are applied over the second and third triangulation, respectively. In figure 2 the fourth refinement is shown.

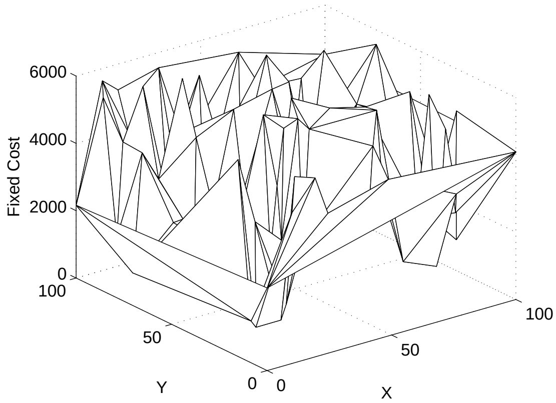

Figure 1: First refinement.

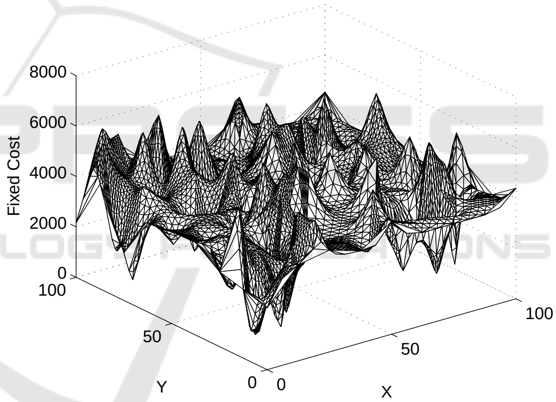

Figure 2: Fourth refinement.

We modified the number of customer nodes from 100 to 1000 customers, i.e., the first 10 experiments have 100 customer nodes, the next 10 experiments have 200 customer nodes, and so on. Each customer location is obtained making random locations, i.e., where $(xc_j, yc_j) \in ([0,100],[0,100])$.

Figure 3 shows the performance profile based on the *performance ratio* of the CPU time for each model (Dolan and Moré, 2002). Considering that t_{pm} is the CPU time for solving the instance p by the model m, we have the *performance ratio*:

$$r_{pm} = \frac{t_{pm}}{\min\{t_{pm} : m \in M\}},$$

where $M = \{DlogCP0, \quad min_{k \in K}\{(SP0(k))\}, min_{k \in K}\{(SCP0(k))\}\}$.

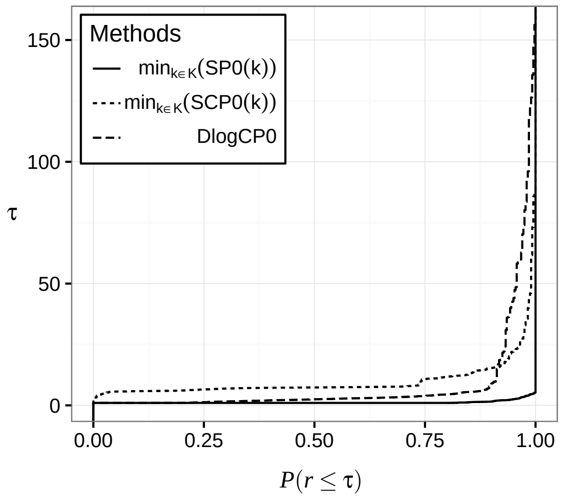

Figure 3: Performance Profile.

We observe in figure 3 that the best model performance is the decomposition formulation, i.e., $min_{k\in K}\{(SP0(k))\}$, because in 80% of the instances has the lowest time, overcome by $(DlogCP0)$, in less than 20% of the instance. The decomposition formulation has the best performance with the greater efficiency, solving all the instances with a $\mathbf{r} \preceq 5$.

There is a pattern in every refinement where $(DlogCP0)$ has the best performance in the instances with a small set of customers nodes, and get outperformed by the decomposition formulation in the rest of the instances. This is shown in table 1, where it shows that the average speedup in the CPU time of the decomposition formulation over $(DlogCP0)$ is greater than $1x$ for all the refinements in the instances with $|J| = 100$. Considering the second and third refinement, $(DlogCP0)$ is better, in average, for the instances with $|J| \leq 200$. For the fourth refinement, $(DlogCP0)$ is better in instances with $|J| \leq 300$ and with an average speedup of over $4x$ when $|J| = 100$.

Table 1: Average Speedup in CPU time of $min_{k\in K}\{(SP0(k))\}$ over $(DlogCP0)$.

	Refinement			
$\lvert J\rvert$	First	Second	Third	Fourth
100	1.556x	2.035x	2.808x	4.125x
200	0.574x	1.185x	1.548x	1.944x
300	0.382x	0.865x	0.840x	1.149x
400	0.320x	0.665x	0.697x	0.494x
500	0.258x	0.528x	0.470x	0.308x
600	0.301x	0.530x	0.410x	0.330x
700	0.226x	0.447x	0.416x	0.099x
800	0.214x	0.387x	0.385x	0.036x
900	0.184x	0.373x	0.367x	0.020x
1000	0.172x	0.309x	0.312x	0.012x

We obtain an average speedup of 7.98x and 7.72x for the decomposition formulation over the decomposition conic formulation and $(DlogCP0)$, respectively.

Our numerical results show that the performance from the conic formulations ($(DlogCP0)$ and the decomposition conic formulation) are sensible to the size of the customer set. This is because the conic formulations create $|J|$ cones and $3|J|$ new variables, so the problem grows faster than the number of customers. For this reason, even that $(DlogCP0)$ can better handle a big set of information nodes, this only is seen with a small set of customers.

The performance of the decomposition formulation, shown in figure 3, is the most stable of the performances of the three solution approaches, i.e., with less difference in the extremes values of its *performance ratio*. This indicates that if the decomposition formulation does not have the best performance in an instance, its CPU time is closer to the better one.

The average improvement in the objective function using the different refinements, compared with the first refinement, are: 0.08% for the second, 0.74% for the third, and 1.29% for the fourth refinement.

We can observe in table 1 that in instances with small number of customers is better to use $(DlogCP0)$, considering that can have a speedup over $4x$ against the decomposition formulation, but it is when the CPU times are lower. For example, in all the instances with $|J| = 100$, although we have a better average of CPU time with $(DlogCP0)$, the worst CPU time for the decomposition formulation does not get over 250 seconds. Considering that the decomposition formulation has a more stable performance with the better overall average in CPU time, and because this solves a strategic decision, we recommend to model the single source Weber problem with fixed cost with the decomposition formulation.

5.2 Discrete Model: Uncapacitated Facility Location Problem

The following experiments where made using the instances previously described in the *test set*, considering the set of information nodes without the refinements. We consider the information nodes as the discrete set of possible locations, modelled by an UFLP.

In table 2 are the average and maximum percentage of the improvement in lowering the value of the objective function of the single-source Weber problem with continuous piecewise fixed cost over the UFLP, and the number of cases where this happened.

Table 2 shows, for the fourth refinement an average improvement of 1.42%. From the total of experiments solved with the fourth refinement, the 67% of

Table 2: Percentage of improvement for the continuous model over the UFLP.

	Refinement			
	First	Second	Third	Fourth
Average	0.13%	0.21%	0.86%	1.42%
Max	1.62%	5.80%	16.83%	18.58%
N° Cases	27	8	25	33

the instances have the same result as the UFLP. But in the 33% where they are different, the average improvement is of 4.297%.

The better solutions found in 23.25% of the instances with the single-source Weber problem over the UFLP is because the UFLP only consider the information nodes as possible locations and not always is consider the global optimum in that set. With the inclusion of more information nodes, i.e. closer to reality, the average savings and the number of better cases tend to grow.

We also observed that in all the instances only one facility is installed. This is in accordance to say that, in an urban area, the fixed cost of an extra DC tends to be bigger than the savings in transportation.

6 CONCLUSIONS

This paper analyses the problem of locating a single DC in an urban area considering the fixed and transportation costs using a single-source Weber problem with continuous piecewise fixed cost. The fixed cost is characterized by a Kriging interpolation method. Using a Delaunay triangulation, we make the fixed cost function convex and tractable. We propose and evaluate three solution approaches to optimally solve the single-source Weber problem with continuous piecewise fixed cost: (*i*) decomposition formulation, (*ii*) decomposition conic formulation, and (*iii*) logarithmic disaggregated convex combination model with a conic formulation.

In the instances we tested, in 23.25% of the time, a better solution is found with the single-source Weber problem with continuous piecewise fixed cost than with the UFLP. We observe two possible explanations:(*i*) the set I is complete, and therefore, the solution of the Weber problem is unfeasible, and (*ii*) the set I is incomplete and requires a more thorough search of feasible locations over the urban area, i.e., the UFLP could have found a sub-optimal solution. To ensure a complete set I in an urban area is expensive and almost impossible. It is possible to improve with the continuous approach, the set of feasible locations focusing in a reduced section of the urban area around the optimal location found, where it is more probable to find the *optimal feasible location*, thus reducing the search effort of feasible points. With this, we can apply an UFLP over the new set I.

The computational results show that the best approach for the single-source Weber problem with continuous piecewise fixed cost, in terms of average CPU time, is the decomposition method, with an average speedup of 7.98x and 7.72x over the decomposition conic method and the conic monolithic reformulation, respectively. The first approach has the best performance and can better handle a bigger set of information nodes only with a small number of customers, but this happens in the instances where the difference between the CPU times are smaller.

There are a number of questions and issues left for future research, such as: (*i*) to apply some Weizfield-like algorithm to improve the performance of the decomposition formulation, given that that was the best one, (*ii*) to use the formulation of a stochastic model of the single-source Weber problem with fixed cost using the variance of the Kriging interpolation method, (*iii*) to consider a location-routing problem, and (*iv*) the extension to a multi-source Weber problem with continuous dependent fixed cost considering the best solution approach we obtain.

REFERENCES

Anselin, L. and Le Gallo, J. (2006). Interpolation of air quality measures in hedonic house price models: spatial aspects. *Spatial Economic Analysis*, 1(1):31–52.

Bongartz, I., Calamai, P. H., and Conn, A. R. (1994). A projection method forl p norm location-allocation problems. *Mathematical Programming*, 66(1):283–312.

Brimberg, J., Drezner, Z., Mladenovic, N., and Salhi, S. (2014). A new local search for continuous location problems. *European Journal of Operational Research*, 232(2):256–265.

Brimberg, J., Hansen, P., Mladenovic, N., and Salhi, S. (2008). Survey of solution methods for the continuous location-allocation problem. *International Journal of Operations Research*, 5(1):1–12.

Brimberg, J., Hansen, P., Mladenovic, N., and Taillard, E. D. (2000). Improvements and comparison of heuristics for solving the uncapacitated multisource weber problem. *Operations Research*, 48(3):444–460.

Brimberg, J., Mladenovic, N., and Salhi, S. (2004). The multi-source weber problem with constant opening cost. *Journal of the Operational Research Society*, 55(6):640–646.

Brimberg, J. and Salhi, S. (2005). A continuous location-allocation problem with zone-dependent fixed cost. *Annals of Operations Research*, 136(1):99–115.

Cellmer, R. (2014). The possibilities and limitations of geostatistical methods in real estate market analyses. *Real Estate Management and Valuation*, 22(3):54–62.

Cellmer, R., Belej, M., Zrobek, S., and Šubic-Kovač, M. (2014). Urban land value maps a methodological approach. *Geodetski vestnik*, 58(3):535–551.

Chen, J.-S., Pan, S., and Ko, C.-H. (2011). A continuation approach for the capacitated multi-facility weber problem based on nonlinear socp reformulation. *Journal of Global Optimization*, 50(4):713–728.

Chen, P.-C., Hansen, P., Jaumard, B., and Tuy, H. (1998). Solution of the multisource weber and conditional weber problems by d.-c. programming. *Operations Research*, 46(4):548–562.

Cooper, L. (1964). Heuristic methods fot location-allocation problems. *SIAM Review*, 6(1):37–53.

Cooper, L. (1972). The transportation-location problem. *Operations Research*, 20(1):94–108.

Dolan, E. D. and Moré, J. J. (2002). Benchmarking optimization software with performance profiles. *Mathematical programming*, 91(2):201–213.

Drezner, Z., Klamroth, K., Schöbel, A., and Wesolowsky, G. O. (2002). 1 the weber problem.

Fernández-Avilés, G., Minguez, R., and Montero, J.-M. (2012). Geostatistical air pollution indexes in spatial hedonic models: the case of madrid, spain. *Journal of Real Estate Research*.

Hansen, P., Mladenovic, N., and Taillard, E. (1998). Heuristic solution of the multisource weber problem as a pmedian problem. *Operations Research Letters*, 22(2-3):55–62.

Helbich, M., Jochem, A., Mücke, W., and Höfle, B. (2013). Boosting the predictive accuracy of urban hedonic house price models through airborne laser scanning. *Computers, environment and urban systems*, 39:81–92.

Hosseininezhad, S. J., Salhi, S., and Jabalameli, M. S. (2015). A cross entropy-based heuristic for the capacitated multi-source weber problem with facility fixed cost. *Computers & Industrial Engineering*, 83:151–158.

Hu, S., Tong, L., Frazier, A. E., and Liu, Y. (2015). Urban boundary extraction and sprawl analysis using landsat images: A case study in wuhan, china. *Habitat International*, 47:183–195.

Kuntz, M. and Helbich, M. (2014). Geostatistical mapping of real estate prices: an empirical comparison of kriging and cokriging. *International Journal of Geographical Information Science*, 28(9):1904–1921.

Larraz, B. and Poblacin, J. (2013). An online real estate valuation model for control risk taking: A spatial approach. *Investment Analysts Journal*, 42(78):83–96.

Liang, H. and Yi, W. (2012). Effect of rent spatial distribution to urban business district planning. In *Advanced Materials Research*, volume 374, pages 2001–2008. Trans Tech Publ.

Luis, M., Salhi, S., and Nagy, G. (2015). A constructive method and a guided hybrid grasp for the capacitated multi-source weber problem in the presence of fixed cost. *Journal of Algorithms & Computational Technology*, 9(2):215–232.

Luo, J. (2004). Modeling urban land values in a gis environment. *University of Wisconsin. Milwaukee, USA*.

Megiddo, N. and Supowit, K. (1984). On the complexity of some common geometric location problems. *SIAM Journal on Computing*, 13(1):182–196.

Michelot, C. and Lefebvre, O. (1987). A primal-dual algorithm for the fermat-weber problem involving mixed gauges. *Mathematical Programming*, 39(3):319–335.

Mladenovic, N., Brimberg, J., Hansen, P., and Moreno-Perez, J. A. (2007). The p-median problem: A survey of metaheuristic approaches. *European Journal of Operational Research*, 179(3):927–939.

Oliver, M. A. and Webster, R. (1990). Kriging: a method of interpolation for geographical information systems. *International Journal of Geographical Information System*, 4(3):313–332.

Planchart, A. and Hurter, A. P. J. (1975). An efficient algorithm for the solution of the weber problem with mixed norms. *SIAM Journal on Control*, 13(3):650665.

Plastria, F. (1987). Solving general continuous single facility location problems by cutting planes. *European Journal of Operational Research*, 29(1):98–110.

Sherali, H. D., Al-Loughani, I., and Subramanian, S. (2002). Global optimization procedures for the capacitated euclidean and l_p distance multifacility location-allocation problems. *Operations Research*, 50(3):433–448.

Vielma, J. P., Ahmed, S., and Nemhauser, G. (2010). Mixed-integer models for nonseparable piecewise linear optimization: Unifying framework and extensions. *Operations Research*, 58(2):303–315.

Weber, A. and Friedrich, C. J. (1929). *Alfred Weber's Theory of the Location of Industries*. The University of Chicago Press, Chicago, Illinois.

Weiszfeld, E. and Plastria, F. (2009). On the point for which the sum of the distances to n given points is minimum. *Annals of Operations Research*, 167(1):4–41.

Xue, G. and Ye, Y. (1997). An efficient algorithm for minimizing a sum of euclidean norms with applications. *SIAM Journal on Optimization*, 7(4):1017–1036.