A New Procedure to Calculate the Owen Value

José Miguel Giménez and María Albina Puente
Department of Mathematics and Engineering School of Manresa, Technical University of Catalonia, Manresa, Spain

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Abstract: In this paper we focus on games with a coalition structure. Particularly, we deal with the Owen value, the coalitional value of the Shapley value, and we provide a computational procedure to calculate this coalitional value in terms of the multilinear extension of the original game.

1 INTRODUCTION

Shapley (Shapley, 1953) (see also (Roth, 1988) and (Owen, 1995)) initiated the value theory for cooperative games. The *Shapley value* applies without restrictions and provides, for every game, a single payoff vector to the players. The restriction of the value to simple games gives rise to the *Shapley–Shubik power index* (Shapley and Shubik, 1954), that was axiomatized in (Dubey, 1975) introducing the transfer property. As a sort of reaction, Banzhaf (Banzhaf, 1965) proposed a different power index that Owen (Owen, 1975) extended to a dummy–independent and somehow "normalized" *Banzhaf value* for all cooperative games. A nice almost common characterization of the Shapley and Banzhaf values would be given in (Feltkamp, 1995).

Games with a *coalition structure* were introduced in (Aumann and Drèze, 1974), who extended the Shapley value to this new framework in such a manner that the game really splits into subgames played by the unions isolatedly from each other, and every player receives the payoff allocated by the restriction of the Shapley value to the subgame he is playing within his union. A second approach was used in (Owen, 1977), when introducing and axiomatically characterizing his coalitional value (*Owen value*). The Owen value is the result of a *two–step procedure*: first, the unions play a *quotient game* among themselves, and each one receives a payoff which, in turn, is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by applying the Shapley value. Further axiomatizations of the Owen value have been given in *e.g.* (Hart and Kurz, 1983), (Peleg, 1989), (Winter, 1992), (Amer and Carreras, 1995) and (Amer and Carreras, 2001), (Vázquez et al., 1997), (Vázquez, 1998), (Hamiache, 1999), (Hamiache, 2001) and (Albizuri, 2002).

Owen applied the same procedure to the Banzhaf value and obtained the *modified Banzhaf value* or *Owen–Banzhaf value* (Owen, 1982). In this case the payoffs at both levels (unions in the quotient game and players within each union) are given by the Banzhaf value.

Alonso and Fiestras suggested to modify the two–step allocation scheme and use the Banzhaf value for sharing in the quotient game and the Shapley value within unions. This gave rise to the *symmetric coalitional Banzhaf value* or *Alonso–Fiestras value* (Alonso and Fiestras, 2002). That same year, Carreras et al. considered a sort of "counterpart" of the Alonso–Fiestras value where the Shapley value is used in the quotient game and the Banzhaf value within unions (Amer et al., 2002). Thus, the possibilities to define a coalitional value by combining the Shapley and Banzhaf values were complete at that moment.

In 1972 Owen introduced the *multilinear extension* (Owen, 1972) and applied it to the calculus of the Shapley value. The computing technique based on the multilinear extension has been applied to many values: in 1975 to the Banzhaf value (Owen, 1975); in 1992 to the Owen value (Owen and Winter, 1992); in 1994 to the Owen–Banzhaf value (Carreras and Magaña, 1994); in 1997 to the quotient game (Carreras and Magaña, 1997); in 2000 to *binomial semivalues* and to *multinomial probabilistic indices* (Puente, 2000); in 2004 to the α*–decisiveness and Banzhaf* α*–indices* (Carreras, 2004); in 2005 to the *Alonso–Fiestras value* (Alonso et al., 2005); in 2011 to *symmetric coalitional binomial semivalues*

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(Carreras and Puente, 2011); in 2011 to semivalues (Carreras and Giménez, 2011); in 2015 to *coalitional multinomial probabilistic values* (Carreras and Puente, 2015).

The present paper focus on giving a new computational procedure for the Owen value by means of the multilinear extension of the game.

The organization of the paper is as follows. In Section 2, a minimum of preliminaries is provided. Section 3 is devoted to give a procedure to compute the Owen value.

2 PRELIMINARIES

2.1 Cooperative Games

Let N be a finite set of *players* and 2^N be the set of its *coalitions* (subsets of N). A *cooperative game* on N is a function $v : 2^N \to \mathbb{R}$, that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$ and *simple* if, moreover, $v(S) = 0$ or 1 for every $S \subseteq N$. A player $i \in N$ is a *dummy* in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$, and *null* in v if, moreover, $v(\{i\}) = 0$. Two players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. Given a nonempty coalition $T \subseteq N$, the restriction to T of a given game v on N is the game $v_{|T}$ on T that we will call a *subgame* of v and is defined by $v_{|T}(S) = v(S)$ for all $S \subseteq T$.

Endowed with the natural operations for real–valued functions, *i.e.* $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set of all cooperative games on N is a vector space $\mathcal{G}_N$. For every nonempty coalition $T \subseteq N$, the *unanimity game* u_T is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$, so that $\dim(\mathcal{G}_N) = 2^n - 1$ if $n = |N|$.

By a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \to \mathbb{R}^N$, that assigns to every game v a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

Well known example of value is the *Shapley value* φ (Shapley (Shapley, 1953)), defined as

$$\varphi_i[v] = \sum_{S \subseteq N \backslash \{i\}} p_s [v(S \cup \{i\}) - v(S)$$

for all $i \in N$, $v \in \mathcal{G}_N$, where $s = |S|$ and $p_s = 1/n\binom{n-1}{s}$.

Notice that this value is defined for each N. In fact, it is defined on cardinalities rather than on specific player sets: this means the weighting vector $\{p_s\}_{s=0}^{n-1}$ defines the Shapley value on all N such that $n = |N|$. When necessary, we shall write $\varphi^{(n)}$ for the Shapley value on cardinality n and p_s^n for its weighting coefficients. $\varphi^{(n)}$ induces values $\varphi^{(t)}$ for all cardinalities $t < n$, recurrently defined by the Pascal triangle (inverse) formula given by Dragan (Dragan, 1997). That is

$$p_s^t = p_s^{t+1} + p_{s+1}^{t+1} \quad \text{for} \quad 0 \leq s < t, \qquad (1)$$

The *multilinear extension* (Owen, 1972) of a game $v \in \mathcal{G}_N$ is the real–valued function defined on $\mathbb{R}^N$ by

$$f_v(X_N) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \in N \backslash S} (1 - x_j) v(S). \qquad (2)$$

where X_N denotes the set of variables x_i for $i \in N$.

As is well known, both the Shapley and Banzhaf values of any game v can be easily obtained from its multilinear extension. Indeed, $\varphi[v]$ can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0,1]^N$ (Owen, 1972)), while the partial derivatives of that multilinear extension evaluated at point $(1/2, 1/2, \ldots, 1/2)$ give $\beta[v]$ (Owen, 1975).

2.2 Games with Coalition Structure

Given $N = \{1, 2, \ldots, n\}$, we will denote by $B(N)$ the set of all partitions of N. Each $B \in B(N)$ is called a *coalition structure* in N, and a *union* each member of B. The so–called *trivial coalition structures* are $B^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ (individual coalitions) and $B^N = \{N\}$ (grand coalition). A *cooperative game with a coalition structure* is a pair $[v; B]$, where $v \in \mathcal{G}_N$ and $B \in B(N)$ for a given N. Each partition B gives a pattern of cooperation among players. We denote by $\mathcal{G}_N^{cs} = \mathcal{G}_N \times B(N)$ the set of all cooperative games with a coalition structure and player set N.

If $[v; B] \in \mathcal{G}_N^{cs}$ and $B = \{B_1, B_2, \ldots, B_m\}$, the *quotient game* v^B is the cooperative game played by the unions or, rather, by the *quotient set* $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^B(R) = v(\bigcup_{r \in R} B_r) \quad \text{for all } R \subseteq M.$$

By a *coalitional value* on $\mathcal{G}_N^{cs}$ we will mean a map $g : \mathcal{G}_N^{cs} \to \mathbb{R}^N$, which assigns to every pair $[v; B]$ a vector $g[v; B]$ with components $g_i[v; B]$ for each $i \in N$.

If f is a value on $\mathcal{G}_N$ and g is a coalitional value on $\mathcal{G}_N^{cs}$, it is said that g is a *coalitional value of* f iff $g[v; B^n] = f[v]$ for all $v \in \mathcal{G}_N$.

2.2.1 The Owen Value

The *Owen value* (Owen (Owen, 1977)) is the coalitional value Φ defined by

$$\Phi_i[v;P] = \sum_{R\subseteq M\setminus\{k\}} \sum_{T\subseteq B_k\setminus\{i\}} p_r^{m-1} p_t^{b_k-1}$$
$$[v(Q\cup T\cup\{i\}) - v(Q\cup T)]$$

for all $i\in N$ and $[v;B]\in \mathcal{G}_N^{cs}$, where $B_k\in B$ is the union such that $i\in B_k$, $Q=\bigcup_{r\in R} B_r$ and

$$p_r^{m-1} = \frac{1}{m}\frac{1}{\binom{m-1}{r}},\, p_t^{b_k-1} = \frac{1}{b_k}\frac{1}{\binom{b_k-1}{t}}.$$

This coalitional value was axiomatically characterized by Owen (Owen, 1977) as the only coalitional value that satisfies the following properties: the natural extensions to this framework of

- *efficiency*
- *additivity*
- *the dummy player property*

and also

- *symmetry within unions*: if $i,j\in B_k$ are symmetric in v then
$$\Phi_i[v;B] = \Phi_j[v;B]$$
- *symmetry in the quotient game*: if $B_r,B_s\in P$ are symmetric in $[v;B]$ then
$$\sum_{i\in B_r}\Phi_i[v;B] = \sum_{j\in B_s}\Phi_j[v;B].$$

Finally, as Φ is defined for any N, the following property makes sense and is also satisfied:

- *quotient game property*: for all $[v;B]\in \mathcal{G}_N^{cs}$,
$$\sum_{i\in B_k}\Phi_i[v;B] = \Phi_k[v^B;B^m] \quad \text{for all } B_k\in B.$$

The Owen value can be viewed as a two–step allocation rule. First, each union B_k receives its payoff in the quotient game according to the Shapley value; then, each B_k splits this amount among its players by applying the Shapley value to a game played in B_k as follows: the worth of each subcoalition T of B_k is the Shapley value that T would get in a "pseudoquotient game" played by T and the remaining unions on the assumption that $B_k\setminus T$ leaves the game, *i.e.* the quotient game after replacing B_k with T. This is the way to bargain within the union: each subcoalition T claims the payoff it would obtain when dealing with the other unions in absence of its partners in B_k.

The Owen value is a *coalitional value of the Shapley value* φ in the sense that $\Phi[v;B^n] = \varphi[v]$ for all $v\in\mathcal{G}_N$. Besides, $\Phi[v;B^N] = \varphi[v]$.

3 A COMPUTATIONAL PROCEDURE TO CALCULATE THE OWEN VALUE

In this section we present a new computational procedure to calculate this coalitional value. Before that, we need two previous results that will be given in Lemma 3.1 and Proposition 3.2.

Lemma 3.1. *Let $[v;B]\in\mathcal{G}_N^{cs}$, $B=\{B_1,B_2,\ldots,B_m\}$ a coalition structure in N. The allocations given by Φ to players belonging to a union B_j can be obtained as a linear combination of the allocations to unanimity games u_T, where $T=V\cup W$, $V\subseteq B_j$ and $W\in 2^{B\setminus B_j}$.*

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Proof Each game $v\in\mathcal{G}_N$ can be uniquely written as linear combination of unanimity games

$$v = \sum_{T\subseteq N:\,T\neq\emptyset}\alpha_T u_T,$$

where $\alpha_T = \alpha_T(v) = \sum_{S\subseteq T}(-1)^{t-s}v(S)$.

By linearity, for all $i\in B_j$,

$$\Phi_i[v;B] = \sum_{T\subseteq N:\,T\neq\emptyset}\alpha_T\Phi_i[u_T]$$

and it suffices consider unanimity games u_T with

$$T = V\cup A_{i_1}\cup A_{i_2}\cup\ldots\cup A_{i_p}$$
$$V\subseteq B_j,\ \{i_1,i_2,\ldots,i_p\}\subseteq M\setminus\{j\}$$
$$\emptyset\neq A_{i_q}\subseteq B_{i_q},\ q=1,\ldots,p.$$

According to the definition of the Owen value it is easy to check that the allocations to players in B_j only depend on the allocations in the unanimity games defined on inside coalitions in B_j and entire unions outside B_j. That is,

$$\Phi_i[u_T;B] = \Phi_i[u_{V\cup A_{i_1}\cup A_{i_2}\cup\ldots\cup A_{i_p}};B]$$
$$= \Phi_i[u_{V\cup B_{i_1}\cup B_{i_2}\cup\ldots\cup B_{i_p}};B]. \qquad \square$$

Notice that the number of unanimity games of this form is $(2^{b_j}-1)2^m$ with $b_j=|B_j|$ and $m=|M|$.

Proposition 3.2. *Let $B=\{B_1,B_2,\ldots,B_m\}$ be a coalition structure in N. Fixed a union B_j, the allocation to a player i belonging to B_j in a unanimity game u_T, $T=V\cup B_{i_1}\cup\cdots\cup B_{i_h}$, $V\subseteq B_j$ and $\{i_1,\ldots,i_h\}\subseteq M\setminus\{j\}$ is given by*

$$\Phi_i[u_T;B] = (\psi/\varphi_j)_i[u_T;B] = \begin{cases} p_h^{h+1}p_{v-1}^v & i\in T\\ 0 & i\notin T\end{cases}$$

where $(p_s^{h+1})_{s=0}^h$ and $(p_s^v)_{s=0}^{v-1}$ are the weighting coefficients of the induced Shapley value and $p_h^{h+1}=\frac{1}{h+1}$ and $p_{v-1}^v=\frac{1}{v}$.

Proof For $i \in T$ we have

$$\Phi_i[u_T;B] = \sum_{R\subseteq M\setminus\{j\}} p_r^m \sum_{S\subseteq B_j\setminus\{i\}} p_s^{b_j}[u_T(Q\cup S\cup\{i\}) - u_T(Q\cup S)]$$

where $Q = \bigcup_{r\in R} B_r$, $b_j = |B_j|$, and $s = |S|$.

Only $u_T(Q\cup S\cup\{i\}) - u_T(Q\cup S)$ does not vanish for coalitions R such that $\{i_1,...,i_h\} \subseteq R \subseteq M\setminus\{j\}$ and for coalitions S such that $V\setminus\{i\} \subseteq S \subseteq B_j\setminus\{i\}$. Then,

$$\Phi_i[u_T;B] = p_h^{h+1} p_{v-1}^v$$

In case of $i \notin T$, all marginal contributions $u_T(Q\cup S\cup\{i\}) - u_T(Q\cup S)$ vanish. □

Example 3.1 On the players set $N = \{1,2,3,4,5,6\}$, let $B = \{\{1,2,3\},\{4,5\},\{6\}\}$ be a coalition structure on N. We will obtain the allocations to players $i \in B_1$ according to Φ for the unanimity games $u_{\{1,2,4,6\}}$ and $u_{\{1,2,4,5,6\}}$. They are

$$\Phi_i[u_{\{1,2,4,6\}};B] = p_2^3 p_1^2 = \frac{1}{3}\frac{1}{2} = \frac{1}{6}, \text{ for } i = 1,2 \text{ and}$$

$$\Phi_3[u_{\{1,2,4,6\}};B] = 0,$$

where $p_2^3 = \frac{1}{3}$ and $p_1^2 = \frac{1}{2}$ are the corresponding weighting coefficient of the induced Shapley value.

In a similar way and according to Lemma 3.1, for $u_{\{1,2,4,5,6\}}$ we obtain

$$\Phi_i[u_{\{1,2,4,5,6\}};B] = p_2^3 p_1^2 = \frac{1}{3}\frac{1}{2} = \frac{1}{6}, \text{ for } i = 1,2 \text{ and}$$

$$\Phi_3[u_{\{1,2,4,5,6\}};B] = 0,$$

Notice that the allocations in both games are the same because coalitions $\{1,2,4,6\}$ and $\{1,2,4,5,6\}$ intersect the same unions B_2 and B_3.

In next theorem we present a new method to compute the Owen value by means of the multilinear extension of the game.

Theorem 3.3. *Let $[v;B] \in \mathcal{G}_N^{cs}$, $B = \{B_1,B_2,\ldots,B_m\}$ a coalition structure in N.*

Then the following steps lead to the Owen value of any player $i \in B_j$ in $[v;B]$.

1. *Obtain the multilinear extension $f(x_1,x_2,\ldots,x_n)$ of game v.*
2. *For every $r \neq j$ and all $h \in B_r$, replace the variable x_h with y_r. This yields a new function of x_k for $k \in B_j$ and y_r for $r \in M\setminus\{j\}$.*
3. *In this new function, reduce to 1 all higher exponents, i.e. replace with y_r each y_r^q such that $q > 1$. This gives a new multilinear function denoted as $g_j((x_k)_{k\in B_j},(y_r)_{r\in M\setminus\{j\}})$ (The modified multilinear extension of union B_j).*
4. *After some calculus, the obtained modified multilinear extension reduces to*
$$g_j((x_k)_{k\in B_j},(y_r)_{r\in M\setminus\{j\}}) = \sum_{V\subseteq B_j}\sum_{W\subseteq M\setminus\{j\}} \lambda_{V\cup W}\prod_{k\in V} x_k \prod_{r\in W} y_r$$
5. *Multiply each product $\prod_{k\in V} x_k$ by $p_{v-1}^{j,v}$ and each product $\prod_{r\in W} y_r$ by p_w^{w+1} obtaining a new multilinear function called $\overline{g}_j$.*
6. *Obtain the partial derivative of $\overline{g}_j$ with respect to x_i evaluated at point $(1,\ldots,1)$ and*
$$\Phi_i[v;B] = \frac{\partial \overline{g}_j}{\partial x_i}(1_{B_j},1_{M\setminus\{j\}}).$$

Proof Steps 1–3 have been already used in many well known works to obtain the modified multilinear extension of union B_j. Step 4 shows the modified multilinear extension as a linear combination of multilinear extensions of unanimity games. Step 5 weights each unanimity game according to Proposition 3.2 so that step 6 gives as usual the marginal contribution of player i and his allocation $\Phi_i[v;B]$ is obtained. □

Example 3.2 Let $v \equiv [68;50,21,20,19,13,9,3]$ be the 7–person weighted majority game and the coalition structure $B = \{\{1\},\{2,3,5\},\{4\},\{6\},\{7\}\}$. We will compute $\Phi[v;B]$.

The set of minimal winning coalitions of the game is

$$W^m(v) = \{\{1,2\},\{1,3\},\{1,4\},\{1,5,6\}\},$$

so that players 2, 3 and 4 on one hand, and 5 and 6 on the other, are symmetric in v. Moreover, player 7 is null and the multilinear extension of v is

$$\begin{aligned} f(X_N) =& x_1x_2 + x_1x_3 + x_1x_4 - x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4 \\ &+ x_1x_5x_6 + x_1x_2x_3x_4 - x_1x_2x_5x_6 - x_1x_3x_5x_6 \\ &- x_1x_4x_5x_6 + x_2x_3x_4x_5 + x_2x_3x_4x_6 - x_1x_2x_3x_4x_5 \\ &- x_1x_2x_3x_4x_6 + x_1x_2x_3x_5x_6 + x_1x_2x_4x_5x_6 \\ &+ x_1x_3x_4x_5x_6 - x_2x_3x_4x_5x_6. \end{aligned}$$

The coalition structure is

$$B = \{\{1\},\{2,3,5\},\{4\},\{6\},\{7\}\}$$

and steps 1–4 in Theorem 3.3 give the modified multilinear extension of each union B_j, for $j = 1,2,3,4$ (notice that player 7 is null in v and it is not necessary to compute g_5).

$$g_1(x_1,y_2,y_3,y_4,y_5) = x_1y_2 + x_1y_3 - 2x_1y_2y_3 + y_2y_3,$$

$$g_2(x_2,x_3,x_5,y_1,y_3,y_4,y_5) = x_2y_1 + x_3y_1 + y_1y_3 - x_2x_3y_1 - x_2y_1y_3 - x_3y_1y_3 + x_5y_1y_4 + x_2x_3y_1y_3 - x_2x_5y_1y_4 - x_3x_5y_1y_4 - x_5y_1y_3y_4 + x_2x_3x_5y_3 + x_2x_3y_3y_4 - x_2x_3x_5y_1y_3 - x_2x_3y_1y_3y_4 + x_2x_3x_5y_1y_4 + x_2x_5y_1y_3y_4 + x_3x_5y_1y_3y_4 - x_2x_3x_5y_3y_4,$$

$$g_3(x_4,y_1,y_2,y_4,y_5) = y_1y_2 + x_4y_1 + x_4y_2 - 2x_4y_1y_2,$$

$$g_4(x_6,y_1,y_2,y_3,y_5) = y_1y_2 + y_1y_3 + y_2y_3 - 2y_1y_2y_3.$$

Step 5 leads to $\overline{g}_j$ for each $j = 1,2,3,4$.

$$\overline{g}_1(x_1,y_2,y_3,y_4,y_5) = p_0^{1,1}p_1^2x_1y_2 + p_0^{1,1}p_1^2x_1y_3 - 2p_0^{1,1}p_2^3x_1y_2y_3 + p_2^3y_2y_3,$$

$$\overline{g}_2(x_2,x_3,x_5,y_1,y_3,y_4,y_5) = p_0^1p_1^2x_2y_1 + p_0^1p_1^2x_3y_1 - p_1^2p_1^2x_2x_3y_1 + p_2^3y_1y_3 - p_0^1p_2^3x_2y_1y_3 - p_0^1p_2^3x_3y_1y_3 + p_0^1p_2^3x_5y_1y_4 + p_1^2p_2^3x_2x_3y_1y_3 - p_1^2p_2^3x_2x_5y_1y_4 - p_1^2p_2^3x_3x_5y_1y_4 - p_0^1p_3^4x_5y_1y_3y_4 + p_2^3p_1^2x_2x_3x_5y_3 + p_1^2p_2^3x_2x_3y_3y_4 - p_2^3p_2^3x_2x_3x_5y_1y_3 - p_1^2p_3^4x_2x_3y_1y_3y_4 + p_2^3p_2^3x_2x_3x_5y_1y_4 + p_1^2p_3^4x_2x_5y_1y_3y_4 + p_1^2p_3^4x_3x_5y_1y_3y_4 - p_2^3p_2^3x_2x_3x_5y_3y_4,$$

$$\overline{g}_3(x_4,y_1,y_2,y_4,y_5) = p_2^3y_1y_2 + p_0^1q_1^2x_4y_1 + p_0^1p_1^2x_4y_2 - 2p_0^1p_2^3x_4y_1y_2,$$

$$\overline{g}_4(x_6,y_1,y_2,y_3,y_5) = p_2^3y_1y_2 + p_2^3y_1y_3 + p_2^3y_2y_3 - 2p_3^4y_1y_2y_3.$$

Finally, step 6 yields

$$\Phi_1[v;B] = 2p_0^1p_1^2 - 2p_0^1p_2^3 = \tfrac{1}{3},$$

$$\Phi_i[v;B] = p_0^1p_1^2 - p_1^2p_1^2 - p_0^1p_2^3 + p_1^2p_2^3 + p_2^3p_1^2 - p_2^3p_2^3 = \frac{5}{36}, \quad \text{for } i = 2,3,$$

$$\Phi_4[v;B] = 2p_0^1p_1^2 - 2p_0^1p_2^3 = \tfrac{1}{3},$$

$$\Phi_5[v;B] = p_0^1p_2^3 - 2p_1^2p_2^3 - p_0^1p_3^4 + p_2^3p_1^2 - p_2^3p_2^3 + 2p_1^2p_3^4 = \frac{1}{18},$$

$\Phi_6[v;B] = 0$ and

$\Phi_7[v;B] = 0.$

4 CONCLUSIONS

As we have said before, the present work is focussed on the calculus of the Owen value. More precisely, the computation of players' allocations are obtained from the multilinear extension of the game. In the context of games with a coalition structure, the multilinear extension technique has been also applied to computing the Owen value in (Owen and Winter, 1992); as well as the Owen–Banzhaf value in (Carreras and Magaña, 1994); in 1997 to the quotient game (Carreras and Magaña, 1997); the Alonso–Fiestras value in (Alonso et al., 2005); the symmetric coalitional binomial semivalues in (Carreras and Puente, 2011); and coalitional multinomial probabilistic values in (Carreras and Puente, 2015). In all these cases, the first three steps of the procedure are the same.

Instead, the consideration of the modified MLE g_j for the union B_j obtained from the initial one has changed the procedure: first, we weight the terms of g_j multiplying each product $\prod_{k\in V} x_k$ by p_{v-1}^v and each product $\prod_{r\in W} y_r$ by q_w^{w+1} obtaining a new multilinear function called $\overline{g}_j$. Second, we obtain players' marginal contributions by partial differentiation of $\overline{g}_j$. This new procedure has an advantage with respect to the traditional method: the allocations given by the Owen value are available since the weighting coefficients p_k^{k-1} and q_k^{k+1} can be always easily obtained.

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