Hyperresolution for Propositional Product Logic

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Abstract: We provide the foundations of automated deduction in the propositional product logic. Particularly, we generalise the hyperresolution principle to the propositional product logic. We propose translation of a formula to an equivalent satisfiable finite order clausal theory, which consists of order clauses - finite sets of order literals of the augmented form: $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is either the truth constant *0* or *1* or a conjunction of powers of propositional atoms, and $\diamond$ is the connective $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the standard equality and strict order on $[0,1]$, respectively. We devise a hyperresolution calculus over order clausal theories, which is refutation sound and complete for the finite case. By means of the translation and calculus, we solve the deduction problem $T \models \phi$ for a finite theory T and a formula ϕ.

1 INTRODUCTION

Automated deduction in fuzzy (many-valued) logics has gradually been receiving an attention from logicians, informaticians, and engineers. The reason is its growing application potential in many fields, spanning from engineering to informatics, such as fuzzy control and optimisation of both discrete and continuous industrial processes, knowledge representation and reasoning, ontology languages, the Semantic Web, the Web Ontology Language (*OWL*), fuzzy description logics and ontologies, multi-step fuzzy (many-valued) inference, fuzzy knowledge/expert systems. An important subclass consists of t-norm fuzzy logics, with the special cases of continuous and left-continuous t-norm (Klement and Mesiar, 2005; Klement et al., 2013). The standard semantics of a t-norm fuzzy logic is formed by the unit interval of real numbers $[0,1]$ equipped with the standard order, supremum, infimum, the t-norm and its residuum. The condition of left-continuity ensures the existence of the unique residuum for a given t-norm. The basic logics of continuous and left-continuous t-norm are the *BL* (basic) (Hájek, 2001) and *MTL* (monodial t-norm) (Esteva and Godo, 2001) ones, respectively. Gödel logic is one of the simplest t-norm fuzzy logics with the (idempotent) minimum t-norm. By the Mostert-Shields theorem (Mostert and Shields, 1957), a t-norm is continuous if and only if it is isomorphic to an ordinal sum (countably many open disjoint subintervals of the unit interval) of the product and Łukasiewicz t-norms, completed by Gödel (minimum) t-norm. This is a useful mathematical characterisation but infinitary, and hence, insufficient for computational purposes. Our objective is to propose logic calculi suitable for automated deduction and underlying procedures/algorithms for (in)finitely summed t-norms and related fuzzy logics. However, even the three fundamental continuous fuzzy logics have not yet been investigated in a systematic way from a computational logic perspective.

Descriptions of real-world problems may become rather complex. So, efficient inference stipulates the methods and techniques of automated deduction. The early research in automated deduction had started in the 1950s, basically focused on theorem proving. The resolution method, devised by Robinson (Robinson, 1965b; Robinson, 1965a), is based on the following inference rules:

(*Binary resolution*)

$$\frac{a \vee B, \quad \neg c \vee D}{(B \vee D)\theta}$$

θ *is a most general unifier of the atoms a and c*;

(*Hyperresolution*)

$$\frac{a_1 \vee B_1, \dots, a_n \vee B_n, \quad \neg c_1 \vee \dots \vee \neg c_n \vee D}{(B_1 \vee \dots \vee B_n \vee D)\theta}$$

θ *is a most general unifier of the atoms a_i and c_i.*

Both the rules/calculi are refutation complete and sound: a clausal theory is unsatisfiable if and only

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if the empty clause can be inferred. A large class of refinements and strategies has been developed (Bachmair and Ganzinger, 1994; Bachmair and Ganzinger, 1998). Another direction in automated deduction constitutes the Davis-Putnam-Logemann-Loveland procedure (*DPLL*) (Davis and Putnam, 1960; Davis et al., 1962) and its refinements, e.g. chronological backtracking is replaced with non-chronological one using so-called conflict-driven clause learning (*CDCL*) (Silva and Sakallah, 1996; Marques-Silva and Sakallah, 1999). Most modern propositional *SAT* solvers are based on the *DPLL* or *CDCL* procedure, improved by various features (Biere et al., 2009; Schöning and Torán, 2013).

In recent years, we have investigated both the propositional and first-order case of Gödel logic. In (Guller, 2010; Guller, 2012a), we have proposed an extension of the *DPLL* procedure. In (Guller, 2012b; Guller, 2016a; Guller, 2014; Guller, 2015a), we have devised an extension of hyperresolution, augmented by truth constants and the equality, $\eqcirc$, strict order, $\prec$, projection, $\boldsymbol{\Delta}$, operators. As a side result, we have shown that unsatisfiable formulae are recursively enumerable (Guller, 2016b; Guller, 2015b).

Our exploration also concerns the propositional product logic with the multiplication t-norm. We have introduced an extension of the *DPLL* procedure (Guller, 2013; Guller, 2016a). In this paper, we examine the resolution counterpart. Particularly, we generalise the hyperresolution principle to the propositional product logic. We propose translation of a formula to an equivalent satisfiable finite order clausal theory, which consists of order clauses - finite sets of order literals of the augmented form: $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is either the truth constant *0* or *1* or a conjunction of powers of propositional atoms, and $\diamond$ is the connective $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the standard equality and strict order on $[0,1]$, respectively. We devise a hyperresolution calculus over order clausal theories, which is refutation sound and complete for the finite case. By means of the translation and calculus, we solve the deduction problem $T \models \phi$ for a finite theory T and a formula ϕ.

The paper is arranged as follows. Section 2 recalls the propositional product logic. Section 3 presents translation to clausal form. Section 4 proposes a hyperresolution calculus. Section 5 brings conclusions.

2 PROPOSITIONAL PRODUCT LOGIC

Throughout the paper, we shall use the common notions and notation of propositional logic. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ designates the set of natural, integer, real numbers, and $=$, $\leq$, $<$ denotes the standard equality, order, strict order on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$. We denote $\mathbb{R}_0^+ = \{c \,|\, 0 \leq c \in \mathbb{R}\}$, $\mathbb{R}^+ = \{c \,|\, 0 < c \in \mathbb{R}\}$, $[0,1] = \{c \,|\, c \in \mathbb{R}, 0 \leq c \leq 1\}$; $[0,1]$ is the unit interval. The set of propositional atoms of the product logic will be denoted as *PropAtom*. We assume truth constants - propositional atoms *0*, *1* $\in$ *PropAtom*; *0* denotes the false and *1* the true in the product logic. By *PropForm* we designate the set of all propositional formulae of the product logic built up from *PropAtom* using the connectives: $\neg$, negation, $\wedge$, conjunction, &, strong conjunction, $\vee$, disjunction, $\rightarrow$, implication, and $\leftrightarrow$, equivalence. We introduce a new unary connective Δ, Delta, and binary connectives $\eqcirc$, equality, $\prec$, strict order. By *OrdPropForm* we designate the set of all so-called order propositional formulae of the product logic built up from *PropAtom* using the connectives: $\neg$, Δ, $\wedge$, &, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\eqcirc$, $\prec$.[1] Note that *PropForm* $\subseteq$ *OrdPropForm*. In the paper, we shall assume that *PropAtom* is countably infinite; hence, both the sets of formulae are countably infinite. Let ε_i, $1 \leq i \leq n$, be either an order formula or a set of order formulae or a set of sets of order formulae, in general. By $atoms(\varepsilon_1, \dots, \varepsilon_n) \subseteq PropAtom$ we denote the set of all atoms occurring in $\varepsilon_1, \dots, \varepsilon_n$. We define the size of order formula $|\phi| : OrdPropForm \longrightarrow \mathbb{N}$ by recursion on the structure of ϕ:

$$|\phi| = \begin{cases} 1 & \textit{if } \phi \in PropAtom, \\ 1 + |\phi_1| & \textit{if } \phi = \diamond\phi_1, \\ 1 + |\phi_1| + |\phi_2| & \textit{if } \phi = \phi_1 \diamond \phi_2. \end{cases}$$

Let $T \subseteq OrdPropForm$ be finite. We define the size of T as $|T| = \sum_{\phi \in T} |\phi|$.

Let X, Y, Z be sets and $f : X \longrightarrow Y$ a mapping. By $\|X\|$ we denote the set-theoretic cardinality of X. The relationship of X being a finite subset of Y is denoted as $X \subseteq_{\mathcal{F}} Y$. Let $Z \subseteq X$. We designate $f[Z] = \{f(z) \,|\, z \in Z\}$; $f[Z]$ is the image of Z under f; $f|_Z = \{(z, f(z)) \,|\, z \in Z\}$; $f|_Z$ is the restriction of f onto Z. Let $\gamma \leq \omega$. A sequence δ of X is a bijection $\delta : \gamma \longrightarrow X$. Recall that X is countable if and only if there exists a sequence of X. Let I be an index set and $S_i \neq \emptyset$, $i \in I$, be sets. A selector $\mathcal{S}$ over $\{S_i \,|\, i \in I\}$ is a mapping $\mathcal{S} : I \longrightarrow \bigcup\{S_i \,|\, i \in I\}$ such that for all $i \in I$, $\mathcal{S}(i) \in S_i$. We denote $\mathcal{S}el(\{S_i \,|\, i \in I\}) = \{\mathcal{S} \,|\, \mathcal{S} \textit{ is a selector over } \{S_i \,|\, i \in I\}\}$. Let $c \in \mathbb{R}^+$. $\log c$ denotes the binary logarithm of c. Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}_0^+$. f is of the order of g, in symbols $f \in O(g)$, iff there exist n_0 and $c^* \in \mathbb{R}_0^+$ such that for all $n \geq n_0$, $f(n) \leq c^* \cdot g(n)$.

[1] We assume a decreasing connective precedence: $\neg$, Δ, &, $\eqcirc$, $\prec$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

The product logic is interpreted by the standard Π-algebra augmented by the operators $\boldsymbol{\eqcirc}$, $\boldsymbol{\prec}$, $\boldsymbol{\Delta}$ for the connectives $\eqcirc$, $\prec$, Δ, respectively.

$$\Pi = ([0,1], \leq, \vee, \wedge, \cdot, \Rightarrow, \overline{}, \boldsymbol{\eqcirc}, \boldsymbol{\prec}, \boldsymbol{\Delta}, 0, 1)$$

where $\vee$, $\wedge$ denotes the supremum, infimum operator on $[0,1]$;

$$a \Rightarrow b = \begin{cases} 1 \ \textit{if}\ a \leq b, \\ \frac{b}{a}\ \textit{else}; \end{cases} \qquad \overline{a} = \begin{cases} 1 \ \textit{if}\ a = 0, \\ 0\ \textit{else}; \end{cases}$$

$$a \boldsymbol{\eqcirc} b = \begin{cases} 1 \ \textit{if}\ a = b, \\ 0\ \textit{else}; \end{cases} \qquad a \boldsymbol{\prec} b = \begin{cases} 1 \ \textit{if}\ a < b, \\ 0\ \textit{else}; \end{cases}$$

$$\boldsymbol{\Delta} a = \begin{cases} 1 \ \textit{if}\ a = 1, \\ 0\ \textit{else}. \end{cases}$$

Recall that Π is a complete linearly ordered lattice algebra; $\vee$, $\wedge$ is commutative, associative, idempotent, monotone; 0, 1 is its neutral element; $\cdot$ is commutative, associative, monotone; 1 is its neutral element; the residuum operator $\Rightarrow$ of $\cdot$ satisfies the condition of residuation:

$$\text{for all } a,b,c \in \Pi,\ a \cdot b \leq c \Longleftrightarrow a \leq b \Rightarrow c; \tag{1}$$

Gödel negation $\overline{}$ satisfies the condition:

$$\text{for all } a \in \Pi,\ \overline{a} = a \Rightarrow 0; \tag{2}$$

$\boldsymbol{\Delta}$ satisfies the condition:[2]

$$\text{for all } a \in \Pi,\ \boldsymbol{\Delta} a = a \boldsymbol{\eqcirc} 1. \tag{3}$$

A valuation $\mathcal{V}$ of propositional atoms is a mapping $\mathcal{V} : \mathit{PropAtom} \longrightarrow [0,1]$ such that $\mathcal{V}(\mathit{0}) = 0$ and $\mathcal{V}(\mathit{1}) = 1$. Let $\phi \in \mathit{OrdPropForm}$ and $\mathcal{V}$ be a valuation. We define the truth value $\|\phi\|^{\mathcal{V}} \in [0,1]$ of ϕ in $\mathcal{V}$ by recursion on the structure of ϕ as follows:

$$\begin{aligned}
&\phi \in \mathit{PropAtom}, && \|\phi\|^{\mathcal{V}} = \mathcal{V}(\phi);\\
&\phi = \neg\phi_1, && \|\phi\|^{\mathcal{V}} = \overline{\|\phi_1\|^{\mathcal{V}}};\\
&\phi = \Delta\phi_1, && \|\phi\|^{\mathcal{V}} = \boldsymbol{\Delta}\,\|\phi_1\|^{\mathcal{V}};\\
&\phi = \phi_1 \diamond \phi_2, && \|\phi\|^{\mathcal{V}} = \|\phi_1\|^{\mathcal{V}} \boldsymbol{\diamond} \|\phi_2\|^{\mathcal{V}},\\
& && \qquad \diamond \in \{\wedge, \&, \vee, \rightarrow, \eqcirc, \prec\};\\
&\phi = \phi_1 \leftrightarrow \phi_2, && \|\phi\|^{\mathcal{V}} = (\|\phi_1\|^{\mathcal{V}} \Rightarrow \|\phi_2\|^{\mathcal{V}}) \cdot\\
& && \qquad (\|\phi_2\|^{\mathcal{V}} \Rightarrow \|\phi_1\|^{\mathcal{V}}).
\end{aligned}$$

An order theory is a set of order formulae. Let $\phi, \phi' \in \mathit{OrdPropForm}$ and $T \subseteq \mathit{OrdPropForm}$. ϕ is true in $\mathcal{V}$, written as $\mathcal{V} \models \phi$, iff $\|\phi\|^{\mathcal{V}} = 1$. $\mathcal{V}$ is a model of T, in symbols $\mathcal{V} \models T$, iff, for all $\phi \in T$, $\mathcal{V} \models \phi$. ϕ is a tautology iff, for every valuation $\mathcal{V}$, $\mathcal{V} \models \phi$. ϕ is equivalent to ϕ', in symbols $\phi \equiv \phi'$, iff, for every valuation $\mathcal{V}$, $\|\phi\|^{\mathcal{V}} = \|\phi'\|^{\mathcal{V}}$.

[2]We assume a decreasing operator precedence: $\overline{}$, $\boldsymbol{\Delta}$, $\cdot$, $\boldsymbol{\eqcirc}$, $\boldsymbol{\prec}$, $\wedge$, $\vee$, $\Rightarrow$.

3 TRANSLATION TO CLAUSAL FORM

We firstly introduce a notion of power of propositional atom and a notion of conjunction of powers of propositional atoms. Let $a \in \mathit{PropAtom} - \{\mathit{0}, \mathit{1}\}$ and $n \geq 1$. The n-th power of the propositional atom a, a raised to the power of n, is the pair (a, n), written as a^n. A power a^1 is denoted as a; if it does not cause the ambiguity with the denotation of the single atom a in a given context. The set of all powers is designated as $\mathit{PropPow}$. Let $a^n \in \mathit{PropPow}$. We define the size of a^n as $|a^n| = n \geq 1$. A conjunction Cn of powers of propositional atoms is a non-empty finite set of powers such that for all $a^m \neq b^n \in \mathit{Cn}$, $a \neq b$. A conjunction $\{a_0^{m_0}, \dots, a_n^{m_n}\}$ is written in the form $a_0^{m_0} \& \cdots \& a_n^{m_n}$. A conjunction $\{p\}$ is called unit and denoted as p; if it does not cause the ambiguity with the denotation of the single power p in a given context. The set of all conjunctions is designated as $\mathit{PropConj}$. Let $p \in \mathit{PropPow}$, $\mathit{Cn}, \mathit{Cn}_1, \mathit{Cn}_2 \in \mathit{PropConj}$, $\mathcal{V}$ be a valuation. The truth value $\|\mathit{Cn}\|^{\mathcal{V}} \in [0,1]$ of $\mathit{Cn} = a_0^{m_0} \& \cdots \& a_n^{m_n}$ in $\mathcal{V}$ is defined by

$$\|\mathit{Cn}\|^{\mathcal{V}} = \underbrace{\|a_0\|^{\mathcal{V}} \cdot \cdots \cdot \|a_0\|^{\mathcal{V}}}_{m_0} \cdot \cdots \cdot \underbrace{\|a_n\|^{\mathcal{V}} \cdot \cdots \cdot \|a_n\|^{\mathcal{V}}}_{m_n}.$$

We define the size of Cn as $|\mathit{Cn}| = \sum_{p \in \mathit{Cn}} |p| \geq 1$. By $p \& \mathit{Cn}$ we denote $\{p\} \cup \mathit{Cn}$ where $p \notin \mathit{Cn}$. Cn_1 is a subconjunction of Cn_2, in symbols $\mathit{Cn}_1 \sqsubseteq \mathit{Cn}_2$, iff, for all $a^m \in \mathit{Cn}_1$, there exists $a^n \in \mathit{Cn}_2$ such that $m \leq n$. Cn_1 is a proper subconjunction of Cn_2, in symbols $\mathit{Cn}_1 \sqsubset \mathit{Cn}_2$, iff $\mathit{Cn}_1 \sqsubseteq \mathit{Cn}_2$ and $\mathit{Cn}_1 \neq \mathit{Cn}_2$.

We finally introduce order clauses in the product logic. l is an order literal iff $l = \varepsilon_1 \diamond \varepsilon_2$, $\varepsilon_i \in \{\mathit{0}, \mathit{1}\} \cup \mathit{PropConj}$, $\diamond \in \{\eqcirc, \prec\}$. The set of all order literals is designated as $\mathit{OrdPropLit}$. Let $l = \varepsilon_1 \diamond \varepsilon_2 \in \mathit{OrdPropLit}$ and $\mathcal{V}$ be a valuation. The truth value $\|l\|^{\mathcal{V}} \in [0,1]$ of l in $\mathcal{V}$ is defined by $\|l\|^{\mathcal{V}} = \|\varepsilon_1\|^{\mathcal{V}} \boldsymbol{\diamond} \|\varepsilon_2\|^{\mathcal{V}}$. Note that $\mathcal{V} \models l$ if and only if either $l = \varepsilon_1 \eqcirc \varepsilon_2$, $\|\varepsilon_1 \eqcirc \varepsilon_2\|^{\mathcal{V}} = 1$, $\|\varepsilon_1\|^{\mathcal{V}} = \|\varepsilon_2\|^{\mathcal{V}}$; or $l = \varepsilon_1 \prec \varepsilon_2$, $\|\varepsilon_1 \prec \varepsilon_2\|^{\mathcal{V}} = 1$, $\|\varepsilon_1\|^{\mathcal{V}} < \|\varepsilon_2\|^{\mathcal{V}}$. We define the size of l as $|l| = 1 + |\varepsilon_1| + |\varepsilon_2|$. An order clause is a finite set of order literals. Since $=$ is symmetric, $\eqcirc$ is commutative; hence, for all $\varepsilon_1 \eqcirc \varepsilon_2 \in \mathit{OrdPropLit}$, we identify $\varepsilon_1 \eqcirc \varepsilon_2$ and $\varepsilon_2 \eqcirc \varepsilon_1 \in \mathit{OrdPropLit}$ with respect to order clauses. An order clause $\{l_0, \dots, l_n\} \neq \emptyset$ is written in the form $l_0 \vee \cdots \vee l_n$. The empty order clause $\emptyset$ is denoted as $\square$. An order clause $\{l\}$ is called unit and denoted as l; if it does not cause the ambiguity with the denotation of the single order literal l in a given context. We designate the set of all order clauses as $\mathit{OrdPropCl}$. Let $l, l_0, \dots, l_n \in \mathit{OrdPropLit}$ and $C, C' \in$

OrdPropCl. We define the size of C as $|C| = \sum_{l \in C} |l|$. By $l_0 \vee \cdots \vee l_n \vee C$ we denote $\{l_0, \ldots, l_n\} \cup C$ where, for all $i, i' \leq n$ and $i \neq i'$, $l_i \notin C$, $l_i \neq l_{i'}$. By $C \vee C'$ we denote $C \cup C'$. C is a subclause of C', in symbols $C \sqsubseteq C'$, iff $C \subseteq C'$. An order clausal theory is a set of order clauses. A unit order clausal theory is a set of unit order clauses. Let $\phi, \phi' \in \mathit{OrdPropForm}$, $T, T' \subseteq \mathit{OrdPropForm}$, $S, S' \subseteq \mathit{OrdPropCl}$, $\mathcal{V}$ be a valuation. C is true in $\mathcal{V}$, written as $\mathcal{V} \models C$, iff there exists $l^* \in C$ such that $\mathcal{V} \models l^*$. $\mathcal{V}$ is a model of S, in symbols $\mathcal{V} \models S$, iff, for all $C \in S$, $\mathcal{V} \models C$. Let $\varepsilon_1 \in \{\phi, T, C, S\}$ and $\varepsilon_2 \in \{\phi', T', C', S'\}$. ε_2 is a propositional consequence of ε_1, in symbols $\varepsilon_1 \models \varepsilon_2$, iff, for every valuation $\mathcal{V}$, if $\mathcal{V} \models \varepsilon_1$, then $\mathcal{V} \models \varepsilon_2$. ε_1 is satisfiable iff there exists a valuation $\mathcal{V}$ such that $\mathcal{V} \models \varepsilon_1$. ε_1 is equisatisfiable to ε_2 iff ε_1 is satisfiable if and only if ε_2 is satisfiable. Let $S \subseteq_{\mathcal{F}} \mathit{OrdPropCl}$. We define the size of S as $|S| = \sum_{C \in S} |C|$. Let $\mathbb{I} = \mathbb{N} \times \mathbb{N}$; a countably infinite index set. Since *PropAtom* is countably infinite, there exist $\mathbb{O}, \tilde{\mathbb{A}} \subseteq \mathit{PropAtom}$ such that $\mathbb{O} \supseteq \{\mathit{0}, \mathit{1}\}$, $\mathbb{O} \cup \tilde{\mathbb{A}} = \mathit{PropAtom}$, $\mathbb{O} \cap \tilde{\mathbb{A}} = \emptyset$, both are countably infinite, $\tilde{\mathbb{A}} = \{\tilde{a}_{\mathbb{i}} \,|\, \mathbb{i} \in \mathbb{I}\}$. Let $A \subseteq \tilde{\mathbb{A}}$. We denote $\mathit{OrdPropForm}_A = \{\phi \,|\, \phi \in \mathit{OrdPropForm}, \mathit{atoms}(\phi) \subseteq \mathbb{O} \cup A\} \subseteq \mathit{OrdPropForm}$ and $\mathit{OrdPropCl}_A = \{C \,|\, C \in \mathit{OrdPropCl}, \mathit{atoms}(C) \subseteq \mathbb{O} \cup A\} \subseteq \mathit{OrdPropCl}$.

From a computational point of view, the worst case time and space complexity will be estimated using the logarithmic cost measurement. Let $\mathcal{A}$ be an algorithm. $\#O_{\mathcal{A}}(\mathit{In}) \geq 1$ denotes the number of all elementary operations executed by $\mathcal{A}$ on an input *In*.

Translation of an order formula or theory to clausal form, is based on the following lemma:

Lemma 1. *Let* $n_\phi, n_0 \in \mathbb{N}$, $\phi \in \mathit{OrdPropForm}_\emptyset$, $T \subseteq \mathit{OrdPropForm}_\emptyset$.

(I) *There exist an index set* $J_\phi \subseteq \{(n_\phi, j) \,|\, j \in \mathbb{N}\} \subseteq \mathbb{I}$ *and* $S_\phi \subseteq_{\mathcal{F}} \mathit{OrdPropCl}_{\{\tilde{a}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}}$ *such that either* $J_\phi = \emptyset$ *or* $J_\phi = \{(n_\phi, j) \,|\, j \leq n_{J_\phi}\}$ *for some* n_{J_ϕ} *(*J_ϕ *is a non-empty interval of indices);*

- (a) $\|J_\phi\| \leq 2 \cdot |\phi|$*;*
- (b) *either* $J_\phi = \emptyset$, $S_\phi = \{\square\}$ *or* $J_\phi = S_\phi = \emptyset$ *or* $J_\phi \neq \emptyset$, $\square \notin S_\phi \neq \emptyset$*;*
- (c) *there exists a valuation* $\mathfrak{A}$ *and* $\mathfrak{A} \models \phi$ *if and only if there exists a valuation* $\mathfrak{A}'$ *and* $\mathfrak{A}' \models S_\phi$*, satisfying* $\mathfrak{A}|_{\mathbb{O}} = \mathfrak{A}'|_{\mathbb{O}}$*;*
- (d) $|S_\phi| \in O(|\phi|)$*; the number of all elementary operations of the translation of* ϕ *to* S_ϕ*, is in* $O(|\phi|)$*; the time and space complexity of the translation of* ϕ *to* S_ϕ*, is in* $O(|\phi| \cdot (\log(1 + n_\phi) + \log|\phi|))$*;*
- (e) *if* $S_\phi \neq \emptyset, \{\square\}$*, then* $J_\phi \neq \emptyset$*, for all* $C \in S_\phi$*,* $\emptyset \neq \mathit{atoms}(C) \cap \tilde{\mathbb{A}} \subseteq \{\tilde{a}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}$*.*

(II) *There exist an index set* $J_T \subseteq \{(i, j) \,|\, i \geq n_0\} \subseteq \mathbb{I}$ *and* $S_T \subseteq \mathit{OrdPropCl}_{\{\tilde{a}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$ *such that*

- (a) *either* $J_T = \emptyset$, $S_T = \{\square\}$ *or* $J_T = S_T = \emptyset$ *or* $J_T \neq \emptyset$, $\square \notin S_T \neq \emptyset$*;*
- (b) *there exists a valuation* $\mathfrak{A}$ *and* $\mathfrak{A} \models T$ *if and only if there exists a valuation* $\mathfrak{A}'$ *and* $\mathfrak{A}' \models S_T$*, satisfying* $\mathfrak{A}|_{\mathbb{O}} = \mathfrak{A}'|_{\mathbb{O}}$*;*
- (c) *if* $T \subseteq_{\mathcal{F}} \mathit{OrdPropForm}_\emptyset$*, then* $J_T \subseteq_{\mathcal{F}} \{(i, j) \,|\, i \geq n_0\}$*,* $\|J_T\| \leq 2 \cdot |T|$*,* $S_T \subseteq_{\mathcal{F}} \mathit{OrdPropCl}_{\{\tilde{a}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$*,* $|S_T| \in O(|T|)$*; the number of all elementary operations of the translation of* T *to* S_T*, is in* $O(|T|)$*; the time and space complexity of the translation of* T *to* S_T*, is in* $O(|T| \cdot \log(1 + n_0 + |T|))$*;*
- (d) *if* $S_T \neq \emptyset, \{\square\}$*, then* $J_T \neq \emptyset$*, for all* $C \in S_T$*,* $\emptyset \neq \mathit{atoms}(C) \cap \tilde{\mathbb{A}} \subseteq \{\tilde{a}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}$*.*

Proof. It is straightforward to prove the following statements:

Let $n_\theta \in \mathbb{N}$ and $\theta \in \mathit{OrdPropForm}_\emptyset$. There exists $\theta' \in \mathit{OrdPropForm}_\emptyset$ such that (4)

- (a) $\theta' \equiv \theta$;
- (b) $|\theta'| \leq 2 \cdot |\theta|$; θ' can be built up from θ via a postorder traversal of θ with $\#O(\theta) \in O(|\theta|)$ and the time, space complexity in $O(|\theta| \cdot (\log(1 + n_\theta) + \log|\theta|))$;
- (c) θ' does not contain $\neg$ and Δ;
- (d) $\theta' \in \{\mathit{0}, \mathit{1}\}$; or for every subformula of θ' of the form $\varepsilon_1 \diamond \varepsilon_2$, $\diamond \in \{\wedge, \&, \vee, \leftrightarrow\}$, $\varepsilon_i \neq \mathit{0}, \mathit{1}$; for every subformula of θ' of the form $\varepsilon_1 \rightarrow \varepsilon_2$, $\varepsilon_1 \neq \mathit{0}, \mathit{1}$, $\varepsilon_2 \neq \mathit{1}$; for every subformula of θ' of the form $\varepsilon_1 \eqcirc \varepsilon_2$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq \{\mathit{0}, \mathit{1}\}$; for every subformula of θ' of the form $\varepsilon_1 \prec \varepsilon_2$, $\varepsilon_1 \neq \mathit{1}$, $\varepsilon_2 \neq \mathit{0}$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq \{\mathit{0}, \mathit{1}\}$.

The proof is by induction on the structure of θ.

Let $n_\theta \in \mathbb{N}$, $\theta \in \mathit{OrdPropForm}_\emptyset - \{\mathit{0}, \mathit{1}\}$, (4c,d) hold for θ; $\mathbb{i} = (n_\theta, j_{\mathbb{i}}) \in \{(n_\theta, j) \,|\, j \in \mathbb{N}\} \subseteq \mathbb{I}$ be an index, $\tilde{a}_{\mathbb{i}} \in \tilde{\mathbb{A}}$. There exist an index set $J = \{(n_\theta, j) \,|\, j_{\mathbb{i}} + 1 \leq j \leq n_J\} \subseteq \{(n_\theta, j) \,|\, j \in \mathbb{N}\} \subseteq \mathbb{I}$ for some n_J, $j_{\mathbb{i}} \leq n_J$, $\mathbb{i} \notin J$, and $S \subseteq_{\mathcal{F}} \mathit{OrdPropCl}_{\{\tilde{a}_{\mathbb{i}}\} \cup \{\tilde{a}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$ such that (5)

- (a) $\|J\| \leq |\theta| - 1$;
- (b) there exists a valuation $\mathfrak{A}$ and $\mathfrak{A} \models \tilde{a}_{\mathbb{i}} \leftrightarrow \theta \in \mathit{OrdPropForm}_{\{\tilde{a}_{\mathbb{i}}\}}$ if and only if there exists a valuation $\mathfrak{A}'$ and $\mathfrak{A}' \models S$, satisfying $\mathfrak{A}|_{\mathbb{O} \cup \{\tilde{a}_{\mathbb{i}}\}} = \mathfrak{A}'|_{\mathbb{O} \cup \{\tilde{a}_{\mathbb{i}}\}}$;
- (c) $|S| \leq 31 \cdot |\theta|$, S can be built up from θ via a preorder traversal of θ with $\#O(\theta) \in O(|\theta|)$;
- (d) for all $C \in S$, $\emptyset \neq \mathit{atoms}(C) \cap \tilde{\mathbb{A}} \subseteq \{\tilde{a}_{\mathbb{i}}\} \cup \{\tilde{a}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$, $\tilde{a}_{\mathbb{i}} \eqcirc \mathit{1}, \tilde{a}_{\mathbb{i}} \prec \mathit{1} \notin S$.

Table 1: Binary interpolation rules for $\wedge$, $\&$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\eqcirc$, $\prec$.

Case		
$\boldsymbol{\theta = \theta_1 \wedge \theta_2}$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \wedge \theta_2)}{\left\{\tilde{a}_{\mathbb{i}_1} \prec \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}_1} \eqcirc \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}} \eqcirc \tilde{a}_{\mathbb{i}_2}, \tilde{a}_{\mathbb{i}_2} \prec \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}} \eqcirc \tilde{a}_{\mathbb{i}_1}, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1, \tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\right\}}$	(6)
$\lvert\text{Consequent}\rvert = 15 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert$		
$\boldsymbol{\theta = \theta_1 \,\&\, \theta_2}$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \,\&\, \theta_2)}{\left\{\tilde{a}_{\mathbb{i}} \eqcirc \tilde{a}_{\mathbb{i}_1} \,\&\, \tilde{a}_{\mathbb{i}_2}, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1, \tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\right\}}$	(7)
$\lvert\text{Consequent}\rvert = 5 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert$		
$\boldsymbol{\theta = \theta_1 \vee \theta_2}$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \vee \theta_2)}{\left\{\tilde{a}_{\mathbb{i}_1} \prec \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}_1} \eqcirc \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}} \eqcirc \tilde{a}_{\mathbb{i}_1}, \tilde{a}_{\mathbb{i}_2} \prec \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}} \eqcirc \tilde{a}_{\mathbb{i}_2}, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1, \tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\right\}}$	(8)
$\lvert\text{Consequent}\rvert = 15 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert$		
$\boldsymbol{\theta = \theta_1 \rightarrow \theta_2, \theta_2 \neq} \mathit{0}$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \rightarrow \theta_2)}{\left\{\tilde{a}_{\mathbb{i}_1} \prec \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}_1} \eqcirc \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}_1} \,\&\, \tilde{a}_{\mathbb{i}} \eqcirc \tilde{a}_{\mathbb{i}_2}, \tilde{a}_{\mathbb{i}_2} \prec \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}} \eqcirc \mathit{1}, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1, \tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\right\}}$	(9)
$\lvert\text{Consequent}\rvert = 17 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert$		
$\boldsymbol{\theta = \theta_1 \leftrightarrow \theta_2}$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \leftrightarrow \theta_2)}{\left\{\begin{array}{l}\tilde{a}_{\mathbb{i}_1} \prec \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}_1} \eqcirc \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}_1} \,\&\, \tilde{a}_{\mathbb{i}} \eqcirc \tilde{a}_{\mathbb{i}_2}, \tilde{a}_{\mathbb{i}_2} \prec \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}_2} \eqcirc \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}_2} \,\&\, \tilde{a}_{\mathbb{i}} \eqcirc \tilde{a}_{\mathbb{i}_1}, \\ \tilde{a}_{\mathbb{i}_1} \prec \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}_2} \prec \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}} \eqcirc \mathit{1}, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1, \tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\end{array}\right\}}$	(10)
$\lvert\text{Consequent}\rvert = 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert$		
$\boldsymbol{\theta = \theta_1 \eqcirc \theta_2, \theta_i \neq} \mathit{0,1}$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \eqcirc \theta_2)}{\left\{\tilde{a}_{\mathbb{i}_1} \eqcirc \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}} \eqcirc \mathit{0}, \tilde{a}_{\mathbb{i}_1} \prec \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}_2} \prec \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}} \eqcirc \mathit{1}, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1, \tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\right\}}$	(11)
$\lvert\text{Consequent}\rvert = 15 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert$		
$\boldsymbol{\theta = \theta_1 \prec \theta_2, \theta_1 \neq} \mathit{0}, \boldsymbol{\theta_2 \neq} \mathit{1}$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \prec \theta_2)}{\left\{\tilde{a}_{\mathbb{i}_1} \prec \tilde{a}_{\mathbb{i}_2} \vee \tilde{a}_{\mathbb{i}} \eqcirc \mathit{0}, \tilde{a}_{\mathbb{i}_2} \prec \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}_2} \eqcirc \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}} \eqcirc \mathit{1}, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1, \tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\right\}}$	(12)
$\lvert\text{Consequent}\rvert = 15 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert + \lvert\tilde{a}_{\mathbb{i}_2} \leftrightarrow \theta_2\rvert$		

The proof is by induction on the structure of θ using the interpolation rules in Tables 1 and 2.

(I) By (4) for n_ϕ, ϕ, there exists $\phi' \in \mathit{OrdPropForm}_\emptyset$ such that (4a–d) hold for n_ϕ, ϕ, ϕ'. We get three cases for ϕ'.

Case 1: $\phi' = \mathit{0}$. We put $J_\phi = \emptyset \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\} \subseteq \mathbb{I}$ and $S_\phi = \{\square\} \subseteq_{\mathcal{F}} \mathit{OrdPropCl}_\emptyset$.

Case 2: $\phi' = \mathit{1}$. We put $J_\phi = \emptyset \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\} \subseteq \mathbb{I}$ and $S_\phi = \emptyset \subseteq_{\mathcal{F}} \mathit{OrdPropCl}_\emptyset$.

Case 3: $\phi' \neq \mathit{0,1}$. We put $j_{\mathbb{i}} = 0$ and $\mathbb{i} = (n_\phi, j_{\mathbb{i}}) \in \{(n_\phi, j) \mid j \in \mathbb{N}\} \subseteq \mathbb{I}$. We get by (5) for n_ϕ, ϕ', $\mathbb{i}$, $\tilde{a}_{\mathbb{i}}$ that there exist $J = \{(n_\phi, j) \mid 1 \leq j \leq n_J\} \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\} \subseteq \mathbb{I}$ for some n_J, $j_{\mathbb{i}} \leq n_J$, $\mathbb{i} \not\in J$, $S \subseteq_{\mathcal{F}} \mathit{OrdPropCl}_{\{\tilde{a}_{\mathbb{i}}\} \cup \{\tilde{a}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, and (5a–d) hold for ϕ', $\tilde{a}_{\mathbb{i}}$, J, S. We put $n_{J_\phi} = n_J$, $J_\phi = \{(n_\phi, j) \mid j \leq n_{J_\phi}\} \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\} \subseteq \mathbb{I}$, $S_\phi = \{\tilde{a}_{\mathbb{i}} \eqcirc \mathit{1}\} \cup S \subseteq_{\mathcal{F}} \mathit{OrdPropCl}_{\{\tilde{a}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}}$. (II) straightforwardly follows from (I). The lemma is proved. $\square$

Theorem 2. *Let* $n_0 \in \mathbb{N}$, $\phi \in \mathit{OrdPropForm}_\emptyset$, $T \subseteq \mathit{OrdPropForm}_\emptyset$. *There exist an index set* $J_T^\phi \subseteq \{(i, j) \mid i \geq n_0\} \subseteq \mathbb{I}$ *and* $S_T^\phi \subseteq \mathit{OrdPropCl}_{\{\tilde{a}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$ *such that*

(i) *there exists a valuation* $\mathfrak{A}$ *and* $\mathfrak{A} \models T$, $\mathfrak{A} \not\models \phi$ *if and only if there exists a valuation* $\mathfrak{A}'$ *and* $\mathfrak{A}' \models S_T^\phi$, *satisfying* $\mathfrak{A}|_{\mathbb{O}} = \mathfrak{A}'|_{\mathbb{O}}$;

(ii) $T \models \phi$ *if and only if* S_T^ϕ *is unsatisfiable;*

Table 2: Unary interpolation rules for $\rightarrow$, $\eqcirc$, $\prec$.

Case		
$\theta = \theta_1 \rightarrow 0$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \rightarrow 0)}{\{\tilde{a}_{\mathbb{i}_1} \eqcirc 0 \vee \tilde{a}_{\mathbb{i}} \eqcirc 0, 0 \prec \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}} \eqcirc 1, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\}}$	(13)
$\lvert\text{Consequent}\rvert = 12 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert$		
$\theta = \theta_1 \eqcirc 0$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \eqcirc 0)}{\{\tilde{a}_{\mathbb{i}_1} \eqcirc 0 \vee \tilde{a}_{\mathbb{i}} \eqcirc 0, 0 \prec \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}} \eqcirc 1, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\}}$	(14)
$\lvert\text{Consequent}\rvert = 12 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert$		
$\theta = \theta_1 \eqcirc 1$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \eqcirc 1)}{\{\tilde{a}_{\mathbb{i}_1} \eqcirc 1 \vee \tilde{a}_{\mathbb{i}} \eqcirc 0, \tilde{a}_{\mathbb{i}_1} \prec 1 \vee \tilde{a}_{\mathbb{i}} \eqcirc 1, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\}}$	(15)
$\lvert\text{Consequent}\rvert = 12 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert$		
$\theta = 0 \prec \theta_1$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (0 \prec \theta_1)}{\{0 \prec \tilde{a}_{\mathbb{i}_1} \vee \tilde{a}_{\mathbb{i}} \eqcirc 0, \tilde{a}_{\mathbb{i}_1} \eqcirc 0 \vee \tilde{a}_{\mathbb{i}} \eqcirc 1, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\}}$	(16)
$\lvert\text{Consequent}\rvert = 12 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert$		
$\theta = \theta_1 \prec 1$	$\dfrac{\tilde{a}_{\mathbb{i}} \leftrightarrow (\theta_1 \prec 1)}{\{\tilde{a}_{\mathbb{i}_1} \prec 1 \vee \tilde{a}_{\mathbb{i}} \eqcirc 0, \tilde{a}_{\mathbb{i}_1} \eqcirc 1 \vee \tilde{a}_{\mathbb{i}} \eqcirc 1, \tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\}}$	(17)
$\lvert\text{Consequent}\rvert = 12 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert \leq 31 + \lvert\tilde{a}_{\mathbb{i}_1} \leftrightarrow \theta_1\rvert$		

(iii) *if* $T \subseteq_{\mathcal{F}} OrdPropForm_{\emptyset}$, *then* $J_T^{\phi} \subseteq_{\mathcal{F}} \{(i,j) \mid i \geq n_0\}$, $\|J_T^{\phi}\| \in O(|T| + |\phi|)$, $S_T^{\phi} \subseteq_{\mathcal{F}} OrdPropCl_{\{\tilde{a}_{\mathbb{j}} \mid \mathbb{j} \in J_T^{\phi}\}}$, $|S_T^{\phi}| \in O(|T| + |\phi|)$*; the number of all elementary operations of the translation of* T *and* ϕ *to* S_T^{ϕ}*, is in* $O(|T| + |\phi|)$*; the time and space complexity of the translation of* T *and* ϕ *to* S_T^{ϕ}*, is in* $O(|T| \cdot \log(1 + n_0 + |T|) + |\phi| \cdot (\log(1 + n_0) + \log|\phi|))$.

Proof. We get by Lemma 1(II) for $n_0 + 1$, T that there exist $J_T \subseteq \{(i,j) \mid i \geq n_0 + 1\} \subseteq \mathbb{I}$, $S_T \subseteq OrdPropCl_{\{\tilde{a}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}}$, and Lemma 1(II a–d) hold for $n_0 + 1$, T, J_T, S_T. By (4) for n_0, ϕ, there exists $\phi' \in OrdPropForm_{\emptyset}$ such that (4a–d) hold for n_0, ϕ, ϕ'. We get three cases for ϕ'.

Case 1: $\phi' = 0$. We put $J_T^{\phi} = J_T \subseteq \{(i,j) \mid i \geq n_0 + 1\} \subseteq \{(i,j) \mid i \geq n_0\} \subseteq \mathbb{I}$ and $S_T^{\phi} = S_T \subseteq OrdPropCl_{\{\tilde{a}_{\mathbb{j}} \mid \mathbb{j} \in J_T^{\phi}\}}$.

Case 2: $\phi' = 1$. We put $J_T^{\phi} = \emptyset \subseteq \{(i,j) \mid i \geq n_0\} \subseteq \mathbb{I}$ and $S_T^{\phi} = \{\square\} \subseteq OrdPropCl_{\emptyset}$.

Case 3: $\phi' \neq 0, 1$. We put $j_{\mathbb{i}} = 0$ and $\mathbb{i} = (n_0, j_{\mathbb{i}}) \in \{(n_0, j) \mid j \in \mathbb{N}\} \subseteq \mathbb{I}$. We get by (5) for n_0, ϕ', $\mathbb{i}$, $\tilde{a}_{\mathbb{i}}$ that there exist $J = \{(n_0, j) \mid 1 \leq j \leq n_J\} \subseteq \{(n_0, j) \mid j \in \mathbb{N}\} \subseteq \mathbb{I}$ for some n_J, $j_{\mathbb{i}} \leq n_J$, $\mathbb{i} \notin J$, $S \subseteq_{\mathcal{F}} OrdPropCl_{\{\tilde{a}_{\mathbb{i}}\} \cup \{\tilde{a}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, and (5a–d) hold for ϕ', $\tilde{a}_{\mathbb{i}}$, J, S. We put $J_T^{\phi} = J_T \cup \{\mathbb{i}\} \cup J \subseteq \{(i,j) \mid i \geq n_0\} \subseteq \mathbb{I}$ and $S_T^{\phi} = S_T \cup \{\tilde{a}_{\mathbb{i}} \prec 1\} \cup S \subseteq OrdPropCl_{\{\tilde{a}_{\mathbb{j}} \mid \mathbb{j} \in J_T^{\phi}\}}$. The theorem is proved. □

4 HYPERRESOLUTION OVER ORDER CLAUSES

In this section, we propose an order hyperresolution calculus operating over order clausal theories, and prove its refutational soundness, completeness. At first, we introduce some basic notions and notation. Let $l \in OrdPropLit$. l is a contradiction iff $l = 0 \eqcirc 1$ or $l = \varepsilon \prec 0$ or $l = 1 \prec \varepsilon$ or $l = \varepsilon \prec \varepsilon$. Let $Cn \in PropConj$ and $C \in OrdPropCl$. We define an auxiliary function *simplify* : $(\{0,1\} \cup PropConj \cup OrdPropLit \cup OrdPropCl) \times PropAtom \times \{0,1\} \longrightarrow \{0,1\} \cup PropConj \cup OrdPropLit \cup OrdPropCl$ as follows:

$$simplify(0,a,\upsilon) = 0;$$
$$simplify(1,a,\upsilon) = 1;$$
$$simplify(Cn,a,0) = \begin{cases} 0 & \text{if } a \in atoms(Cn), \\ Cn & \text{else}; \end{cases}$$
$$simplify(Cn,a,1) = \begin{cases} 1 & \text{if } \exists n^*\, Cn = a^{n^*}, \\ Cn - a^{n^*} & \text{if } \exists n^*\, a^{n^*} \in Cn \neq a^{n^*}, \\ Cn & \text{else}; \end{cases}$$
$$simplify(l,a,\upsilon) = simplify(\varepsilon_1,a,\upsilon) \diamond simplify(\varepsilon_2,a,\upsilon) \quad \text{if } l = \varepsilon_1 \diamond \varepsilon_2;$$
$$simplify(C,a,\upsilon) = \{simplify(l,a,\upsilon) \,|\, l \in C\}.$$

For an input expression, atom, truth constant, *simplify* replaces every occurrence of the atom by the truth constant in the expression, and returns a simplified expression according to laws holding in Π. Let $Cn_1, Cn_2 \in PropConj$ and $l_1, l_2 \in OrdPropLit$. Another auxiliary function $\odot : (\{0,1\} \cup PropConj) \times (\{0,1\} \cup PropConj) \longrightarrow \{0,1\} \cup PropConj$ is defined as follows:

$$0 \odot \varepsilon = \varepsilon \odot 0 = 0;$$
$$1 \odot \varepsilon = \varepsilon \odot 1 = \varepsilon;$$
$$Cn_1 \odot Cn_2 = \{a^{m+n} \,|\, a^m \in Cn_1, a^n \in Cn_2\} \cup \{a^n \,|\, a^n \in Cn_1, a \notin atoms(Cn_2)\} \cup \{a^n \,|\, a^n \in Cn_2, a \notin atoms(Cn_1)\}.$$

For two input expressions, $\odot$ returns the product of them. It can be extended to $\{0,1\} \cup OrdPropLit$ component-wisely. $\odot : (\{0,1\} \cup OrdPropLit) \times (\{0,1\} \cup OrdPropLit) \longrightarrow \{0,1\} \cup OrdPropLit$ is defined as

$$0 \odot \varepsilon = \varepsilon \odot 0 = 0;$$
$$1 \odot \varepsilon = \varepsilon \odot 1 = \varepsilon;$$
$$l_1 \odot l_2 = (\varepsilon_1 \odot \varepsilon_2) \diamond (\upsilon_1 \odot \upsilon_2) \text{ if } l_i = \varepsilon_i \diamond_i \upsilon_i,$$
$$\diamond = \begin{cases} \eqsim \text{ if } \diamond_1 = \diamond_2 = \eqsim, \\ \prec \text{ else}. \end{cases}$$

Note that $\odot$ is a binary commutative, associative operator. We denote $l^n = \underbrace{l \odot \cdots \odot l}_{n}$, $n \geq 1$, and say that l^n is the n-th power of l. Let $I \subseteq_{\mathcal{F}} \mathbb{N}$, $l_i \in OrdPropLit$, $\alpha_i \geq 1$, $i \in I$. We define by recursion on I:

$$\bigodot_{i \in I} l_i^{\alpha_i} = \begin{cases} 1 & \text{if } I = \emptyset, \\ l_{i^*}^{\alpha_{i^*}} \odot \big(\bigodot_{i \in I - \{i^*\}} l_i^{\alpha_i} \big) & \text{if } \exists i^* \in I. \end{cases}$$

Let $S \subseteq OrdPropCl$. The basic order hyperresolution calculus is defined as follows. The first rule is the central order hyperresolution one.

(*Order hyperresolution rule*) (18)

$$\frac{0 \prec a_0, \dots, 0 \prec a_m, a_0 \prec 1, \dots, a_m \prec 1, \; l_0 \vee C_0, \dots, l_n \vee C_n \in S_{\kappa-1}}{\bigvee_{i=0}^{n} C_i \in S_\kappa};$$

$atoms(l_0, \dots, l_n) = \{a_0, \dots, a_m\} \subseteq PropAtom - \{0,1\}$,
$l_i = Cn_1^i \diamond^i Cn_2^i, Cn_j^i \in PropConj$,
there exist $\alpha_i^* \geq 1, i = 0, \dots, n, J^* \subseteq \{j \,|\, j \leq m\}$,
$\beta_j^* \geq 1, j \in J^*$, *such that*
$\big(\odot_{i=0}^{n} l_i^{\alpha_i^*}\big) \odot \big(\odot_{j \in J^*} (a_j \prec 1)^{\beta_j^*}\big)$ *is a contradiction.*

If there exists a product of powers of the input order literals $l_0, \dots, l_n$ and of some so-called literals-guards $a_j \prec 1$, $j \in J^*$, which is a contradiction of the form $\varepsilon \prec \varepsilon$, then we can derive the output order clause $\bigvee_{i=0}^{n} C_i$ consisting of the remainder order clauses C_i, $i \leq n$. We say that $\bigvee_{i=0}^{n} C_i$ is an order hyperresolvent of $0 \prec a_1, \dots, 0 \prec a_m, a_1 \prec 1, \dots, a_m \prec 1, l_0 \vee C_0, \dots, l_n \vee C_n$.

(*Order contradiction rule*) (19)

$$\frac{l \vee C \in S_{\kappa-1}}{C \in S_\kappa};$$

l is a contradiction.

If the order literal l is a contradiction, then it can be removed from the input order clause $l \vee C$. C is an order contradiction resolvent of $l \vee C$.

(*Order 0-simplification rule*) (20)

$$\frac{a \eqsim 0, C \in S_{\kappa-1}}{simplify(C,a,0) \in S_\kappa};$$

$a \in atoms(C)$.

If a so-called literal-guard $a \eqsim 0$ is in the antecedent order clausal theory and the input order clause C contains the atom a, then C can be simplified using the auxiliary function *simplify*. $simplify(C,a,0)$ is an order 0-simplification resolvent of $a \eqsim 0$ and C. Analogously, C can be simplified with respect to a literal-guard $a \eqsim 1$.

(*Order 1-simplification rule*) (21)

$$\frac{a \eqsim 1, C \in S_{\kappa-1}}{simplify(C,a,1) \in S_\kappa};$$

$a \in atoms(C)$.

$simplify(C,a,1)$ is an order 1-simplification resolvent of $a \eqsim 1$ and C.

(*Order 0-contradiction rule*) (22)

$$\frac{a_0^{\alpha_0} \& \cdots \& a_n^{\alpha_n} \eqsim 0 \vee C, 0 \prec a_0, \dots, 0 \prec a_n \in S_{\kappa-1}}{C \in S_\kappa}.$$

C is an order *0*-contradiction resolvent of $a_0^{\alpha_0} \& \cdots \& a_n^{\alpha_n} \eqcirc 0 \vee C, 0 \prec a_0, \dots, 0 \prec a_n$.

(*Order 1-contradiction rule*) (23)

$$\frac{a_0^{\alpha_0} \& \cdots \& a_n^{\alpha_n} \eqcirc 1 \vee C, a_i \prec 1 \in S_{\kappa-1}}{C \in S_\kappa};$$

$i \le n$.

C is an order *1*-contradiction resolvent of $a_0^{\alpha_0} \& \cdots \& a_n^{\alpha_n} \eqcirc 1 \vee C$ and $a_i \prec 1$. The last two rules detect a contradictory set of order literals of the form either $\{a_0^{\alpha_0} \& \cdots \& a_n^{\alpha_n} \eqcirc 0, 0 \prec a_0, \dots, 0 \prec a_n\}$ or $\{a_0^{\alpha_0} \& \cdots \& a_n^{\alpha_n} \eqcirc 1, a_i \prec 1\}$, $i \le n$. In either case, the remainder order clause C can be derived. Note that all the rules are sound; for every rule, the consequent order clausal theory is a propositional consequence of the antecedent one.

Let $S_0 = \emptyset \subseteq OrdPropCl$. Let $\mathcal{D} = C_1, \dots, C_n$, $C_\kappa \in OrdPropCl, n \ge 1$. $\mathcal{D}$ is a deduction of C_n from S by order hyperresolution iff, for all $1 \le \kappa \le n$, $C_\kappa \in S$, or there exist $1 \le j_k^* \le \kappa - 1$, $k = 0, \dots, m$, such that C_κ is an order resolvent of $C_{j_0^*}, \dots, C_{j_m^*} \in S_{\kappa-1}$ using Rule (18)–(23) with respect to $S_{\kappa-1}$; S_κ is defined by recursion on $1 \le \kappa \le n$ as follows:

$$S_\kappa = S_{\kappa-1} \cup \{C_\kappa\} \subseteq OrdPropCl.$$

$\mathcal{D}$ is a refutation of S iff $C_n = \square$. We denote

$$clo^{\mathcal{H}}(S) = \{C \mid \textit{there exists a deduction of } C \textit{ from } S \textit{ by order hyperresolution}\} \subseteq OrdPropCl.$$

Lemma 3. *Let $S \subseteq_{\mathcal{F}} OrdPropCl$. $clo^{\mathcal{H}}(S) \subseteq_{\mathcal{F}} OrdPropCl$.*

Proof. Straightforward. □

Lemma 4. *Let $A = \{a_i \mid 1 \le i \le m\} \subseteq PropAtom - \{0,1\}$, $S_1 = \{0 \prec a_i \mid 1 \le i \le m\} \cup \{a_i \prec 1 \mid 1 \le i \le m\} \subseteq OrdPropCl$, $S_2 = \{Cn_1^i \diamond^i Cn_2^i \mid Cn_j^i \in PropConj, 1 \le i \le n\} \subseteq OrdPropCl$, $atoms(S_2) = A$, $S = S_1 \cup S_2 \subseteq OrdPropCl$, there not exist an application of Rule* (18) *with respect to S. S is satisfiable.*

Proof. S is unit. Note that an application of Rule (18) with respect to S would derive □. We denote $PropConj_A = \{Cn \mid Cn \in PropConj, atoms(Cn) \subseteq A\} \subseteq PropConj$. Let $Cn_1, Cn_2 \in PropConj_A$ and $Cn_2 \sqsubset Cn_1$. We define

$$cancel(Cn_1, Cn_2) = \{a^{r-s} \mid a^r \in Cn_1, a^s \in Cn_2, r > s\} \cup \{a^r \mid a^r \in Cn_1, a \notin atoms(Cn_2)\} \in PropConj_A.$$

We further denote

$$gen = \{Cn_1 \eqcirc Cn_2 \mid Cn_i \in PropConj_A, \textit{ there exist } \emptyset \ne I^* \subseteq \{i \mid 1 \le i \le n\}, \alpha_i^* \ge 1, i \in I^*, Cn_1 \eqcirc Cn_2 = \bigodot_{i \in I^*} (Cn_1^i \diamond^i Cn_2^i)^{\alpha_i^*}\} \cup$$
$$\{Cn_1 \prec Cn_2 \mid Cn_i \in PropConj_A, \textit{ there exist } \emptyset \ne I^* \subseteq \{i \mid 1 \le i \le n\}, \alpha_i^* \ge 1, i \in I^*, J^* \subseteq \{j \mid 1 \le j \le m\}, \beta_j^* \ge 1, j \in J^*, Cn_1 \prec Cn_2 = \big(\bigodot_{i \in I^*} (Cn_1^i \diamond^i Cn_2^i)^{\alpha_i^*}\big) \odot \big(\bigodot_{j \in J^*} (a_j \prec 1)^{\beta_j^*}\big)\}$$
$$\subseteq OrdPropLit,$$

$$cnl = \{Cn_1 \diamond Cn_2 \mid Cn_i \in PropConj_A, \textit{ there exist } Cn_1^* \diamond Cn_2^* \in gen, Cn^* \in PropConj_A, Cn^* \sqsubset Cn_i^*, Cn_i = cancel(Cn_i^*, Cn^*)\} \subseteq OrdPropLit,$$

$$clo = gen \cup cnl \subseteq OrdPropLit.$$

Then $S_2 \subseteq gen \subseteq clo$.

For all $Cn \in PropConj_A, Cn \prec Cn \notin gen, clo$. (24)

The proof is straightforward; we have that there does not exist an application of Rule (18) with respect to S.

$A \cap \{0,1\} \subseteq (PropAtom - \{0,1\}) \cap \{0,1\} = \emptyset$. Let $\{0,1\} \subseteq X \subseteq \{0,1\} \cup A$. A partial valuation $\mathcal{V}$ is a mapping $\mathcal{V} : X \longrightarrow [0,1]$ such that $\mathcal{V}(0) = 0$ and $\mathcal{V}(1) = 1$. We denote $dom(\mathcal{V}) = X$, $\{0,1\} \subseteq dom(\mathcal{V}) \subseteq \{0,1\} \cup A$. We define a partial valuation $\mathcal{V}_\iota$ by recursion on $\iota \le m$ in Table 3.

For all $\iota \le \iota' \le m$, $\mathcal{V}_\iota$ is a partial valuation, $dom(\mathcal{V}_\iota) = \{0,1\} \cup \{a_1, \dots, a_\iota\}$, $\mathcal{V}_\iota \subseteq \mathcal{V}_{\iota'}$. (25)

The proof is by induction on $\iota \le m$.

For all $\iota \le m$, for all $a \in dom(\mathcal{V}_\iota) - \{0,1\}$, $Cn_1, Cn_2 \in PropConj_A$ and $atoms(Cn_i) \subseteq dom(\mathcal{V}_\iota) - \{0,1\}$,
$0 < \mathcal{V}_\iota(a) < 1$;
if $Cn_1 \eqcirc Cn_2 \in clo$, then $\|Cn_1\|^{\mathcal{V}_\iota} = \|Cn_2\|^{\mathcal{V}_\iota}$;
if $Cn_1 \prec Cn_2 \in clo$, then $\|Cn_1\|^{\mathcal{V}_\iota} < \|Cn_2\|^{\mathcal{V}_\iota}$. (26)

The proof is by induction on $\iota \le m$.

$atoms(S_1) = \{0,1\} \cup A$ and $atoms(S) = atoms(S_1) \cup atoms(S_2) = \{0,1\} \cup A$. We put $\mathcal{V} = \mathcal{V}_m$, $dom(\mathcal{V}) \overset{(25)}{=\!=} \{0,1\} \cup \{a_1, \dots, a_m\} = \{0,1\} \cup A = atoms(S)$.

For all $a \in A, Cn_1, Cn_2 \in PropConj_A$,
$0 < \mathcal{V}(a) < 1$;
if $Cn_1 \eqcirc Cn_2 \in clo$, then $\|Cn_1\|^{\mathcal{V}} = \|Cn_2\|^{\mathcal{V}}$;
if $Cn_1 \prec Cn_2 \in clo$, then $\|Cn_1\|^{\mathcal{V}} < \|Cn_2\|^{\mathcal{V}}$. (27)

Table 3: A partial valuation $\mathcal{V}_\iota$.

$$\mathcal{V}_0 = \{(\mathit{0},0),(\mathit{1},1)\};$$

$$\mathcal{V}_\iota = \mathcal{V}_{\iota-1} \cup \{(a_\iota,\lambda_\iota)\} \quad (1 \le \iota \le m),$$

$$\mathbb{E}_{\iota-1} = \left\{ \left(\frac{\|Cn_2\|^{\mathcal{V}_{\iota-1}}}{\|Cn_1\|^{\mathcal{V}_{\iota-1}}} \right)^{\frac{1}{\alpha}} \,\middle|\, \begin{array}{l} Cn_1 \,\&\, a_\iota^\alpha \eqcirc Cn_2 \in clo, \\ atoms(Cn_i) \subseteq dom(\mathcal{V}_{\iota-1}) \end{array} \right\} \cup \left\{ \left(\|Cn_2\|^{\mathcal{V}_{\iota-1}} \right)^{\frac{1}{\alpha}} \,\middle|\, \begin{array}{l} a_\iota^\alpha \eqcirc Cn_2 \in clo, \\ atoms(Cn_2) \subseteq dom(\mathcal{V}_{\iota-1}) \end{array} \right\},$$

$$\mathbb{D}_{\iota-1} = \left\{ \left(\frac{\|Cn_2\|^{\mathcal{V}_{\iota-1}}}{\|Cn_1\|^{\mathcal{V}_{\iota-1}}} \right)^{\frac{1}{\alpha}} \,\middle|\, \begin{array}{l} Cn_2 \prec Cn_1 \,\&\, a_\iota^\alpha \in clo, \\ atoms(Cn_i) \subseteq dom(\mathcal{V}_{\iota-1}) \end{array} \right\} \cup \left\{ \left(\|Cn_2\|^{\mathcal{V}_{\iota-1}} \right)^{\frac{1}{\alpha}} \,\middle|\, \begin{array}{l} Cn_2 \prec a_\iota^\alpha \in clo, \\ atoms(Cn_2) \subseteq dom(\mathcal{V}_{\iota-1}) \end{array} \right\},$$

$$\mathbb{U}_{\iota-1} = \left\{ \left(\frac{\|Cn_2\|^{\mathcal{V}_{\iota-1}}}{\|Cn_1\|^{\mathcal{V}_{\iota-1}}} \right)^{\frac{1}{\alpha}} \,\middle|\, \begin{array}{l} Cn_1 \,\&\, a_\iota^\alpha \prec Cn_2 \in clo, \\ atoms(Cn_i) \subseteq dom(\mathcal{V}_{\iota-1}) \end{array} \right\} \cup \left\{ \left(\|Cn_2\|^{\mathcal{V}_{\iota-1}} \right)^{\frac{1}{\alpha}} \,\middle|\, \begin{array}{l} a_\iota^\alpha \prec Cn_2 \in clo, \\ atoms(Cn_2) \subseteq dom(\mathcal{V}_{\iota-1}) \end{array} \right\},$$

$$\lambda_\iota = \begin{cases} \dfrac{\bigvee \mathbb{D}_{\iota-1} + \bigwedge \mathbb{U}_{\iota-1}}{2} & \mathit{if}\ \mathbb{E}_{\iota-1} = \emptyset, \\ \bigvee \mathbb{E}_{\iota-1} & \mathit{else}. \end{cases}$$

The proof is by (26) for m.

We put $\mathfrak{A} = \mathcal{V} \cup \{(a,0) \,|\, a \in \mathit{PropAtom} - \mathit{dom}(\mathcal{V})\}$; $\mathfrak{A}$ is a valuation. Let $l \in S$. Then $l \in \mathit{OrdPropLit}$ and $atoms(l) \subseteq atoms(S) = dom(\mathcal{V})$. We get two cases for l.

Case 1: $l \in S_1$, either $l = \mathit{0} \prec a$ or $l = a \prec \mathit{1}$. Hence, $a \in A$, by (27) for a, either $\mathfrak{A}(\mathit{0}) = \mathcal{V}(\mathit{0}) = 0 < \mathcal{V}(a) = \mathfrak{A}(a)$ or $\mathfrak{A}(a) = \mathcal{V}(a) < 1 = \mathcal{V}(\mathit{1}) = \mathfrak{A}(\mathit{1})$, $\mathfrak{A} \models l$.

Case 2: $l \in S_2$, either $l = Cn_1 \eqcirc Cn_2$ or $l = Cn_1 \prec Cn_2$. Hence, $l \in S_2 \subseteq clo$, either $Cn_1 \eqcirc Cn_2 \in clo$ or $Cn_1 \prec Cn_2 \in clo$, $Cn_1, Cn_2 \in \mathit{PropConj}_A$, by (27) for Cn_1, Cn_2, either $\|Cn_1\|^{\mathfrak{A}} = \|Cn_1\|^{\mathcal{V}} = \|Cn_2\|^{\mathcal{V}} = \|Cn_2\|^{\mathfrak{A}}$ or $\|Cn_1\|^{\mathfrak{A}} = \|Cn_1\|^{\mathcal{V}} < \|Cn_2\|^{\mathcal{V}} = \|Cn_2\|^{\mathfrak{A}}$, $\mathfrak{A} \models l$.

So, in both Cases 1 and 2, $\mathfrak{A} \models l$; $\mathfrak{A} \models S$; S is satisfiable. □

Lemma 5 (Reduction Lemma). *Let* $A = \{a_i \,|\, i \le m\} \subseteq \mathit{PropAtom} - \{\mathit{0},\mathit{1}\}$, $S_1 = \{\mathit{0} \prec a_i \,|\, i \le m\} \cup \{a_i \prec \mathit{1} \,|\, i \le m\} \subseteq \mathit{OrdPropCl}$, $S_2 = \{(\bigvee_{j=0}^{k_i} Cn1_j^i \diamond_j^i Cn2_j^i) \vee C_i \,|\, Cn1_j^i, Cn2_j^i \in \mathit{PropConj}, i \le n\} \subseteq \mathit{OrdPropCl}$, $atoms(S_2) = A$, $S = S_1 \cup S_2 \subseteq \mathit{OrdPropCl}$ *such that for all* $\mathcal{S} \in Sel(\{\{j \,|\, j \le k_i\}_i \,|\, i \le n\})$, *there exists an application of Rule* (18) *with respect to* $S_1 \cup \{Cn1_{\mathcal{S}(i)}^i \diamond_{\mathcal{S}(i)}^i Cn2_{\mathcal{S}(i)}^i \,|\, i \le n\} \subseteq \mathit{OrdPropCl}$. *There exists* $\emptyset \neq I^* \subseteq \{i \,|\, i \le n\}$ *such that* $\bigvee_{i \in I^*} C_i \in clo^{\mathcal{H}}(S)$.

Proof. Analogous to the one of Proposition 2, (Guller, 2009). □

Let $S \subseteq \mathit{OrdPropCl}$. S is a guarded order clausal theory iff, for all $a \in atoms(S) - \{\mathit{0},\mathit{1}\}$, either $a \eqcirc \mathit{0} \in S$ or $\mathit{0} \prec a, a \prec \mathit{1} \in S$ or $a \eqcirc \mathit{1} \in S$. Let $l \in \mathit{OrdPropLit}$ and $a \in \mathit{PropAtom} - \{\mathit{0},\mathit{1}\}$. l is a guard iff either $l = a \eqcirc \mathit{0}$ or $l = \mathit{0} \prec a$ or $l = a \prec \mathit{1}$ or $l = a \eqcirc \mathit{1}$. We denote $guards(S) = \{l \,|\, l \in S \text{ is a guard}\} \subseteq S$.

Lemma 6 (Normalisation Lemma). *Let* $S \subseteq_{\mathcal{F}} \mathit{OrdPropCl}$ *be guarded. There exists* $S^* \subseteq_{\mathcal{F}} clo^{\mathcal{H}}(S)$ *such that there exist* $A = \{a_i \,|\, 1 \le i \le m\} \subseteq \mathit{PropAtom} - \{\mathit{0},\mathit{1}\}$ *for some* m, $S_1 = \{\mathit{0} \prec a_i \,|\, 1 \le i \le m\} \cup \{a_i \prec \mathit{1} \,|\, 1 \le i \le m\} \subseteq \mathit{OrdPropCl}$, $S_2 = \{\bigvee_{j=1}^{k_i} Cn1_j^i \diamond_j^i Cn2_j^i \,|\, Cn1_j^i, Cn2_j^i \in \mathit{PropConj}, 1 \le i \le n\} \subseteq \mathit{OrdPropCl}$ *for some* n; *and* $atoms(S_2) = A$, $S^* = S_1 \cup S_2$, $guards(S^*) = S_1$, S^* *is guarded;* S^* *is equisatisfiable to* S.

Proof. Let $B_\mathit{0} = \{b \,|\, b \eqcirc \mathit{0} \in guards(S)\} \subseteq atoms(S) - \{\mathit{0},\mathit{1}\}$ and $B_\mathit{1} = \{b \,|\, b \eqcirc \mathit{1} \in guards(S)\} \subseteq atoms(S) - \{\mathit{0},\mathit{1}\}$. Then, for all $b \in B_\mathit{0}$, $clo^{\mathcal{H}}(S)$ is closed with respect to applications of Rule (20); for all $b \in B_\mathit{1}$, $clo^{\mathcal{H}}(S)$ is closed with respect to applications of Rule (21); $clo^{\mathcal{H}}(S)$ is closed with respect to applications of Rule (19); $clo^{\mathcal{H}}(S)$ is closed with respect to applications of Rule (22); $clo^{\mathcal{H}}(S)$ is closed with respect to applications of Rule (23); the order clausal theory in the antecedent is equisatisfiable to the one in the consequent of every Rule (19), (20)–(23). By Lemma 3 for S, $clo^{\mathcal{H}}(S) \subseteq_{\mathcal{F}} \mathit{OrdPropCl}$. We put $S_2 = \{C \,|\, C = \bigvee_{j=1}^{k} Cn_{1j} \diamond_j Cn_{2j} \in clo^{\mathcal{H}}(S), Cn_{1j}, Cn_{2j} \in \mathit{PropConj}, atoms(C) \cap (B_\mathit{0} \cup B_\mathit{1}) = \emptyset\} \subseteq_{\mathcal{F}} clo^{\mathcal{H}}(S)$, $A = atoms(S_2) \subseteq \mathit{PropAtom} - \{\mathit{0},\mathit{1}\}$, $S_1 = \{\mathit{0} \prec a \,|\, a \in A, \mathit{0} \prec a \in guards(S)\} \cup \{a \prec \mathit{1} \,|\, a \in A, a \prec \mathit{1} \in guards(S)\} \subseteq S \subseteq_{\mathcal{F}} clo^{\mathcal{H}}(S)$, $S^* = S_1 \cup S_2 \subseteq_{\mathcal{F}} clo^{\mathcal{H}}(S)$. Hence, $guards(S^*) = S_1$, S^* is guarded; S^* is equisatisfiable to S. □

Theorem 7 (Refutational Soundness and Completeness). *Let* $S \subseteq_{\mathcal{F}} \mathit{OrdPropCl}$ *be guarded.* $\square \in clo^{\mathcal{H}}(S)$ *if and only if* S *is unsatisfiable.*

Proof. ($\Longrightarrow$) Let $\mathfrak{A}$ be a model of S and $C \in clo^{\mathcal{H}}(S)$. Then $\mathfrak{A} \models C$. The proof is by complete induction on the length of a deduction of C from S by order hyperresolution. Let $\square \in clo^{\mathcal{H}}(S)$ and $\mathfrak{A}$ be a model of S. Hence, $\mathfrak{A} \models \square$, which is a contradiction; S is unsatisfiable.

($\Longleftarrow$) Let $\square \notin clo^{\mathcal{H}}(S)$. Then, by Lemma 6 for S, there exists $S^* \subseteq_{\mathcal{F}} clo^{\mathcal{H}}(S)$ such that there exist $A = \{a_i \,|\, 1 \leq i \leq m\} \subseteq PropAtom - \{0,1\}$ for some m, $S_1 = \{0 \prec a_i \,|\, 1 \leq i \leq m\} \cup \{a_i \prec 1 \,|\, 1 \leq i \leq m\} \subseteq OrdPropCl$, $S_2 = \{\bigvee_{j=1}^{k_i} Cn1_j^i \diamond_j^i Cn2_j^i \,|\, Cn1_j^i, Cn2_j^i \in PropConj, 1 \leq i \leq n\} \subseteq OrdPropCl$ for some n; and $atoms(S_2) = A$, $S^* = S_1 \cup S_2$, S^* is equisatisfiable to S; $\square \notin clo^{\mathcal{H}}(S^*)$. We get two cases for S^*.

Case 1: $S^* = \emptyset$. Then S^* is satisfiable, and S is satisfiable.

Case 2: $S^* \neq \emptyset$. Then $m,n \geq 1$, for all $1 \leq i \leq n$, $k_i \geq 1$, by Lemma 5 for S^*, there exists $\mathcal{S}^* \in \mathcal{S}el(\{\{j \,|\, 1 \leq j \leq k_i\}_i \,|\, 1 \leq i \leq n\})$ such that there does not exist an application of Rule (18) with respect to $S_1 \cup \{Cn1_{\mathcal{S}^*(i)}^i \diamond_{\mathcal{S}^*(i)}^i Cn2_{\mathcal{S}^*(i)}^i \,|\, 1 \leq i \leq n\} \subseteq OrdPropCl$. We put $S_2' = \{Cn1_{\mathcal{S}^*(i)}^i \diamond_{\mathcal{S}^*(i)}^i Cn2_{\mathcal{S}^*(i)}^i \,|\, 1 \leq i \leq n\} \subseteq OrdPropCl$, $A' = atoms(S_2') \subseteq_{\mathcal{F}} PropAtom - \{0,1\}$, $S_1' = \{0 \prec a \,|\, 0 \prec a \in S_1, a \in A'\} \cup \{a \prec 1 \,|\, a \prec 1 \in S_1, a \in A'\} \subseteq_{\mathcal{F}} OrdPropCl$, $S' = S_1' \cup S_2' \subseteq OrdPropCl$. Hence, $atoms(S_2') = A'$, $S_1' \subseteq S_1$, $S' = S_1' \cup S_2' \subseteq S_1 \cup S_2'$, there does not exist an application of Rule (18) with respect to S'; by Lemma 4 for S', S' is satisfiable; $S_1 \cup S_2'$ is satisfiable; S^* is satisfiable; S is satisfiable.

So, in both Cases 1 and 2, S is satisfiable. The theorem is proved. $\square$

Let $S \subseteq S' \subseteq OrdPropCl$. S' is a guarded extension of S iff S' is guarded and minimal with respect to $\subseteq$.

Theorem 8 (Satisfiability Problem)**.** *Let $S \subseteq_{\mathcal{F}} OrdPropCl$. S is satisfiable if and only if there exists a guarded extension $S' \subseteq_{\mathcal{F}} OrdPropCl$ of S which is satisfiable.*

Proof. ($\Longrightarrow$) Let S be satisfiable and $\mathfrak{A}$ be a model of S. Then $atoms(S) \subseteq_{\mathcal{F}} PropAtom$. We put $S_1 = \{a \eqcirc 0 \,|\, a \in atoms(S) - \{0,1\}, \mathfrak{A}(a) = 0\} \cup \{0 \prec a \,|\, a \in atoms(S) - \{0,1\}, 0 < \mathfrak{A}(a) < 1\} \cup \{a \prec 1 \,|\, a \in atoms(S) - \{0,1\}, 0 < \mathfrak{A}(a) < 1\} \cup \{a \eqcirc 1 \,|\, a \in atoms(S) - \{0,1\}, \mathfrak{A}(a) = 1\} \subseteq_{\mathcal{F}} OrdPropCl$ and $S' = S_1 \cup S \subseteq_{\mathcal{F}} OrdPropCl$. Hence, S' is a guarded extension of S, for all $l \in S_1$, $\mathfrak{A} \models l$; $\mathfrak{A} \models S_1$; $\mathfrak{A} \models S'$; S' is satisfiable.

($\Longleftarrow$) Let there exist a guarded extension $S' \subseteq_{\mathcal{F}} OrdPropCl$ of S which is satisfiable. Then $S \subseteq S'$ is satisfiable. The theorem is proved. $\square$

Corollary 9. *Let $n_0 \in \mathbb{N}$, $\phi \in OrdPropForm_{\emptyset}$, $T \subseteq_{\mathcal{F}} OrdPropForm_{\emptyset}$. There exist $J_T^{\phi} \subseteq_{\mathcal{F}} \{(i,j) \,|\, i \geq n_0\}$ and $S_T^{\phi} \subseteq_{\mathcal{F}} OrdPropCl_{\{\tilde{a}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^{\phi}\}}$ such that $T \models \phi$ if and only if, for every guarded extension $S' \subseteq_{\mathcal{F}} OrdPropCl$ of S_T^{ϕ}, $\square \in clo^{\mathcal{H}}(S')$.*

Proof. An immediate consequence of Theorems 2, 7, and 8. $\square$

We illustrate the solution to the deduction problem with an example. We show that $\phi = (0 \prec c) \,\&\, (a \,\&\, c \prec b \,\&\, c) \rightarrow a \prec b \in OrdPropForm$ is a tautology using the proposed translation to clausal form and the order hyperresolution calculus. Let $\mathcal{V}$ be a valuation. Let there exist $p^* \in \{a,b,c\}$ and $\mathcal{V}(p^*) \in \{0,1\}$. Then $\mathcal{V} \models \phi$. Hence, it suffices to examine the case that for all $p \in \{a,b,c\}$, $0 < \mathcal{V}(p) < 1$. We put $S_0 = \{0 \prec a, a \prec 1, 0 \prec b, b \prec 1, 0 \prec c, c \prec 1\}$. Let there exist $p^* \in \{\tilde{a}_5, \dots, \tilde{a}_7, \tilde{a}_{10}, \dots, \tilde{a}_{13}\}$ and $\mathcal{V}(p^*) \in \{0,1\}$. Then $\mathcal{V} \not\models S_0 \cup S^{\phi}$. Hence, it suffices to examine the case that for all $p \in \{\tilde{a}_5, \dots, \tilde{a}_7, \tilde{a}_{10}, \dots, \tilde{a}_{13}\}$, $0 < \mathcal{V}(p) < 1$. We put $S_1 = S_0 \cup \{0 \prec \tilde{a}_i \,|\, i \in \{5, \dots, 7, 10, \dots, 13\}\} \cup \{\tilde{a}_i \prec 1 \,|\, i \in \{5, \dots, 7, 10, \dots, 13\}\}$. Let $\mathcal{V}(\tilde{a}_0) = 1$. Then $\mathcal{V} \not\models \{[1]\} \subseteq S_1 \cup S^{\phi}$. Let $\mathcal{V}(\tilde{a}_0) < 1$. Then, from [16] and [17], $\mathcal{V}(\tilde{a}_2) \in \{0,1\}$, from [6], $\mathcal{V}(\tilde{a}_3) = 1$, from [4], $\mathcal{V}(\tilde{a}_1) = \mathcal{V}(\tilde{a}_4)$, from [8] and [9], $\mathcal{V}(\tilde{a}_4) \in \{0,1\}$, $\mathcal{V}(\tilde{a}_1) \in \{0,1\}$, from [3], $\mathcal{V}(\tilde{a}_2) < \mathcal{V}(\tilde{a}_1)$, $\mathcal{V}(\tilde{a}_2) = 0$, $\mathcal{V}(\tilde{a}_4) = \mathcal{V}(\tilde{a}_1) = 1$, from [2], $\mathcal{V}(\tilde{a}_1) \cdot \mathcal{V}(\tilde{a}_0) = \mathcal{V}(\tilde{a}_2)$, $\mathcal{V}(\tilde{a}_0) = \mathcal{V}(\tilde{a}_2) = 0$. We put $S_2 = S_1 \cup \{\tilde{a}_0 \eqcirc 0, \tilde{a}_1 \eqcirc 1, \tilde{a}_2 \eqcirc 0, \tilde{a}_3 \eqcirc 1, \tilde{a}_4 \eqcirc 1\}$. In Table 4, we derive [21], [23] from $S_2 \cup S^{\phi}$. Let $\mathcal{V}(\tilde{a}_8) = 0$. Then $\mathcal{V} \not\models \{0 \prec \tilde{a}_{10}, 0 \prec \tilde{a}_{11}, [10]\} \subseteq S_2 \cup S^{\phi}$. Let $\mathcal{V}(\tilde{a}_8) = 1$. Then $\mathcal{V} \not\models \{\tilde{a}_{10} \prec 1, \tilde{a}_{11} \prec 1, [10]\} \subseteq S_2 \cup S^{\phi}$. Let $\mathcal{V}(\tilde{a}_9) = 0$. Then $\mathcal{V} \not\models \{0 \prec \tilde{a}_{12}, 0 \prec \tilde{a}_{13}, [13]\} \subseteq S_2 \cup S^{\phi}$. Let $\mathcal{V}(\tilde{a}_9) = 1$. Then $\mathcal{V} \not\models \{\tilde{a}_{12} \prec 1, \tilde{a}_{13} \prec 1, [13]\} \subseteq S_2 \cup S^{\phi}$. We put $S_3 = S_2 \cup \{0 \prec \tilde{a}_8, 0 \prec \tilde{a}_9, \tilde{a}_8 \prec 1, \tilde{a}_9 \prec 1\}$. In Table 4, we get a refutation of $S_3 \cup S^{\phi}$. We conclude that there exists a refutation of every guarded extension of S^{ϕ}; by Corollary 9 for ϕ, S^{ϕ}, ϕ is a tautology.

5 CONCLUSIONS

We have generalised the hyperresolution principle to the propositional product logic. We have proposed translation of a formula to an equivalent satisfiable finite order clausal theory. Order clauses are finite sets of order literals of the augmented form: $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is either the truth constant 0 or 1 or a conjunction of powers of propositional atoms, and $\diamond$ is the connective $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the standard equality and strict order on $[0,1]$, respectively. We have devised a hyperresolution calculus over order clausal theories. The calculus is refutation sound

Table 4: $\phi = (0 \prec c) \,\&\, (a \,\&\, c \prec b \,\&\, c) \rightarrow a \prec b$.

$$\phi = (0 \prec c) \,\&\, (a \,\&\, c \prec b \,\&\, c) \rightarrow a \prec b$$

$$\Big\{ \tilde{a}_0 \prec 1, \tilde{a}_0 \leftrightarrow \big(\underbrace{(0 \prec c) \,\&\, (a \,\&\, c \prec b \,\&\, c)}_{\tilde{a}_1} \rightarrow \underbrace{a \prec b}_{\tilde{a}_2} \big) \Big\} \qquad (9)$$

$$\Big\{ \tilde{a}_0 \prec 1, \tilde{a}_1 \prec \tilde{a}_2 \vee \tilde{a}_1 = \tilde{a}_2 \vee \tilde{a}_1 \,\&\, \tilde{a}_0 = \tilde{a}_2, \tilde{a}_2 \prec \tilde{a}_1 \vee \tilde{a}_0 = 1,$$
$$\tilde{a}_1 \leftrightarrow (\underbrace{0 \prec c}_{\tilde{a}_3}) \,\&\, (\underbrace{a \,\&\, c \prec b \,\&\, c}_{\tilde{a}_4}), \tilde{a}_2 \leftrightarrow \underbrace{a}_{\tilde{a}_5} \prec \underbrace{b}_{\tilde{a}_6} \Big\} \qquad (7), (12)$$

$$\Big\{ \tilde{a}_0 \prec 1, \tilde{a}_1 \prec \tilde{a}_2 \vee \tilde{a}_1 = \tilde{a}_2 \vee \tilde{a}_1 \,\&\, \tilde{a}_0 = \tilde{a}_2, \tilde{a}_2 \prec \tilde{a}_1 \vee \tilde{a}_0 = 1,$$
$$\tilde{a}_1 = \tilde{a}_3 \,\&\, \tilde{a}_4, \tilde{a}_3 \leftrightarrow 0 \prec \underbrace{c}_{\tilde{a}_7}, \tilde{a}_4 \leftrightarrow \underbrace{a \,\&\, c}_{\tilde{a}_8} \prec \underbrace{b \,\&\, c}_{\tilde{a}_9},$$
$$\tilde{a}_5 \prec \tilde{a}_6 \vee \tilde{a}_2 = 0, \tilde{a}_6 \prec \tilde{a}_5 \vee \tilde{a}_6 = \tilde{a}_5 \vee \tilde{a}_2 = 1, \tilde{a}_5 = a, \tilde{a}_6 = b \Big\} \qquad (16), (12)$$

$$\Big\{ \tilde{a}_0 \prec 1, \tilde{a}_1 \prec \tilde{a}_2 \vee \tilde{a}_1 = \tilde{a}_2 \vee \tilde{a}_1 \,\&\, \tilde{a}_0 = \tilde{a}_2, \tilde{a}_2 \prec \tilde{a}_1 \vee \tilde{a}_0 = 1,$$
$$\tilde{a}_1 = \tilde{a}_3 \,\&\, \tilde{a}_4, 0 \prec \tilde{a}_7 \vee \tilde{a}_3 = 0, \tilde{a}_7 = 0 \vee \tilde{a}_3 = 1, \tilde{a}_7 = c,$$
$$\tilde{a}_8 \prec \tilde{a}_9 \vee \tilde{a}_4 = 0, \tilde{a}_9 \prec \tilde{a}_8 \vee \tilde{a}_9 = \tilde{a}_8 \vee \tilde{a}_4 = 1, \tilde{a}_8 \leftrightarrow \underbrace{a}_{\tilde{a}_{10}} \,\&\, \underbrace{c}_{\tilde{a}_{11}}, \tilde{a}_9 \leftrightarrow \underbrace{b}_{\tilde{a}_{12}} \,\&\, \underbrace{c}_{\tilde{a}_{13}},$$
$$\tilde{a}_5 \prec \tilde{a}_6 \vee \tilde{a}_2 = 0, \tilde{a}_6 \prec \tilde{a}_5 \vee \tilde{a}_6 = \tilde{a}_5 \vee \tilde{a}_2 = 1, \tilde{a}_5 = a, \tilde{a}_6 = b \Big\} \qquad (7)$$

$S^{\phi} = \Big\{$

$\tilde{a}_0 \prec 1$ [1]

$\tilde{a}_1 \prec \tilde{a}_2 \vee \tilde{a}_1 = \tilde{a}_2 \vee \tilde{a}_1 \,\&\, \tilde{a}_0 = \tilde{a}_2$ [2]

$\tilde{a}_2 \prec \tilde{a}_1 \vee \tilde{a}_0 = 1$ [3]

$\tilde{a}_1 = \tilde{a}_3 \,\&\, \tilde{a}_4$ [4]

$0 \prec \tilde{a}_7 \vee \tilde{a}_3 = 0$ [5]

$\tilde{a}_7 = 0 \vee \tilde{a}_3 = 1$ [6]

$\tilde{a}_7 = c$ [7]

$\tilde{a}_8 \prec \tilde{a}_9 \vee \boxed{\tilde{a}_4 = 0}$ [8]

$\tilde{a}_9 \prec \tilde{a}_8 \vee \tilde{a}_9 = \tilde{a}_8 \vee \tilde{a}_4 = 1$ [9]

$\boxed{\tilde{a}_8 = \tilde{a}_{10} \,\&\, \tilde{a}_{11}}$ [10]

$\boxed{\tilde{a}_{10} = a}$ [11]

$\boxed{\tilde{a}_{11} = c}$ [12]

$\boxed{\tilde{a}_9 = \tilde{a}_{12} \,\&\, \tilde{a}_{13}}$ [13]

$\boxed{\tilde{a}_{12} = b}$ [14]

$\boxed{\tilde{a}_{13} = c}$ [15]

$\tilde{a}_5 \prec \tilde{a}_6 \vee \tilde{a}_2 = 0$ [16]

$\tilde{a}_6 \prec \tilde{a}_5 \vee \tilde{a}_6 = \tilde{a}_5 \vee \boxed{\tilde{a}_2 = 1}$ [17]

$\boxed{\tilde{a}_5 = a}$ [18]

$\boxed{\tilde{a}_6 = b} \Big\}$ [19]

Rule (21) : [8]$[\tilde{a}_4 = 1]$:

$\tilde{a}_8 \prec \tilde{a}_9 \vee \boxed{1 = 0}$ [20]

Rule (19) : [20] :

$\boxed{\tilde{a}_8 \prec \tilde{a}_9}$ [21]

Rule (20) : [17]$[\tilde{a}_2 = 0]$:

$\tilde{a}_6 \prec \tilde{a}_5 \vee \tilde{a}_6 = \tilde{a}_5 \vee \boxed{0 = 1}$ [22]

Rule (19) : [22] :

$\boxed{\tilde{a}_6 \prec \tilde{a}_5 \vee \tilde{a}_6 = \tilde{a}_5}$ [23]

repeatedly **Rule (18)** : $\{0 \prec p \mid p \in \{a, b, c, \tilde{a}_5, \tilde{a}_6, \tilde{a}_8, \dots, \tilde{a}_{13}\}\}$,
$\{p \prec 1 \mid p \in \{a, b, c, \tilde{a}_5, \tilde{a}_6, \tilde{a}_8, \dots, \tilde{a}_{13}\}\}$;
[10][11][12][13][14][15][21]; [18][19][23] :

$\square$ [24]

and complete for finite guarded order clausal theories. A clausal theory is satisfiable if and only if there exists a satisfiable guarded extension of it. So, the *SAT* problem of a finite order clausal theory can be reduced to the *SAT* problem of a finite guarded order clausal theory. By means of the translation and calculus, we have solved the deduction problem $T \models \phi$ for a finite theory T and a formula ϕ.

REFERENCES

Baaz, M., Ciabattoni, A., and Fermüller, C. G. (2012). Theorem proving for prenex Gödel logic with Delta: checking validity and unsatisfiability. *Logical Methods in Computer Science*, 8(1).

Baaz, M. and Fermüller, C. G. (2010). A resolution mechanism for prenex Gödel logic. In *CSL 2010*, volume 6247 of *Lecture Notes in Computer Science*, pages 67–79. Springer.

Bachmair, L. and Ganzinger, H. (1994). Rewrite-based equational theorem proving with selection and simplification. *J. Log. Comput.*, 4(3):217–247.

Bachmair, L. and Ganzinger, H. (1998). Ordered chaining calculi for first-order theories of transitive relations. *J. ACM*, 45(6):1007–1049.

Biere, A., Heule, M. J., van Maaren, H., and Walsh, T. (2009). *Handbook of Satisfiability*, volume 185 of *Frontiers in Artificial Intelligence and Applications*. IOS Press.

Bongini, M., Ciabattoni, A., and Montagna, F. (2016). Proof search and co-NP completeness for many-valued logics. *Fuzzy Sets and Systems*, 292:130–149.

Davis, M., Logemann, G., and Loveland, D. (1962). A machine program for theorem-proving. *Commun. ACM*, 5(7):394–397.

Davis, M. and Putnam, H. (1960). A computing procedure for quantification theory. *J. ACM*, 7(3):201–215.

Esteva, F. and Godo, L. (2001). Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy sets and systems*, 124(3):271–288.

Guller, D. (2009). On the refutational completeness of signed binary resolution and hyperresolution. *Fuzzy Sets and Systems*, 160(8):1162 – 1176.

Guller, D. (2010). A DPLL procedure for the propositional Gödel logic. In *ICFC 2010*, pages 31–42. SciTePress.

Guller, D. (2012a). On the satisfiability and validity problems in the propositional Gödel logic. In *Computational Intelligence - IJCCI 2010*, volume 399 of *Studies in Computational Intelligence*, pages 211–227. Springer.

Guller, D. (2012b). An order hyperresolution calculus for Gödel logic - General first-order case. In *FCTA 2012*, pages 329–342. SciTePress.

Guller, D. (2013). A DPLL procedure for the propositional product logic. In *FCTA 2013*, pages 213–224. SciTePress.

Guller, D. (2014). An order hyperresolution calculus for Gödel logic with truth constants. In *FCTA 2014*, pages 37–52. SciTePress.

Guller, D. (2015a). An order hyperresolution calculus for Gödel logic with truth constants and equality, strict order, Delta. In *FCTA 2015*, pages 31–46. SciTePress.

Guller, D. (2015b). Unsatisfiable formulae of Gödel logic with truth constants and $\eqcirc$, $\prec$, Δ are recursively enumerable. In *BRICS Congress, CCI 2015, Part III*, volume 9142 of *Lecture Notes in Computer Science*, pages 242–250. Springer.

Guller, D. (2016a). On the deduction problem in Gödel and Product logics. In *Computational Intelligence - IJCCI 2013*, volume 613 of *Studies in Computational Intelligence*, pages 299–321. Springer.

Guller, D. (2016b). Unsatisfiable formulae of Gödel logic with truth constants and Delta are recursively enumerable. In *Computational Intelligence - IJCCI 2014*, volume 620 of *Studies in Computational Intelligence*, pages 213–234. Springer.

Hájek, P. (2001). *Metamathematics of Fuzzy Logic*. Trends in Logic. Springer.

Klement, E. and Mesiar, R. (2005). *Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms*. Elsevier.

Klement, E., Mesiar, R., and Pap, E. (2013). *Triangular Norms*. Trends in Logic. Springer.

Marchioni, E. and Metcalfe, G. (2010). Interpolation properties for uninorm based logics. In *ISMVL 2010*, pages 205–210. IEEE Computer Society.

Marques-Silva, J. P. and Sakallah, K. A. (1999). Grasp: A search algorithm for propositional satisfiability. *Computers, IEEE Transactions on*, 48(5):506–521.

Mostert, P. S. and Shields, A. L. (1957). On the structure of semigroups on a compact manifold with boundary. *Annals of Mathematics*, pages 117–143.

Novák, V., Perfilieva, I., and Močkoř, J. (1999). *Mathematical Principles of Fuzzy Logic*. The Springer International Series in Engineering and Computer Science. Springer.

Robinson, J. A. (1965a). Automatic deduction with hyperresolution. *Internat. J. Comput. Math.*, 1(3):227–234.

Robinson, J. A. (1965b). A machine-oriented logic based on the resolution principle. *J. ACM*, 12(1):23–41.

Schöning, U. and Torán, J. (2013). *The Satisfiability Problem: Algorithms and Analyses*. Mathematik für Anwendungen. Lehmanns Media.

Silva, J. P. M. and Sakallah, K. A. (1996). GRASP - a new search algorithm for satisfiability. In *ICCAD 1996*, pages 220–227.